Leslie Matrices

Teacher Notes & Answers

7 8 9 10 11 **12**

TI-Nspire™ Investigation Student 45 min

Teacher Notes:

Teachers are encouraged to watch the webinar on Leslie Matrices. The first question requires students to do the calculations for each transition by hand, without using matrices. The idea here is to show students that the Leslie matrix is a powerful tool that can be used to expedite and simplify calculations, rather than a disconnected 'thing' to learn. The Leslie matrix is another example of a recursive calculation, an extension of the singular dimensioned recursive sequences previously studied.

In this investigation students do not have to set up an entire Leslie matrix, rather edit an existing matrix, with the view to building an understanding of how the matrix is created. The use of the Notes application makes it quick for students to explore small changes in the original population and transition conditions.

The extension section is not compulsory as Eigenvalues and Eigenvectors are not specifically mentioned on the VCE General Mathematics course, however, they are introduced here through commands only and at a simplistic level. The purpose is to show students that additional numerical methods are available to determine the general behaviour of a population being described by a Leslie matrix.

Introduction

Leslie matrices are a type of transition matrix, designed specifically to model the change in age distribution over time for a population. This activity explores the age distribution of a particular dung beetle. For the purpose of this activity, the discrete one-year age categories are:

- Juvenile (J)
- Adult (A)
- Senior (S)

Juvenile dung beetles do not breed and only have a 40% likelihood of maturing to become an Adult beetle. Adult beetles can breed, producing on average 2 Juvenile beetles per Adult, they have only a 35% probability of maturing to become a Senior beetle. Senior beetles are also capable of breeding, producing on average 1.5 Juveniles per Adult. Senior beetles do not live beyond a year.

Dung Beetle – Warm Up

Dung beetles, as their name suggests, live off the faeces of other animals. As gross as this may sound, in so doing, they perform a very important role, reducing piles of excrement into organic matter, one mouthful at a time. The dung beetle collects a pile of excrement, rolls it up into a ball and then rolls that ball back to its nest, a hole in the ground. The female beetle lays eggs inside this ball. As the eggs hatch the Juvenile beetles eat their way out and eventually grow into Adult beetles, capable of breeding and rolling their own balls of excrement. Needless to say, farmers generally welcome the presence of dung beetles as they clean up and fertilise all at the same time.

Author: P. Fox

Question: 1.

Imagine you're a farmer and want to have these little dung wonders clean up around the stables. Suppose you buy a pile of excrement containing 150 Juvenile beetles only.

a) Use the information provided above to help determine the quantity of Juvenile, Adult and Senior beetles after:

i) One year ii) Two years iii) Three years iv) Four years

Answers:

Yr	Juvenile (J_0 = 150)	Adult (A_0 = 0)	Senior (S_0 = 0)
1	J_1 = 0 x **150** + 2 x **0** + 1.5 x **0** = 0	A_1 = 0.4 x **150** + 0 x **0** + 0 x **0** = 60	S_1 = 0 x **150** + 0.35 x **0** + 0 x **0** = 0
2	J_2 = 0 x **0** + 2 x **60** + 1.5 x **0** = 120	A_2 = 0.4 x **0** + 0 x **60** + 0 x **0** = 0	S_2 = 0 x **0** + 0.35 x **60** + 0 x **0** = 21
3	J_3 = 0 x **120** + 2 x **0** + 1.5 x **21** = 31.5	A_3 = 0.4 x **120** + 0 x **0** + 0 x **21** = 48	S_3 = 0 x **120** + 0.35 x **0** + 0 x **21** = 0
4	J_3 = 0 x **31.5** + 2 x **48** + 1.5 x **0** = 96	A_4 = 0.4 x **31.5** + 0 x **48** + 0 x **21** = 12.6	S_3 = 0 x **31.5** + 0.35 x **0** + 0 x **21** = 16.8

b) What do you think will happen to the farmer's population of dung beetles in the long run? Will it thrive, demise or reach some sort of equilibrium?

Answer: Based on the information students have calculated for the first 4 years, it is reasonable for students to think that the dung beetle population may disappear since the total population is still less than the original amount. There is however an enormous variation in the population and population distribution in the first couple of weeks.

Teacher Note: The purpose of the question is to build the desire to generate more data.

c) Suppose your ball of excrement came with 150 Juvenile beetles and 100 Adult beetles. Determine the quantity of beetles after one, two, three and four years and comment on your findings.

Answers:

Yr	Juvenile (J_0 = 150)	Adult (A_0 = 100)	Senior (S_0 = 0)
1	J_1 = 0 x **150** + 2 x **100** + 1.5 x **0** = 200	A_1 = 0.4 x **150** + 0 x **100** + 0 x **0** = 60	S_1 = 0 x **150** + 0.35 x **100** + 0 x **0** = 35
2	J_2 = 0 x **200** + 2 x **60** + 1.5 x **35** = 172.5	A_2 = 0.4 x **200** + 0 x **60** + 0 x 35 = 80	S_2 = 0 x **200** + 0.35 x **60** + 0 x 35 = 21
3	J_3 = 0 x **172.5** + 2 x **80** + 1.5 x **21** = 191.5	A_3 = 0.4 x **172.5** + 0 x **80** + 0 x **21** = 69	S_3 = 0 x **172.5** + 0.35 x **80** + 0 x **21** = 28
4	J_3 = 0 x **191.5** + 2 x **69** + 1.5 x **28** = 180	A_4 = 0.4 x **191.5** + 0 x **69** + 0 x **28** = 76.6	S_3 = 0 x **191.5** + 0.35 x **69** + 0 x **28** = 24.15

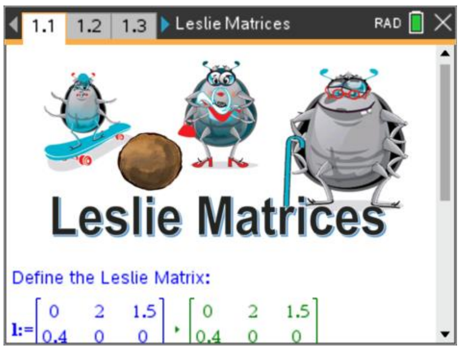
Comment: Students should note there are a lot more dung beetles in this scenario. The inclusion of some adult beetles sees the immediate re-population of the Juvenile beetles.

Creating a Leslie Matrix

The previous questions provide some clues as to the repetitive and recursive calculations required to track population growth and distribution.

Open the TI-Nspire™ file: "**Leslie Matrices**".

Page 1.1 consists of a Notes Application. Scroll down to see the Leslie matrix (L) and the population vector or matrix (S).



Author: P. Fox

Examine the top row of the Leslie matrix and compare it to the data provided on the previous page.

$$\begin{bmatrix} 0 & 2 & 1.5 \\ 0.4 & 0 & 0 \\ 0 & 0.35 & 0 \end{bmatrix}$$

The **first row**, often referred to as the fecundity row, refers to the reproduction values.

Column 1: Reproduction rate from Juveniles.
Juveniles too young to reproduce new Juveniles

Column 2: Reproduction rate from Adults.
Each adult produces 2 Juveniles

Column 3: Reproduction rate from Seniors.
Each senior produces 1.5 Juveniles

These are the factors that effect the next population of Juveniles.

Now examine the **second row** of the Leslie matrix.

$$\begin{bmatrix} 0 & 2 & 1.5 \\ 0.4 & 0 & 0 \\ 0 & 0.35 & 0 \end{bmatrix}$$

The second row represents the transition options into Adult beetles.

Column 1: 40% of Juvenile beetles transition and become Adults

Column 2: 0% of Adult beetles remain as Adults

Column 3: 0% of Senior beetles transition to Adults.

Question: 2.

Study the Leslie matrix on Page 1.1 and identify the meaning of the third row.

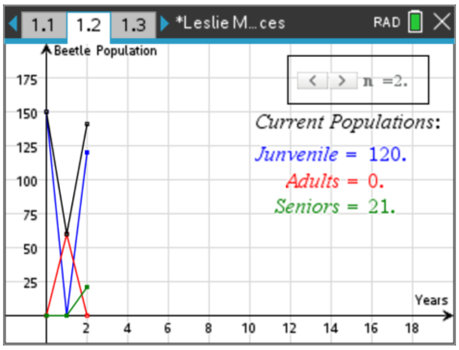
Answer: The third row represents the transition row to "Senior" beetles. Juveniles do not contribute to the next generation of Seniors, 35% of Adults survive to contribute to the next generation of Senior beetles, and Senior beetles do not contribute to the next generation of Senior beetles.

Navigate to Page 1.2 in the TI-Nspire Leslie Matrix document.

The slider (n) can be used to navigate through time periods where n = 0 represents the initial population for each dung beetle stage.

Use the slider to determine the population of each dung beetle stage from n = 0 through to n = 4.

Check the results against the values you calculated in Question 1.



Question: 3.

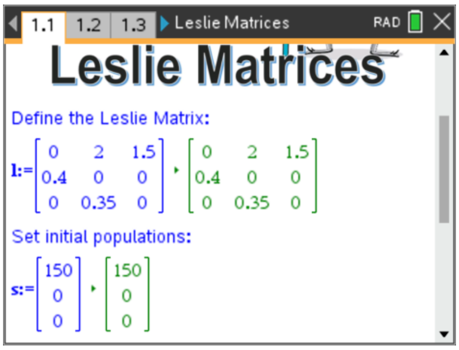
Continue clicking on the slider to see what the dung beetle population will be 20 years from the initial population of 150 Juvenile dung beetles. What is happening to the population?

Answer: The proportion of each beetle generation fluctuates enormously in the early stages. After approximately 12 years each generation has reached an almost steady state: Juveniles: 73 (65%), Adults: 29 (26%) and Seniors: 10 (9%). The overall population is almost constant by the 12th with almost no change (111.5 to 114.4) in the final 8 years.

Author: P. Fox

Navigate back to Page 1.1 of the TI-Nspire Leslie Matrix document.

Leslie Matrix and initial population vector are assigned in Maths Boxes. Navigate to the population vector and change the original quantity of Juvenile dung beetles from 150 up to 200.



Question: 4.

Suppose the farmer buys a bigger pile of excrement containing 200 pile of excrement. What happens to this population? Compare the percentage of each age grouping after 20 years and discuss your observations.

Answer: The patterns for 200 Juvenile beetles is very similar to the initial population of 150. The proportion of each beetle generation fluctuates enormously in the early stages. After approximately 12 years each generation has reached an almost steady state: Juveniles: 98 (65%), Adults: 37 (26%) and Seniors: 14 (9%). The overall population is almost constant. In the 12th year, the overall population is 148.7. In the 20th year the overall population is 152.5, an increase of just 4 beetles in 8 years!

Question: 5.

The farmer sources another pile of excrement from a different supplier. This time he has 150 Juvenile beetles and 50 adults. Edit the population vector (matrix S) in Page 1.1 and explore the impact on Page 1.2.

a) Determine the proportions for each age type after 20 years.

Answer: The 'steady' state is reached much quicker (graphs not as variable at the start). The final proportions are the same as before: 65% Juvenile, 26% Adult and 9% Senior.

b) Comment on the overall population after 20 years.

Answer: The overall population after 20 years is now greater than the initial population growing from 200 to 210. [Steady state is reached much quicker allowing the population to grow naturally]

Question: 6.

This time the farmer places a special order: 130 Juveniles, 50 Adults and 20 Seniors. Edit the population vector on Page 1.1 and explore this result.

a) How does the Population v Time graph compare to previous graphs? Explain

Answer: The population time graph shows almost no variation. The farmer has ordered proportions closely aligned to the long-term outcomes, therefore the 'stabilisation' process is bypassed allowing the Dung Beetle population to expand naturally.

b) Compare the overall population after 20 years with an alternative order of 200 Juveniles.

Answer: The 130:50:20 order reaches a population of 217 after 20 years compared with 152 when 200 Juveniles are ordered with no Adults or Seniors. The 200 Juvenile order takes approximately 10 years to stabalise; the natural order is stable from the outset.

Question: 7.

A new biologically sound spray has been created to help keep the juvenile dung beetles safe from predators. The spray boosts the survival rate up slightly from 40% up to 50%. [Set the order to 200 Juveniles only.]

a) Determine the proportions for each age type after 20 years.

Answer: Juvenile: 62.6%, Adults: 28.5% and Seniors: 8.8%

b) Comment on the overall population after 20 years and the potential long-term dung beetle population.

Answer: There has been a population explosion! The small increase in survival rates of the Juvenile dung beetles results in a population of 1085 after 20 years! The population explosion on the farmer's property would mean the farmer wouldn't need to continue purchasing Dung Beetles.

Extension

The biological balance can be quite delicate. Relatively small changes in the ecosystem can quickly and easily blow out to massive changes, so much so they are sometimes referred to as explosions. Altering the dung beetle age distribution or transition proportions support this notion. Furthermore, taking a small snap shot in time is not always indicative of long-term outcomes. Consider for example the fluctuations in the dung beetle population in the early stages of colonisation, particularly where the age group proportions are not in line with long term predictions.

There are mathematical methods that help predict long-term outcomes, Eigen Values and Eigen Vectors, both can be obtained from the calculator.

Navigate back to Page 1.1 of the TI-Nspire Leslie Matrix document.

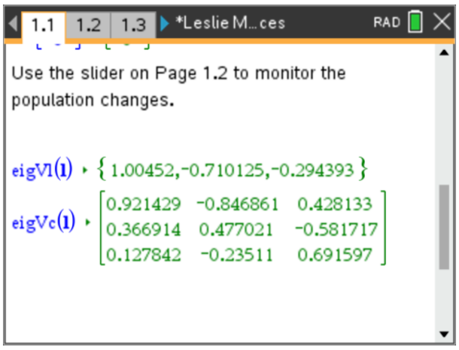
Return the initial conditions to the Leslie Matrix and population vector.

Scroll down to the Maths boxes for:

- Eigen Values (eigVl)
- Eigen Vector (eigVc)

The largest Eigenvalue for the **original** transition information is 1.0045.

The corresponding Eigenvector is located in the corresponding column in the matrix: 0.9214, 0.3669 and 0.1278.



Navigate to Page 1.2 and set n = 30.

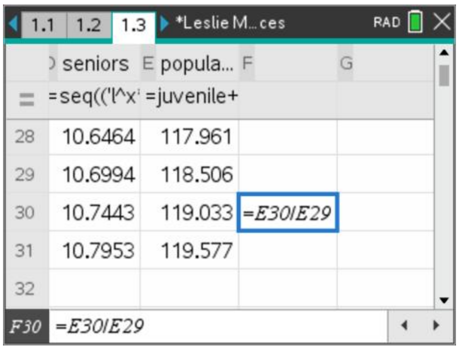
Now navigate to page 1.3. This page is a spreadsheet application containing all the raw data for the dung beetles.

Press Ctrl + G and enter the cell reference: F30.

Column E contains the total dung beetle population. Notice that it is increasing slightly each year. To see this grow rate, enter the formula:

= E30/E29

Find the growth rate across E31 and E30.



The Spreadsheet Application on page 1.3 contains a lot of calculations. To force all cells to be recalculated press: **Ctrl + R** (re-calculate). This may be required from time to time to ensure all calculations reflect the most up to date status of each variable.

Question: 8.

Compare the calculated growth rates for the 29th and 30th years with the largest Eigenvalue on Page 1.1, comment on your findings.

Answer: The growth rate for the 29th and 30th years are almost identical to the Eigenvalue

Teacher Note: Students are not expected to know, or even understand, how the Eigenvalue is obtained. The purpose is to simply show students that such tools exist and are possible even without running the simulation.

Question: 9.

Navigate to cell H1. This group of cells contain the proportion of Juvenile, Adult and Senior beetles in the n^{th} year. Compare these calculated values to the Eigenvector on Page 1.1

Example: From the Eigenvector, Juvenile proportion = $\dfrac{0.9214}{0.0214+0.3669+0.1278} \approx ?$

Answer: The population proportions are almost the same (3 decimal places) after 20 years for each category.

Question: 10.

Change the transition rate from Juvenile to Adult beetles to 0.35 on Page 1.1

a) What do you notice about the largest Eigenvalue?

Answer: The largest Eigenvalue is now less than one.

b) What do you notice about the overall population on Page 1.2?

Answer: The population is declining quite rapidly.

Question: 11.

Change the transition rate from Juvenile to Adult beetles to 0.45 on Page 1.1

a) What do you notice about the largest Eigenvalue?

Answer: The largest Eigenvalue is greater than one, and greater than before.

b) What do you notice about the overall population on Page 1.2?

Answer: The population is 'exploding'!

c) Comment on your findings from Questions 9, 10 and 11 with regards to the largest Eigenvalues and Eigenvectors.

Answer: The Eigenvalues provide information about decay / growth (dilation of vector). If the maximum Eigenvalues is greater than 1, the population is be growing, if the Eigenvalue is less than 1, the population is shrinking. Eigenvectors provide the relative populations for the long-term outcomes.
Note: In the first example in this activity, the Eigenvalue was 1.004, a very small growth factor which made it look like the beetle population was not growing, (steady state) where as the populations were in fact the lines were actually increasing slightly.

Author: P. Fox