Binomial Pdf Eye Color

IB Aligned

Math Objectives

- Students will identify a situation involving binomial trials.
- Students will observe that a binomial distribution is a function of both sample size and the probability of a success.
- Students will interpret a table of binomial probabilities.
- Student will answer probability questions using the graph of a binomial model for a random variable involving binomial trials.

Vocabulary

- binomial distribution
- expected value
- probability

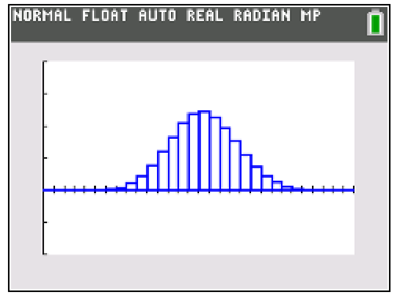
About the Lesson

- This lesson is aligning with the curriculum of IB Mathematics Applications and Interpretations SL/HL and IB Mathematics Approaches and Analysis SL/HL
- This falls under the IB Mathematics Core Content Topic 4 Statistics and Probability:

  4.8 Binomial Distribution as a model and found using technology. Mean and variance of a binomial distribution.

- As a result, students will:
  - For a binomial distribution with a given sample size and probability of success
    - find the probability of specific outcomes from a spreadsheet of values.
    - find the probability of specific outcomes from the graph.
    - Find the probability of mutually exclusive events from a graph or spreadsheet.
  - Change the sample size and the probability of a success and find the probability of specific outcomes.

Analyze how the probabilities change as the sample size increases and as the probability of a success increases.



Tech Tips:

- This activity includes screen captures taken from the TI-84 Plus CE. It is also appropriate for use with the rest of the TI-84 Plus family Slight variations to these directions given within may be required if using other calculator models.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Files:

Student Activity

Binomial_Pdf_Eye_Color_Student-84.pdf

Binomial_Pdf_Eye_Color_Student-84.doc

Create File

Binomial_Pdf_Eye_Color_Create-84.doc

This lesson also involves binomial trials, distributions, and probabilities. Students can create the binomial distribution following the steps in Binomial_Pdf_Eye_Color_Create-84 document.

Teacher Preparation and Notes

- Students should be familiar with the concept of a binomial distribution.
- Notation and Terminology to be familiar with: In a sequence of n independent trials of an experiment in which there are exactly two outcomes "success" and "failure" with constant probabilities P(success) = *p*, P(failure) = 1 – *p*, if *X* denotes the discrete random variable equal to the number of successes in *n* trials, then the probability distribution function of *X* is

$P(X = x) = C_x^n \cdot p^x \cdot (1-p)^{n-x}, x \in \{0, 1, 2, \dots, n\}$

These facts are summarized with the words "*X* is distributed binomially with parameters *n* and *p*" and with symbols $X \sim B(n, p)$.

Activity Materials

- Compatible TI Technologies:
  TI-84 Plus*, TI-84 Plus Silver Edition*, TI-84 Plus C Silver Edition, TI-84 Plus CE

Discussion Points and Possible Answers

Tech Tip: Using the **trace** function, moving the cursor over a bar in a histogram will display the values in the interval spanned by the bar and the probability of those values.

Several sources indicate that the probability of a person in the United States having blue eyes is approximately 30%. Suppose you randomly sampled 50 people in the United States. One basic question to answer is whether an underlying probability model might describe the probability of the possible numbers of blue-eyed people in your sample.

1. This situation involves binomial trials, so the first step is to check whether the requirements for binomial trials are met: two outcomes per trial; a fixed number of trials; a constant probability of success; and independent trials. Verify that these conditions are met in this context.

Teacher Tip: A binomial distribution is sometimes related to Bernoulli trials— according to some interpretations, a situation involving Bernoulli trials has the same requirements as binomial trials.

Answer: The problem is formulated with two outcomes and exactly 50 trials. A success is a person with blue-eyes. Any other eye color would be classified as not a success. The probability of being blue-eyed is assumed to be constant for the population based on a variety of sources. Under the assumption that the random sample of people has been selected independently, the conditions for binomial trials are met.

2. a. Reflect on the binomial distribution you built. State two inputs that determined the binomial Distribution.

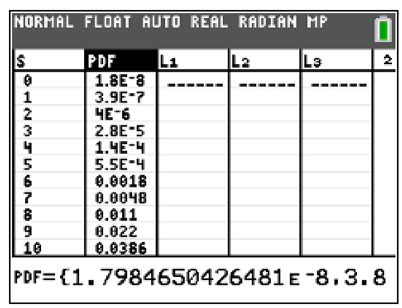
Answer: The probability 0.3 and the sample size, *n*.

b. For a random sample of 50 people, state what you would expect to be the most likely number of blue-eyed people. Explain your reasoning.

Answer: The most likely number of blue-eyed people will probably be the expected value (mean), which is (0.3)(50) or 15.

3. In terms of the context,

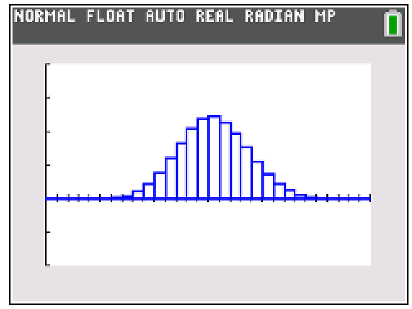
a. state what the zero represents in the first cell under column heading S.



Answer: The zero indicates the outcome of having no blue-eyed people in the sample of 50 people.

b. state what the value in the 5th cell under the column heading PDF represents.

Answer: The value is 0.00014, the probability of getting exactly 4 blue-eyed people in the sample of 50 people.



If the conditions for binomial trials are met, a binomial model can be used to represent the probabilities, which is the distribution you created.

4. a. Describe the distribution in terms of shape, center, and spread.

Answer: The distribution is mound shaped but not quite symmetric. The center seems to be at 15, and the most likely outcomes seem to be between 10 and 21.

> **Teacher Tip:** You might ask students how their answer to 2b relates to their graph. Some important points to discuss include: What are the end points of the graph? How is this distribution similar to and different from a normal distribution? This would help students recognize that the probabilities of values far from 15 do exist but are very small and that the binomial distribution is a discrete distribution while the normal curve is continuous. Here you cannot have "0.5" success.

b. Describe how the context of the eye color for a sample of randomly chosen people relates to the graph.

Answer: A success is being blue-eyed, and the probability of a success is given as 0.30. Each bin represents the probability of having that number of blue-eyed people in a sample of 50 randomly-chosen people.

c. Saundra wonders whether the probability of getting 4 blue-eyed people in the sample of 50 was 0. State what you would tell her.

Sample Answers: One response would be to return to the table they created and look for the probability associated with 30 blue-eyed people in the sample, 0.000008. The graph does not show the results for such very small probabilities. Some students might calculate the value using the binomial distribution formula.

5. Suppose *x* represents the number of blue-eyed people. Describe how the graph helps you answer each of the following:

a. The most likely outcomes.

Sample Answers: Some students might suggest the outcomes represented by the tallest bin, or 15: P(x = 15) = 0.1223. Others might combine the tallest bars, for example from 14 to 16 blue-eyed people in the sample of 50; P(14 ≤ x < 16) = 0.1147 + 0.1223 + 0.1189 = 0.3559.

b. The least likely outcomes.

Sample Answers: Depending on how students think about "least likely," the answers will vary. Some students might use the outcomes in the tails, P(x = 7) = 0.0048 and P(x = 23) = 0.0067. Others might include the outcomes whose probabilities are so small they did not show up in the distribution, any outcome less than 7 or greater than 23; P(x < 7 or x > 23).

c. If you add the probabilities of every possible outcome, state what the result should be.

Answer: 1 because the probabilities of all possible outcomes sum to 1.

6. a. State the approximate probability that exactly 10 people will have blue eyes, P(x = 10).
Answer: Approximately, 0.0386.

b. State the approximate probability that 10 or 11 people will have blue eyes, P(x = 10,11).

Answer: The sum of the probability of each outcome, 10 and 11, will be approximately 0.0386 + 0.0602 = 0.0988.

> **Teacher Tip:** Students should be aware that you can add the probabilities only because the events are mutually exclusive. You might ask them why this is important in a class discussion and could refer to a Venn diagram for events that are not mutually exclusive (the probability of a blonde haired person having blue eyes, for example).

c. State the approximate probability that fewer than 10 people will have blue eyes, P(x < 10).

Answer: P(0 ≤ x ≤ 9) where P(0 ≤ x ≤ 6) is very small, so approximately
0 + 0.0048 + 0.0220 + 0.0110 = 0.0378.

d. State the approximate probability that at least 10 people will have blue eyes, P(x ≥ 10). Describe how you could use your answer to part **b** to find the answer to this question.

Answer: You can subtract the probability for P(x < 10) from 1, the sum of all the probabilities; P($x \geq$ 10) is approximately 1 - 0.0378 = 0.9622.

7. Suppose a researcher was checking the number of candies of a certain color chosen from randomly selected bags where each bag has exactly 50 candies. The manufacturer claimed that the probability of getting a green candy was approximately 0.3.

   a. Describe how this context is alike or different from the context involving eye color above.

   Answer: While the context is different, the two parameters for a binomial distribution are the same: n = 50 and p = 0.3.

   b. In a sample size of 50 candy pieces, if the researcher found the number of green candies in the sample indicated below, explain if he should be surprised or not.
   i. 12

   Sample Answers: No because the chance that this would occur is 10.5%, which is not too unusual. Other students might think this is a likely event.

   > **Teacher Tip:** Deciding what is "unlikely" can provide an opportunity to consider different benchmarks and raise the question: where would you begin to think an outcome was not due to chance – 1 out of 4; 1 out of 10; 1 out of 20? The discussion can lay the foundation for later work with confidence levels.

   ii. 8

   Answer: Yes because this would occur randomly only 1% of the time.

   c. If the researcher were surprised, state the conclusions he might draw.

   Sample Answers: The outcome might just be due to chance, or it could be that the manufacturer's claim about the number of candies was suspect.

   > **Teacher Tip:** This is an informal approach to inference and hypothesis testing.

Extension

In the extension, we will change both the sample size and the probability of a success. Follow your teacher's instructions for the following:

Teacher Tip: You might want students to compare distributions with the same sample size for complementary probabilities (i.e., 0.05 and 0.95; 0.10 and 0.90).

Teacher Tip: You might want to relate the formula (listed on page 2) for finding p(*x*) where *x* is a random variable in a binomial distribution to the outcomes visible in the graph to help students explore how changing the values of ***n*** and ***p*** affect the probabilities.

8. The probability of having green or light brown eyes with green specks is about 12%. In a random sample of 100 people, find the probability of the following:
   a. Exactly 6 people with that eye color.

   Answer: Approximately, 0.0215

   b. At least 16 people with that eye color.

   Answer: Approximately, $P(x \geq 16) \approx 0.141$

   c. Anywhere from 7 to 15, including 7 and 15, people with that eye color.

   Answer: Approximately, $P(7 \leq x \leq 15) \approx 0.822$

9. a. Predict how your answers to question 8 might change if the sample size were 30 people.

   Answer: Students might suspect that the probability of getting at least 16 blue-eyed people will decrease considerably because the expected value is (0.12)(30)=3.6 or about 4, and 16 seems to be farther in the right tail of the distribution than it would be when the expected value is 12.

   b. Check your prediction by changing the sample size to 30 and using the resulting distribution.

   Answer: The probability of exactly 6 would be approximately 0.0825 or 8%; an outcome at least 16 is very close to 0; from 7 to 15 inclusive is about 6%.

10. a. Predict how your answers to question 7 would change if the sample size remained 100 but the probability of brown eyes was 0.40 or 40%.

Answer: The distribution will shift and center around 40, so outcomes below 20 (including two asked for in problem 7, exactly 6 and from 7 to 15) and above 55 will be very unlikely. The probability of an outcome at least 16 is nearly certain.

b. Check your prediction by changing *n* and *p* accordingly and using the resulting distribution.

Answer: The probability of exactly 6 would be close to 0; at least 16 is about 99.999%; from 7 to 15 inclusive is close to 0.

11. a. Describe how the distribution changes as the sample size goes from 100 to 50 people for *p*= 0.4.

Answer: The center shifts from 40 to 20, and the range of the probabilities not close to zero decreases from about 30 to about 20, which means the distribution has a higher peak for the smaller *n.*

b. State how your answer to part **a** would change if the probability of a success was a different value.

Answer: The center would shift to the expected value, but in general, as the sample size decreases, the distribution becomes tighter.

c. Fix the sample size at *n* = 50. Change the values of *p*, and describe what happens.

Answer: For very small *p*-values, the distribution is skewed right and becomes symmetric. For very large *p*-values, the distribution is skewed left.

Teacher Tip: Students might need to further investigate what happens to the probabilities of an outcome as the sample size increases in general and what happens to the probability of an outcome if the sample size remains constant and the probability of a success decreases. This should lead them to notice that for small probabilities the distributions are not symmetric.

Wrap Up

Upon completion of the lesson, the teacher should ensure that students are able to understand:

- A binomial distribution is determined by the sample size and the probability of a success.
- How to find the probability of mutually exclusive events by adding the probabilities of each event or by using the fact that the probability of all the outcomes is 1.
- How to use a graph of the binomial distribution to find the probability of an outcome for a given sample size and given probability of a success.

Assessment

Select the probability of a success and a sample size. Make up three questions and find the answers – exchange questions with your partner, then check each other's answers.

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