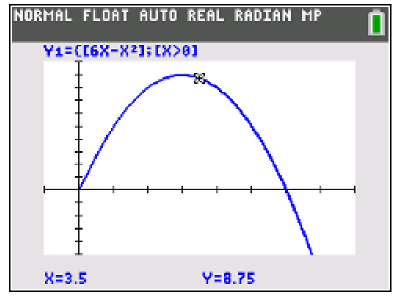
Distance vs. Displacement: A Particle's Journey



Math Objectives

- Students will further discuss the graphical and algebraic relationships between derivatives, integration and particle motion while using kinematics.
- Students will then apply this particle knowledge to real world situations.
- Students will try to make a connection with how to understand these topics in IB Mathematics courses and on their final assessments.

Vocabulary

- Displacement
- Velocity
- Acceleration
- Kinematics
- Derivative
- Integration

About the Lesson

- This lesson is aligning with the curriculum of IB Mathematics Applications and Interpretations HL and IB Mathematics Approaches and Analysis SL/HL
- This falls under the IB Mathematics Content Topic 5 Calculus:
  5.9 (AA only) Kinematics problems involving displacement *s*, velocity *v*, acceleration *a,* and total distance travelled.
  5.13 (AI HL only) Kinematics problems involving displacement *s*, velocity *v*, and acceleration *a*.
  As a result, students will:
  - Apply this information to real world situations.

Teacher Preparation and Notes.

- This activity is done with the use of the TI-84 family as an aid to the problems.

Activity Materials

- Compatible TI Technologies: TI-84 Plus*, TI-84 Plus Silver Edition*, TI-84 Plus C Silver Edition, TI-84 Plus CE

** with the latest operating system (2.55MP) featuring MathPrint™ functionality.*

Tech Tips:

- This activity includes screen captures taken from the TI-84 Plus CE. It is also appropriate for use with the rest of the TI-84 Plus family. Slight variations to these directions may be required if using other calculator models.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Files:
Student Activity
Distance vs. Displacement A particles journey_Student-84CE.pdf
Distance vs. Displacement A particles journey_Student-84CE.doc

To get this activity started, a quick review on Kinematics will be beneficial. Simply stated, Kinematics is the features or properties of the motion of an object. Three words associated with Kinematics are displacement, velocity, and acceleration. *s* is the displacement of the object (or particle) from a fixed origin at time *t*. *v* is the velocity of the particle at time *t*. *a* is the acceleration of the particle at time *t.*

Students should already know the relationship between a particle's displacement, velocity and acceleration, but we will review those skills through this activity.

Review

(a) Discuss with a classmate how using a particle's displacement equation can help you find a particle's velocity equation. Also, discuss how you using a particle's velocity equation can help you find the particle's acceleration equation. Share your results with the class.

Possible discussion points: By taking the derivative of the particle's displacement equation, you can find the velocity equation and by taking the derivative of the particle's velocity equation, you can find its acceleration equation. Its Velocity is the rate of change of the particle's displacement. Its acceleration is the rate of change of the particle's velocity.

(b) Discuss with a classmate how using a particle's acceleration equation and a boundary condition can help find the velocity equation. Also, discuss how you using a particle's velocity equation and a boundary condition can help find the displacement equation. Share your results with the class.

Possible discussion points: By integrating the acceleration equation, you can find the velocity equation and by integrating the velocity equation, you can find the displacement equation. Also, by find the area between the curve and the x-axis, you can find displacement and total distance.

After your results have been shared, fill in the following blanks:

$s(t)$	$s(t) = \int \boldsymbol{v(t)\,dt}$
$v(t) = \boldsymbol{s'(t)}$	$\boldsymbol{v(t)} = \int \boldsymbol{a(t)\,dt}$
$a(t) = \boldsymbol{s''(t)} = \boldsymbol{v'(t)}$	$a(t)$

What is the difference between displacement and distance traveled? Let us explore that now!

Teacher Tip: This is a good place to start the discussion of displacement and distance, but a full description can be made within Problems 1 and 2.

Problem 1

Particle A travels in a straight line such that its displacement, *s* meters, from a fixed origin after *t* seconds is given by $s(t) = 6t - t^2$, for $0 \leq t \leq 8$.
Particle A starts at the origin and passes through the origin again when *t* = *q*. Particle A changes direction when *t* = *r*. The total distance travelled by particle A is given by *d*.

(a) (i) Find the position, velocity and acceleration of particle A after 1 second. With a classmate, discuss what each of these answers mean with respect to particle A. Share your results with the class.

Solution: $s(t) = 6t - t^2$ $\quad s(1) = 6(1) - (1)^2 = 5\,m$

Possible discussion: after 1 sec, the particle is *5 m* to the right of its starting point (origin)

$s'(t) = v(t) = 6 - 2t \quad v(1) = 6 - 2(1) = 4\,ms^{-1}$

Possible discussion: after 1 sec, the particle is travelling at *4* ms^{-1} to the right of the origin

$s''(t) = v'(t) = a(t) = -2 \quad a(1) = -2\,ms^{-2}$

Possible discussion: after 1 sec, the velocity of the particle is changing in a negative direction

(ii) Determine if particle A is speeding up or slowing down at *t = 1*. Explain your reasoning.

Solution: The particle is slowing down at *t = 1*. Since the velocity is positive and the acceleration is negative, the opposite signs will cause the particle to slow down.

(b) Find the value of *q*. With a classmate, discuss the significance of what this value of *q* means about particle A and the math used to find the value of *q*. Share your results with the class.

Solution: Since the particle passes through the origin again at the value of *q*, you will set the displacement equal to zero and solve for *t*. $s(t) = 0 \quad 6t - t^2 = 0 \quad t(6 - t) = 0 \quad t = 0 \text{ and } t = 6$
At zero seconds and at six seconds, the particle has a displacement of zero.

(c) Find the intervals on which particle A is speeding up and the intervals on which it is slowing down. With a classmate, discuss the math used to come to your conclusion. Share your results with the class.

Solution: You will need to find the places at which the acceleration changes signs and compare these signs to the signs of the velocity at each interval on which one of them changes. If the signs are the same, the particle is speeding up, if the signs are different, the particle is slowing down. See the chart:

t	t < 3	t > 3
v	+	-
a	-	-

From $0 < t < 3$ the particle is slowing down and from $3 < t < 8$ the particle is speeding up.

(d) (i) Find the value of *r*. With a classmate, discuss the significance of what this value of *r* means about the particle and the math used to find the value of *r*. Share your results with the class.

Solution: Since the particle is not moving/changing direction at *r*, set the velocity equal to zero and solve or find the local maximum of the displacement equation (vertex of the quadratic).

$$v(t) = 0 \qquad 6 - 2t = 0 \quad t = 3 \;\therefore\; r = 3$$

(ii) Find the displacement of particle A from the origin when *t* = *r*.

Solution: $s(3) = 6(3) - 3^2 = 9\,m$ or $\int_0^3 v(t)\,dt = \int_0^3 (6 - 2t)\,dt = [6t - t^2]_0^3 = 9\,m$

(e) Find the distance of particle A from the origin when *t = 8*.

Solution: $s(8) = 6(8) - (8)^2 = -16 = 16\,m$ (to the left of the origin)

(f) Find the value of *d*. Discuss with a classmate what the difference is between particle displacement and a particle's total distance. Share your results with the class.

Solution: To find the total distance, find $\int_0^8 |v(t)|\,dt \;\therefore\; \int_0^8 |6 - 2t|\,dt = 34\,m$
You could also find the distance from the origin to its turning point (*t = 3*), which represents the particle going forward, double it going back to the origin, and then combine that with *s(8)*, *9 + 9 + 16 = 34 m.*
Discussion points: Displacement is measured with reference to a specific point, it is a straight line from the starting point (origin) to the end point. It is the shortest distance between two points. Total distance measures the actual ground covered (forward/backward) and is always positive.

(g) A second particle B, travels along the same straight line such that the velocity is given by $v(t) = 8 - 2t$, for $t \geq 0$. When *t* = *p*, the distance traveled by B is equal to *d*. Find the value of *p*. Discuss with a classmate how you would use velocity to help find the distance traveled. Share your results with the class.

Solution: To find the time of *p* when the particle distances are equal, you will need to integrate the absolute value of the velocity of particle B and set it equal to the answer to part (f) (34 *m*), but since we do not know how long particle B travels to reach that distance, we will have to split up the absolute value integral into two parts and solve for that time *p*. First, find any change in direction by setting the velocity equal to zero and solving for *t*. $8-2t=0 \quad t=4$

$$\int_0^4 (8-2t)dt + \int_4^p (2t-8)dt = 34$$
$$16 + p^2 - 8p + 32 = 34$$
$$p^2 - 8p - 2 = 0$$
$$\boldsymbol{p = 4 + 3\sqrt{2}}$$

Reflection

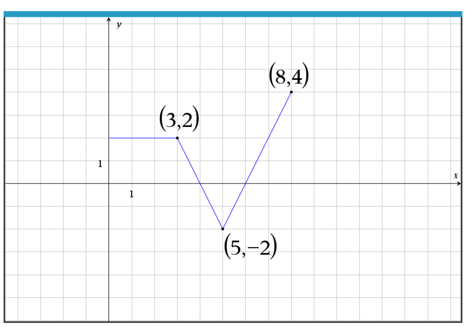
Discuss with a classmate how your TI-84 Plus CE could have helped or did help you through the process of answering **Problem 1**. Share your results with the class.

Possible Discussion Points: Using the math button or the alpha, window buttons to find derivatives, definite integrals, absolute values; using graphs to find maximums and minimums; also using graphs to find where the particle changes direction or passes through the origin by finding the zeros.

Teacher Tip: Problem 1 is a big review of Kinematics from both a derivative perspective and an integral perspective. It will be useful if the teacher demonstrates how to do some of these parts using the 84 Plus CE and not just wait for the reflection. This will be a big time saver come examination time.

Problem 2

The velocity, $v\ ms^{-1}$, of a particle moving along a line, for $0 \le t \le 8$, is shown in the following diagram:



(a) Find the acceleration of the particle when *t* = 4. Discuss with a classmate how you would find this given the velocity graph. Share your results with the class.

Solution: Knowing that acceleration is the rate of change of the velocity, use the velocity graph and find the slope of the line that passes through the time at 4 seconds.
$slope\ (rate\ of\ change) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{-2-2} = \frac{2}{-4} = -\frac{1}{2}$

(b) Write down the interval(s) on which the particle is travelling to the right. Discuss with a classmate how you can tell by the velocity graph. Share your results with the class.

Solution: The particle moves to the right when the velocity is positive or above the x-axis. Looking at the graph these intervals are $0 < t < 4\ and\ 6 < t < 8$ or in interval notation $(0, 4)\ and\ (6, 8)$.

(c) Write down a definite integral that represents the displacement of the particle after 8 seconds. Find this displacement.

Solution: $\int_0^8 v(t)dt = Area_1 - Area_2 + Area_3 = 7 - 2 + 4 = 9\ m$

(d) Write down a definite integral that represents the total distance travelled for $0 \leq t \leq 8$. Find this total distance.

Solution: $\int_0^8 |v(t)|\ dt = Area_1 + Area_2 + Area_3 = 7 + 2 + 4 = 13\ m$

(e) Discuss with a classmate the difference between distance travelled and displacement. Share your results with the class.

Solution: As mentioned in problem 1, displacement is measured with reference to a specific point. It is a straight line from the starting point (origin) to the end point. It is the shortest distance between two points. Distance measures the actual ground travelled (forward/backward) and can only be positive.

Problem 3

The velocity, $v\ ms^{-1}$, of a particle moving in a straight line is given by $v(t) = t^2 - 25$, where $t \geq 0$ seconds.

(a) Find the acceleration of the particle at *t* = 3.

Solution: $a(t) = 2t \qquad a(3) = 2(3) = 6\ ms^{-2}$

(b) The initial displacement of the particle is 8 m. Find an expression, *s*, for the displacement of the particle at time *t*.

Solution: $\int v(t)dt = s(t)$

$$\int (t^2 - 25)\, dt = \frac{1}{3}t^3 - 25t + C$$

To solve for *C*, use the initial condition of an *8 m* displacement.

$$8 = \frac{1}{3}(0)^3 - 25(0) + C$$
$$C = 8$$
$$s(t) = \frac{1}{3}t^3 - 25t + 8$$

(c) Find the distance travelled between times 3 seconds and 6 seconds.

Solution: $\int_3^6 |t^2 - 25|\, dt = 22.66666 \ldots \approx 26.7\ m$

<u>Further Applications Beyond Particles</u>

1.) A child's sling shot launches water balloons in the air. The height, *h* m, of a water balloon which is launched into the air with an initial velocity v_o and an initial height of h_o can be modelled by the function $h(t) = h_o + v_o(t) - 4.9t^2$ where t is time in seconds that have passed since the balloon was launched. A water balloon is launched from the ground with $v_o = 50\ ms^{-1}$. Find the maximum height the balloon reaches and the time that passes before it hits the ground again.

Solution:

<u>Method 1</u>

Once you substitute into the height function, find its derivative and set this equal to zero and solve.

$$h(t) = 0 + 50t - 4.9t^2$$
$$h'(t) = 50 - 9.8t$$
$$0 = 50 - 9.8t$$
$$t \approx 5.10 \quad \therefore \ \max height \approx 128\ m$$

<u>Method 2</u>

Graph the function, $h(t)$, and using your Nspire CX II to find the maximum under menu, Analyze Graph, maximum.

$$(5.10, 128) \quad \therefore \ \max height \ \approx 128\ m$$

2.) During the diving championships, a team member jumps from a diving board above a swimming pool. At a time, t seconds after leaving the board, the team member's height above the surface of the pool, s meters, can be modelled by the function $s(t) = 12 + 4t - t^2$. Find:

(a) The height of the diving board above the surface of the pool.

Solution: $s(0) = 12 + 4(0) - (0)^2 = 12\ m$

(b) The time between the person leaving the board and hitting the water.

Solution: $0 = 12 + 4t - t^2$

$0 = t^2 - 4t - 12$

$0 = (t-6)(t+2)$

$t = 6\ and -2$

Therefore it takes *6* seconds to hit the water.

(c) The velocity and acceleration of the diver upon impact with the water. Interpret these in the context of the problem.

Solution: $s(t) = 12 + 4t - t^2$ $\quad v(t) = 4 - 2t$ $\quad a(t) = -2$

$v(6) = 4 - 2(6)$ $\quad a(6) = -2\ ms^{-2}$

$v(6) = -8\ ms^{-1}$

With a velocity of $-8\ ms^{-1}$, the diver is going in a downward direction and hits the water at $8\ ms^{-1}$. With an acceleration of $-2\ ms^{-2}$, the diver is speeding up at impact as the signs of the velocity and acceleration are the same at 6 seconds.

Teacher Tip: Please know that in this activity there is a lot of time dedicated to students talking with one another and sharing their thoughts with the class. The goal here is to not only review Kinematics in calculus, but also to generate discussion.