Math Objectives

- Students will identify the percent of area under any normal curve that is 1, 2, or 3 standard deviation units from the mean.
- Students will recognize that for any member of the family of normal curves, the areas whose boundaries are the same number of standard deviations from the respective means of the curves are always equal.
- Students will reason abstractly and quantitatively (CCSS Mathematical Practices).

Vocabulary

- *z*-score
- probability density function
- mean
- symmetry
- standard deviation
- standard normal curve

About the Lesson

- This lesson involves finding the area under the standard normal curve with mean 0 and standard deviation 1 for a given distance from the mean and compare this to the area under the curve for another member of the family of normal curves.
- As a result, students will:
  - Play a *Match the Area* game, and learn that the areas 1, 2, and 3 standard deviations from the mean are approximately 68%, 95%, and 99.7%, respectively, of the total area under the curve regardless of which normal curve is being examined (the empirical rule).
  - Change the mean and standard deviation to learn that the area between the boundaries determined by any given distance from the mean, when measured in standard deviation units, is equal for any member of the family of normal curves.

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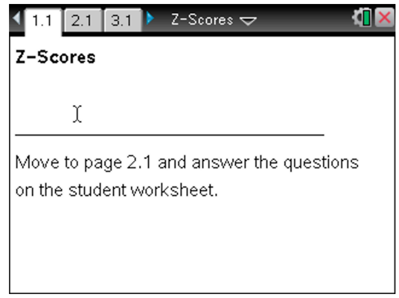
- Send the .tns file to students.
- Use Screen Capture to examine the values of x that match the area under the curve of the bottom graph to the top graph.
- Use Quick Poll questions to adjust the pace of the lesson according to student understanding.



TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point

Tech Tips:

- Make sure the font size on your TI-Nspire handhelds is set to Medium.
- You can hide the function entry line by pressing ctrl G.

Lesson Materials:

Student Activity

- Z-Scores_Student.pdf
- Z-Scores_Student.doc

TI-Nspire document

- Z-Scores.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

Discussion Points and Possible Answers

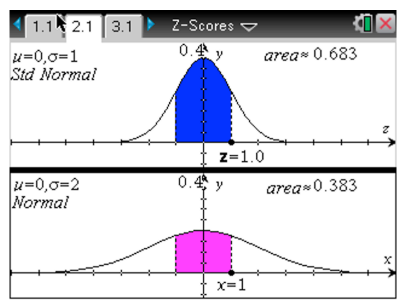
Tech Tip: If students experience difficulty dragging a point, check to make sure that they have moved the cursor (arrow) until it becomes a hand ( ) getting ready to grab the point. Also, be sure that the word *point* appears. Then press ctrl to grab the point and close the hand ( ). When finished moving the point, press esc to release the point.

Teacher Tip: The family of normal curves is defined by the function:

$$p(x) = \frac{1}{s\sqrt{2\pi}} e^{\frac{-(x-m)^2}{2s^2}}$$

. Each different value of μ and σ determines a different function in this family. Each of these normal functions is a valid probability density for a random variable. The area under the function and within a given interval gives the probability that the random variable will take on a value within that interval. In this activity, students explore the relationship between these probabilities for different normal densities.

Move to page 2.1.



1. The curve in the top graph is called the standard normal curve.
   a. What are the mean and standard deviation of the curve?

      Answer: The mean is 0; the standard deviation is 1.

   b. Slide the point on the *z*-axis, and describe what changes.

      Answer: The shaded area changes, increasing as the point is dragged to the right and the *z*-value on the horizontal axis increases.

   c. Slide the point on the *z*-axis back to $z = 1$. What is the area of the shaded region?

      Answer: 0.683 square unit

> **Teacher Tip:** Be sure students recognize that the areas are all approximate and are accurate only to three decimal places. As the boundary moves to the right, the area may appear to be 1, but it is only approaching 1, as the curve is asymptotic to the horizontal axis.

2. Consider the normal curve in the bottom graph with mean 0 and standard deviation 2.

a. How does this curve compare to the curve in the top graph?

Answer: The bottom graph is the normal curve having mean 0 and standard deviation 2. The region under the curve and between $z = 0.5$ and $z = –0.5$ has been shaded, and a point on the horizontal axis can be moved.

b. *Match the Area:* Slide the point on the x-axis until the value for area in the bottom graph matches the value for area in the top graph as closely as possible. How is the *x*-value related to the *z*-value when the area values are similar? Explain why this is reasonable.

Answer: The new endpoint is at $x = 2$, two standard deviations from the mean, which is twice the distance from the mean of the comparable endpoint in the top figure of the standard normal curve. The curve in the bottom graph is flatter and more spread out, so it makes sense that you would have to move farther than 1 along the *x*-axis to have the same area under both curves.

c. Repeat *Match the Area* in the top graph using each *z*-value below. Describe the area in each case, and explain how the boundaries of corresponding regions appear to be related.

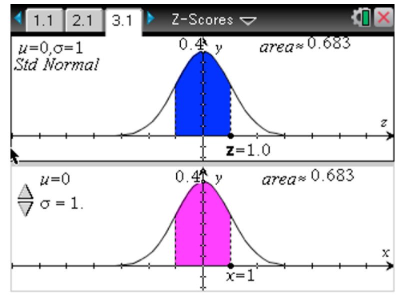
(i) $z = 2$

Answer: When $z = 2$ in the standard normal curve, the corresponding *x*-value that has the same area, 0.954 square unit, in the bottom graph is at 4.

(ii) $z = 3$

Answer: When $z = 3$, to have the same area, 0.997 square unit, in the bottom graph as in the top graph, the *x*-value in the bottom graph is at 6.

> **TI-Nspire Navigator Opportunity:** ***Screen Capture***
> **See Note 1 at the end of this lesson.**



Move to page 3.1.

3. The top graph on this page is a copy of the top graph on page 2.1 but with $z = 1.0$. The bottom graph on this page is the normal curve with a mean of 0 and standard deviation controlled by the slider σ.
   a. Adjust the slider so that the standard deviation is 1.5, and play *Match the Area* again. How is the *x*-value you obtained related to the *z*-value shown in the top graph?
   Answer: The new boundary is 1.5 units from the mean and is 1.5 times the *z*-value, which is 1, in the standard normal curve.

   b. Repeat *Match the Area* in the top graph using each *z*-value below. Compare your answers to those you found in Question 2 part c.
   (i) $z = 2$

   Answer: When $z = 2$ in the standard normal curve, the corresponding *x*-value that has the same area, 0.954 square units, in the bottom graph is at 3, twice the standard deviation, 1.5, of that curve.

   (ii) $z = 3$

   Answer: When $z = 3$, to have the same area, 0.997 square units, the *x*-value in the bottom graph is at 4.5, or 3 times the standard deviation, 1.5.

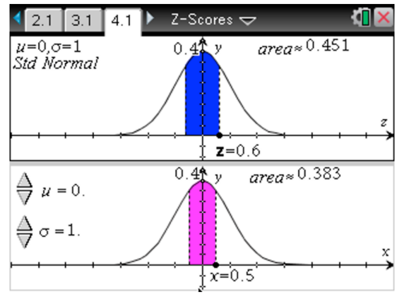
   Teacher Tip: Students may not recognize the fact that the boundary value is the product of the standard deviation and the number of standard deviations from the mean at this stage but should do so after they answer Question 4.

TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 2 at the end of this lesson.

4. Repeat *Match the Area* with several more choices for σ. Summarize your conclusions about the relationship between corresponding boundary values and standard deviations.

   Answer: The area, when the boundary line is 1 standard deviation away from the mean on any member of the family of normal curves with mean = 0, seems to be about 0.683 square unit; the area, when the boundary line is 2 standard deviations away from the mean, is always about 0.954 square units; and the area with a boundary line 3 standard deviations from the mean is always about 0.997 square units.

Move to page 4.1.



5. This page shows the standard normal curve (top) and a normal curve having the mean controlled by slider μ and the standard deviation controlled by slider σ (bottom).
   a. Adjust the sliders so that the mean is 1.5 and the standard deviation is 1. Describe as carefully as possible how the top and bottom graphs are related.

   Answer: The two graphs are congruent. The normal curve on the bottom has been translated or shifted to the right 1.5 units, but the shape has remained the same.

   b. Based on your description in part a, predict the value for the right-hand boundary of the shaded region in the bottom graph when the area is as close as possible to 0.683 square units. Slide the right endpoint to verify your prediction. Explain why this makes sense.

   Answer: The right boundary of the shaded region should be at 2.5 because the whole graph has just been shifted over 1.5 units. Thus, the boundary line 1 standard deviation away from the mean in the standard normal curve would be shifted 1.5 units to the right to 2.5.

   c. Repeat *Match the Area* with the settings for μ and *σ* as in part a, using the *z*-values given below in the standard normal curve. In each case, how far is the final right boundary from the mean?
   (i) $z = 2$

   Answer: The area 2 standard deviations from the mean, about 0.954 square units, will have the right boundary line at 3.5.

   (ii) $z = 3$

   Answer: The area 3 standard deviations from the mean, about 0.997 square units, will have the right boundary line at 4.5.

6. Adjust the mean and standard deviation sliders to values of your own choosing. For each new setting, predict the right endpoint values of *x* needed to obtain equal areas when the endpoints of the top graph are $z = 1$, $z = 2$, and $z = 3$.
   a. Explain how you obtained your predicted values. Play *Match the Area* to verify your prediction.

   Answer: The final right boundary lines will always be the mean plus 1, 2, or 3 standard deviations.

b. What is the relationship among the shape of the normal curve, the standard deviation of the curve, and the boundary lines for areas within 1, 2, and 3 standard deviations from the mean?

Answer: The shape of the normal curve is determined by the standard deviation; when the standard deviation is larger than 1, the curve begins to flatten out. Thus, to have the same amount of area as the standard normal curve between two boundaries, the boundaries have to be located farther along the horizontal axis. Shifting the mean changes the location of the boundary lines for a given area, but the area is congruent to the area between the original boundary lines for that standard deviation in the standard normal curve.

7. Summarize the results of your work thus far. Be sure to specify particular values of area and associated boundaries and how these relate to the mean and standard deviation of the distribution you are investigating.

Answer: Students should note that you can find areas under any normal curve that match a given area under the standard normal curve by using the boundary lines for the area determined by the number of standard deviations from the mean in the standard normal curve and multiplying that by the standard deviation of the normal curve under consideration. The mean changes the location of the boundary lines for a given area, but the area is equal to the area between the original boundary lines for that standard deviation in the standard normal curve.

8. a. If you know the right boundary line at 6.5 for a normal curve is 2 standard deviations from the mean and the left boundary line is at –4.5, explain how you would find the mean.

Answer: Because the normal curve is symmetric and the mean occurs at the line of symmetry in the curve, the mean will be exactly halfway between the two boundary lines, or at 1.

b. What is the area between the two boundary lines in part b?

Answer: The area between the two boundary lines is about 0.954 square unit.

Teacher Tip: If students have trouble with b and c, have them use the .tns file.

9. Using an area of 0.451 square unit, repeat *Match the Area*.

a. Identify the *z*-value for the right boundary line in the standard normal curve, and explain what it tells you about the location of the boundary line.

Answer: The *z*-value is 0.6, which means the point is located $\frac{6}{10}$ of a standard deviation from the mean.

b. For a mean of 0.5 and a standard deviation of 2.5, find the right boundary line in the curve in the bottom graph that would give an area of 0.451 square unit. How are the mean and standard deviation related to the boundary lines in the standard normal curve?

Answer: The right boundary in the bottom curve is at $x = 2.0$, or 1.5 units from the mean of 0.5. Note that 1.5 is 0.6 times the standard deviation of 2.5.

c. Choose a new μ and σ. Answer the questions in part b again.

Sample Answers: Student answers will vary, but each time they should find the boundaries when the bottom curve has an area of about 0.451 square units are the product of the standard deviation for the new curve and the distance from the mean of the standard normal curve plus the mean they selected for their bottom graph.

d. Choose a new area and answer the questions in parts a through c again.

Answer: Students should reach the same conclusion as they did in part c for any area they choose: the boundaries when the bottom curve has matching area will be the product of the standard deviation for the new curve and the distance from the mean of the standard normal curve plus the mean, i.e., boundary = $(\sigma)z + \mu$.

10. Summarize the results of your explorations. In your summary, include details of how corresponding distances to the mean are related.

Answer: In general, for any given standard deviation, z, from the mean of the standard normal curve, the area determined by the boundary line through z will be equal to the area of any member of the family of normal curves with boundary lines through $\mu + (\sigma)z$ and $\mu - (\sigma)z$.

Wrap Up

Upon completion of the discussion, the teacher should ensure that students are able to understand:

- The 68%, 95%, 99.7% empirical rule for area 1, 2, and 3 standard deviations from the mean for any normal curve.
- That regions under two normal curves will always have the same area if their boundary lines are equal distances from their means when measured in standard deviation units.

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Note 1

Question 2, *Screen Capture*

Use Screen Capture to see if every student found the correct *x*-value, 2 part a, 4 and 6 for part c. Discuss the connection to the standard deviation.

Note 2

Question 3, *Quick Poll*

Use an open response Quick Poll to see what value of *x* each student found when z = 1, 2, and 3. Discuss any deviations from the correct *x*-value.

Extension

Questions 10 through 12 are not included on the student handout but are follow-up questions that would help students make the connection between the exploration and the numerical ways to describe the relationship between the means and the standard deviations for any member of the family of normal curves to the standard normal curve that leads to the formula for *z*-scores.

10. Suppose you know that the boundaries of a central region having some desired area under the standard normal curve are –1.8 and 1.8. Predict the boundaries for a central region having the same area under the normal curve with mean μ and standard deviation σ.

    Answer: The boundaries would be $\mu + 1.8\,\sigma$ and $\mu - 1.8\,\sigma$.

    > **Teacher Tip:** If students struggle with finding the boundaries, suggest they experiment with page 3.1 to find the impact of *s* and page 4.1 to find how both μ and σ affect the outcomes. For example, they might set the standard normal to +1.8 and –1.8 on 3.1 and guess at what they think the correct boundary would be when the mean is 0 and the standard deviation is 2. Then they can check and try again when the standard deviation is 3. Once they note that the boundaries become multiples of the standard deviation, move to page 4.1.

11. Suppose a friend has been playing *Match the Area* using a normal curve with mean μ and standard deviation σ. The boundaries of the resulting symmetric central region for some particular area were found to be –4.2 and 12.6.

    a. If possible, determine the value of μ. If it is not possible, explain why not.

       Answer: μ = 4.2 (the average of the endpoints)

b. If possible, determine the value of σ. If it is not possible, explain why not.

Answer: The value of σ cannot be determined without knowing the friend's area.

c. What values do you think appeared in your friend's "top graph"—the standard normal curve—for his particular area? Give numerical values if possible. If not, write your answers in terms of μ and σ.

Answer: The endpoints on the standard normal graph are the same distance from their mean (0) as those on the bottom graph when measured in SD units. So, if *z* is the right endpoint in the standard normal curve, then *z* (in 1σ) must be 8.4 (in σ s), so $\frac{z}{1} = \frac{8.4}{\sigma}$.

Values along the horizontal axis of the standard normal graph are usually called ***z*-scores**. Thus, 1.8 is the *z*-score for the right endpoint found in Question 8. Likewise, –1.8 is the *z*-score for the left endpoint found there.

12. Write a formula relating a *z*-score to its corresponding value, *x*, in a normal distribution having mean μ and standard deviation σ.

Answer: $x = \sigma z + \mu$ or $z = \frac{x - \bar{x}}{\sigma}$

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