Concepts

If a function f has derivatives of all orders, then under certain conditions we can write f as a Taylor series centered at a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

In the case where $a = 0$, the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

This is called a Maclaurin series.

This expression means that $f(x)$ is the limit of the sequence of partial sums. The partial sums are

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$
$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The expression for $T_n(x)$ is a polynomial of degree n. $T_n(x)$ is called the nth-degree Taylor polynomial of f at $x = a$, or centered at a.

Course and Exam Description Unit

Section 10.11: Finding Taylor Polynomial Approximations of Functions
Section 10.14: Finding Taylor or Maclaurin Series for a Function

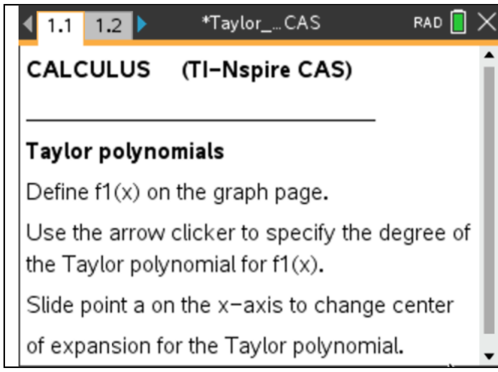
Calculator Files

Taylor_Polynomials_CAS.tns

Using the Document

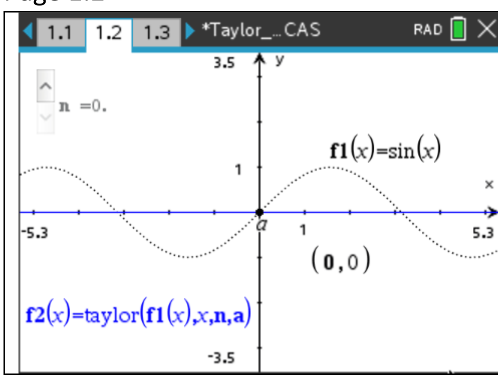
Taylor_Polynomials_CAS: This calculator file provides a tool for generating and graphing Taylor polynomials. The degree of the Taylor polynomial is changed using the arrow clicker for n, and the value for a can be changed by dragging the point on the x-axis or by entering a new x-coordinate in the ordered pair displayed on the graph screen.

Page 1.1



This introductory screen provides information for constructing and graphing a Taylor polynomial for a function $f1(x)$. This function, $f1(x)$, is defined on Page 1.2, a Graphs page. Once the function is defined, the degree of the Taylor polynomial is set using the arrow clicker. The value of a, the center of the Taylor polynomial, is set by moving the corresponding point on the x-axis or by entering a new x-coordinate in the ordered pair displayed on the graph screen.

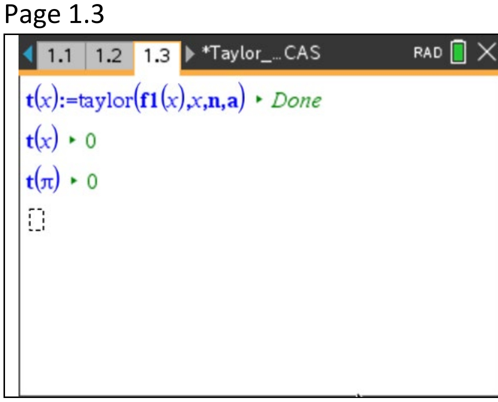
Page 1.2



The default settings are $f1(x) = \sin x$, $a = 0$, and $n = 0$. The function $f1$ is plotted as a dotted curve, and the Taylor polynomial, $T_n(x)$, is plotted in blue.

Change the value of n by using the arrow clicker, and change the value of a by dragging the corresponding point on the x-axis or by entering a new x-coordinate in the ordered pair displayed on the graph screen.

Page 1.3



The function t is defined to be the expression for the nth-degree Taylor polynomial centered at a. The polynomial is displayed by the second Math Box. The last Math Box is used to evaluate this Taylor polynomial at a specific value.

Suggested Applications and Extensions

1. (a) Find the Taylor polynomials up to degree 7 for $f(x) = \sin x$ centered at $a = 0$. Examine these graphs as n increases.

   (b) Evaluate f and these Taylor polynomials at $x = \frac{\pi}{4}, \frac{\pi}{2}$, and π.

   (c) Explain how the Taylor polynomials converge to $f(x)$.

2. Find the Taylor polynomial $T_5(x)$ for the function f centered at the number a. Observe how the graphs of the Taylor polynomials change as n increases, and find an interval in which the Taylor polynomial is a good approximation to f.

   (a) $f(x) = e^x$, $a = -1$

   (b) $f(x) = \cos x$, $a = \frac{\pi}{6}$

   (c) $f(x) = \ln x$, $a = 1$

   (d) $f(x) = x \sin x$, $a = \frac{\pi}{2}$

   (e) $f(x) = x \tan^{-1} x$, $a = -\frac{\pi}{4}$

   (f) $f(x) = x^2 e^{-x}$, $a = \frac{1}{2}$

3. Find the Taylor polynomial $T_5(x)$ for the function f centered at 0. Observed how the graphs change as n increases, find an interval in which the Taylor polynomial is a good approximation to f, and find $T_5(b)$.

   (a) $f(x) = (1 - x)^{-3}$ $b = -\frac{1}{4}$

   (b) $f(x) = \ln(1 + x)$ $b = \frac{1}{2}$

   (c) $f(x) = e^{-x/2}$, $b = 2$

   (d) $f(x) = 3^x$, $b = -\frac{1}{2}$

   (e) $f(x) = x \tan x$, $b = \frac{\pi}{4}$

   (f) $f(x) = \frac{1}{1 + x^2}$, $b = 1$

4. Find the Taylor polynomial $T_5(x)$ for the function $f(x) = x^5 - 3x^3 + x$ centered at $a = 1$. Explain this result.

5. (a) Find the Taylor polynomial $T_3(x)$ for the function $f(x) = e^{x^2}$ centered at $a = 0$.

   (b) Find the Taylor polynomial $T_3(x)$ for the function $g(x) = \ln(x^2 + 1)$ centered at $a = 0$.

   (c) Find the Taylor polynomial $T_3(x)$ for the function $h(x) = e^{x^2} \ln(x^2 + 1)$ centered at $a = 0$. Explain how this Taylor polynomial is related to those found in parts (a) and (b).