Concepts

The purpose of this document is to investigate several methods to estimate the area under a curve using known geometric figures. A Riemann sum of the continuous function f over the interval $[a,b]$ with partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

where $\Delta x_i = x_i - x_{i-1}$ is the length of the ith subinterval and $x_i^* \in [x_{i-1}, x_i]$ is an arbitrary value in the ith subinterval. For a given partition P, the way in which we select the values x_i^* determines the Riemann sum.

Let S be the region in the plane bounded above by the graph of a nonnegative, continuous function $y = f(x)$, below by the x-axis, on the left by the line $x = a$, and on the right by the line $x = b$. The following Riemann sums can be used to estimate the area of the region S.

Left Riemann Sum

Let x_i^* be the left endpoint of each subinterval, $x_i^* = x_{i-1}$ for all i. Then

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x_i$$

Right Riemann Sum

Let x_i^* be the right endpoint of each subinterval, $x_i^* = x_i$ for all i. Then

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x_i$$

Midpoint Riemann Sum

Let x_i^* be the midpoint of each subinterval, $x_i^* = \overline{x}_i = \frac{1}{2}(x_i + x_{i-1})$ for all i. Then

$$M_n = \sum_{i=1}^{n} f(\overline{x}_i)\Delta x_i$$

We can also use trapezoids to estimate the area of the region S.

Trapezoidal Sum

$$T_n = \sum_{i=1}^{n} \frac{1}{2}[f(x_{i-1} + f(x_i)]\Delta x_i$$

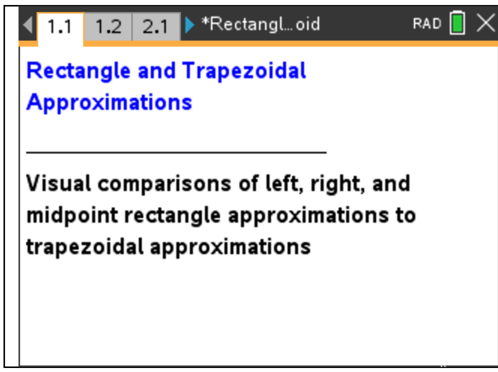
For a regular partition, all of the subintervals are of equal length, $\Delta x_i = \Delta x = \dfrac{b-a}{n}$.

Course and Exam Description Unit
Section 6.2: Approximating Areas with Reimann Sums

Calculator Files
RectangleTrapezoid.tns

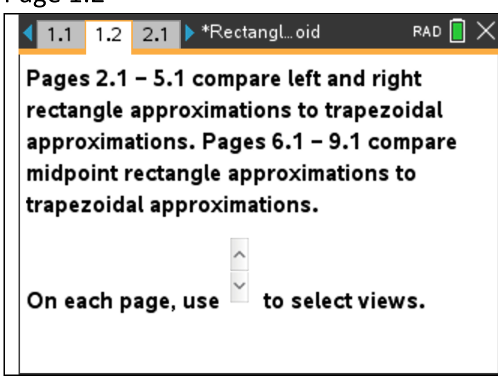
Using the Document
RectangleTrapezoid.tns: This file is used to graphically compare left, right, and midpoint rectangle approximations, and trapezoidal approximations to the area under a curve. Various examples are presented (using a single subinterval and single rectangle and/or trapezoid) to help determine when each of these approximations is an overestimate or underestimate, and to suggest which estimate produces the smallest error in estimating the area under a curve. Given certain characteristics of the curve, the graphs can be used to order the approximations. And the examples allow the user to describe how concavity affects the midpoint and trapezoidal approximations.

Page 1.1



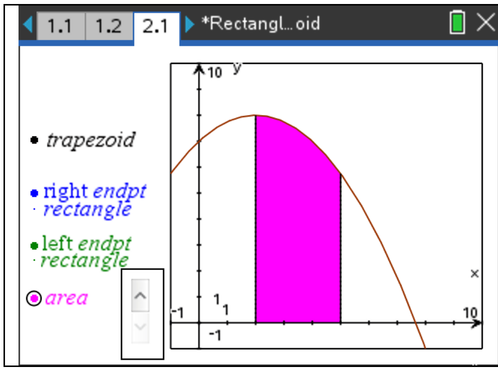
This welcome screen provides the intent of this activity; to visually, or graphically, compare left, right, and midpoint rectangle approximations, and trapezoidal approximations.

Page 1.2



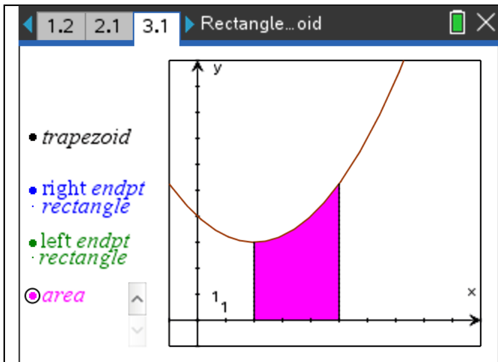
On pages 2.1 – 5.1, the user can compare rectangle approximations to trapezoidal approximations. Each curve exhibits different characteristics. On pages 6.1 – 9.1, the user can compare a midpoint approximation to the corresponding trapezoidal approximations. And each curve exhibits different characteristics. The sliders are used to toggle between views.

Page 2.1

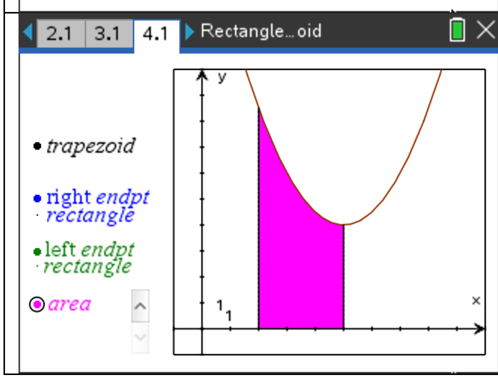


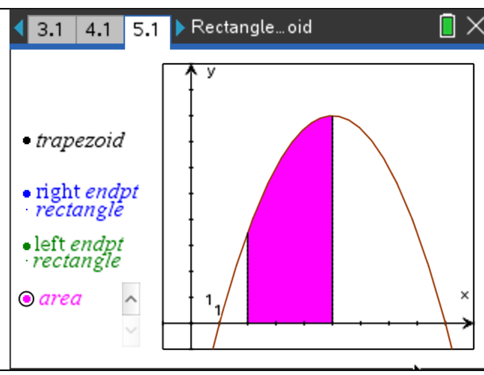
The default view shows the graph of a function f where the shaded area represents the area under the curve over one subinterval. The slider is used to toggle between different (shaded region) views: the exact area under the curve, an estimate of the area under the curve using a rectangle constructed using the left endpoint of the subinterval, an estimate of the area under the curve using a rectangle constructed using the right endpoint, and an estimate of the area under the curve using a trapezoid.

Pages 3.1 – 5.1

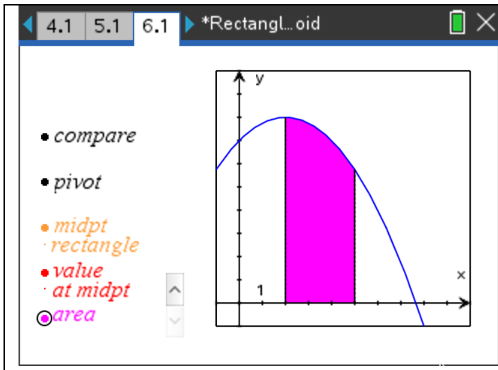


These three calculator pages are similar to Page 2.1. However, each shows the graph of a function f with different characteristics. The same four views are available using the slider: exact area, left endpoint rectangle, right endpoint rectangle, and trapezoid.



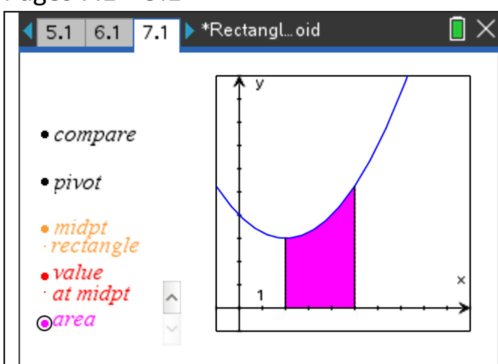


Page 6.1

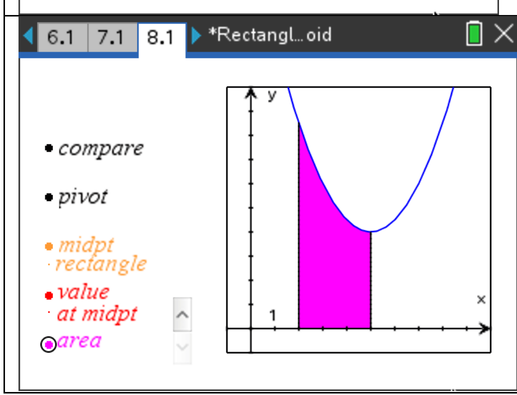


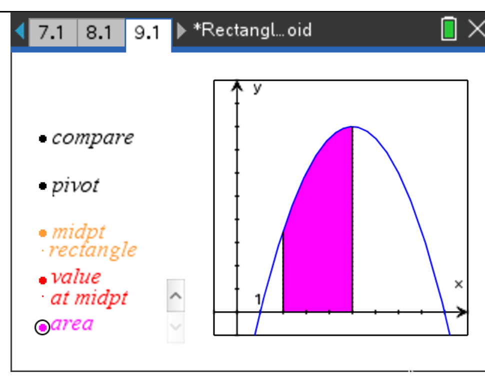
The default view shows the graph of a function f where the shaded area represents the area under the curve over one subinterval. The slider is used to toggle between different views: the shaded region that represents the exact area under the curve, the point on the graph of f used to construct a midpoint rectangle (value at midpoint), the shaded region that represents an estimate of the area under the curve using the midpoint of the subinterval, and the shaded region that represents an estimate of the area under the curve using a trapezoid. Using the slider repeatedly, the pivot option shows the top of the midpoint rectangle rotating to a position parallel to the top of a trapezoid (used to estimate the area under the curve). The compare option shows the trapezoid and a slice of the pivoted midpoint rectangle.

Pages 7.1 – 9.1



These three calculator pages are similar to Page 6.1. However each shows the graph of a function f with different characteristics. The same four views are available using the slider: exact area, value at the midpoint, midpoint rectangle, pivot, and compare (with a trapezoid).





Suggested Applications and Extensions

Page 2.1

Use the slider arrow to view the shaded regions that represent the exact area under the curve and the estimates of the area under the curve.

1. Explain how each estimate region is constructed: left endpoint rectangle, right endpoint rectangle, and trapezoid.
2. Which of these three estimates is an overestimate for the exact area? Which of these is an underestimate? Which estimate do you think is the closest to the exact area?
3. Do you think that your answers to Problem 2 will be true for any function? Explain you reasoning.

Page 3.1

1. Use the slider to observe the estimates of the shaded area under the curve. Which of these three estimates is an overestimate for the exact area? Which is an underestimate? Are these results consistent with your answer to Problem 3 on Page 2.1? If so, explain why? If not, how would you adjust your prediction for overestimates and underestimates?
2. When will the trapezoid be an overestimate? When will it be an underestimate? When will it be a better estimate than the left endpoint rectangle and the right endpoint rectangle?

Pages 4.1 and 5.1

1. Use the slider to view the estimates of the shaded are under the curve, on both pages. Use your observations from Pages 2.1-5.1 to explain when a left endpoint rectangle is an underestimate or overestimate, when a right rectangle is an underestimate or overestimate, and when a trapezoid is an underestimate or overestimate.
2. Will a left rectangle ever provide the exact area under the curve? If so, how? If not, why not? Answer these same questions for a right rectangle and for a trapezoid.

Page 6.1

1. Use the slider to view the exact are under the curve, the point on the curve use to construct the midpoint rectangle, and the midpoint rectangle. In your own words, explain how a midpoint rectangle is constructed. Why is it difficult to determine whether the area of a midpoint rectangle is an overestimate or underestimate?
2. Use the slider to select pivot. Click on pivot several times.
   (a) Explain how the midpoint rectangle changes.
   (b) As the top segment of the midpoint rectangle pivots, or rotates, how does the shaded area compare to the original midpoint rectangle? Justify your answer.
   (c) Once the segment is done pivoting, a dashed line appears. What does the dashed line represent?
   (d) Is the area of the midpoint rectangle an overestimate or an underestimate? Explain your reasoning.

3. In this case, which approach provides a more accurate estimate of the exact area under the curve: a midpoint rectangle or a trapezoid? Explain your reasoning.

Pages 7.1, 8.1, and 9.1

1. The functions presented on these pages are all nonnegative (over the subinterval considered), but may be increasing or decreasing, and may be concave up or concave down. Observe the estimates in each case and use your results from pages 3.1-9.1 to determine whether each approximation (L: left rectangle, R: right rectangle, M: midpoint rectangle, T: trapezoid) to the exact area under the curve is an underestimate or overestimate.
   (a) The function is nonnegative, increasing, and concave up.
   (b) The function is nonnegative, decreasing, and concave up.
   (c) The function is nonnegative, increasing, and concave down.
   (d) The function is nonnegative, decreasing and concave down.
2. How do your answers in Problem 1 change if *nonnegative* is replaced by *negative*? Explain.

In the next Problems, use your observations about a single subinterval to draw conclusions about the relationship among left, right, midpoint, and trapezoidal sums.

1. Suppose the function f is positive, continuous, increasing, and concave up on an interval $[a,b]$. Determine whether the inequality is true. Justify you answers.
   (a) $L_n \leq M_n \leq R_n$
   (b) $L_n \leq T_n \leq M_n$
   (c) $M_n < \dfrac{L_n + R_n}{2}$
   (d) $L_{2n} > L_n$

2. Suppose the function f is positive, continuous, decreasing, and concave down on an interval $[a,b]$. Determine whether the inequality is true. Justify you answers.
   (a) $L_n \leq M_n \leq T_n \leq R_n$
   (b) $L_n \geq M_n \geq T_n \geq R_n$
   (c) $L_n \geq T_n \geq M_n \geq R_n$
   (d) $L_n \geq R_n \geq M_n \geq T_n$

3. Let A be the area of the region bounded above by the graph of $y = e^{-x}$, below by the x-axis, on the left by the line $x = 0$, and on the right by the line $x = 8$.
   (a) List the values L_n, R_n, and A in increasing order. Justify your answer.
   (b) Is T_n or M_n a better estimate of A? Justify your answer.