Monday Night Calculus

Integration by Parts

Exercises

1. Use integration by parts to evaluate the integral $\displaystyle\int_0^1 x^2 \tan^{-1} x\, dx$

$$u = \tan^{-1} x \qquad dv = x^2\, dx$$

$$du = \frac{1}{1+x^2}\, dx \qquad v = \int x^2\, dx = \frac{x^3}{3}$$

$$\int x^2 \tan^{-1} x\, dx = \tan^{-1} x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{1+x^2}\, dx \qquad \text{Integration by parts}$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3} \int \left(x - \frac{x}{1+x^2} \right) dx \qquad \text{Long division}$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3} \left(\frac{x^2}{2} - \frac{1}{2} \ln|1+x^2| \right) + C \qquad \text{Antiderivatives}$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{6}x^2 + \frac{1}{6} \ln(1+x^2) + C \qquad \text{Distribute}$$

$$\int_0^1 x^2 \tan^{-1} x\, dx = \left[\frac{x^3 \tan^{-1} x}{3} - \frac{1}{6}x^2 + \frac{1}{6} \ln(1+x^2) \right]_0^1$$

$$= \left[\frac{1^3 \tan^{-1} 1}{3} - \frac{1}{6}1^2 + \frac{1}{6} \ln(1+1^2) \right] - \left[\frac{0^3 \tan^{-1} 0}{3} - \frac{1}{6}0^2 + \frac{1}{6} \ln(1+0^2) \right]$$

$$= \left[\frac{\pi}{12} - \frac{1}{6} + \frac{1}{6} \ln 2 \right] - 0$$

$$= \frac{\pi}{12} - \frac{1}{6} + \frac{1}{6} \ln 2$$

2. (a) Evaluate the integral $\int \sin^2 x\,dx$ by using the trigonometric identity $\sin^2 x = \dfrac{1-\cos 2x}{2}$ and the substitution $u = 2x$.

$$\begin{aligned}\int \sin^2 x\,dx &= \int \left(\frac{1-\cos 2x}{2}\right) dx \\ &= \frac{1}{2}\left[\int 1\,dx - \int \cos 2x\,dx\right] \\ &= \frac{x}{2} - \frac{\sin 2x}{4} + C\end{aligned}$$

(b) Evaluate the integral $\int \sin^2 x\,dx$ by using integration by parts with $u = \sin x$ and $dv = \sin x\,dx$.

$$\begin{aligned} u &= \sin x & dv &= \sin x\,dx \\ du &= \cos x\,dx & v &= \int \sin x\,dx = -\cos x\end{aligned}$$

$$\begin{aligned}\int \sin^2 x\,dx &= \sin x \cdot (-\cos x) - \int -\cos x \cdot \cos x\,dx && \text{Integration by parts} \\ &= -\sin x \cos x + \int \cos^2 x\,dx && \text{Simplify} \\ &= -\sin x \cos x + \int (1-\sin^2 x)\,dx && \text{Trig identity} \\ &= -\sin x \cos x + x - \int \sin^2 x\,dx && \text{Antiderivative}\end{aligned}$$

$$2\int \sin^2 x\,dx = -\sin x \cos x + x \qquad \text{Same integral on both sides}$$

$$\int \sin^2 x\,dx = \frac{1}{2}(x - \sin x \cos x) + C \qquad \text{Solve for } \int \sin^2 x\,dx$$

Question: Is this really the same result as in part (a)?
Hint: Consider a double angle formula.

3. (a) Use integration by parts to show that

$$\int x^n \ln x \, dx = \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) \qquad \text{for } n \neq -1$$

$$u = \ln x \qquad dv = x^n \, dx$$

$$du = \frac{1}{x}\, dx \qquad v = \int x^n \, dx = \frac{x^{n+1}}{n+1} \text{ provided } n \neq -1$$

$$\int x^n \ln x \, dx = \ln x \cdot \frac{x^{n+1}}{n+1} - \int \frac{x^{n+1}}{n+1} \cdot \frac{1}{x}\, dx \qquad \text{Integration by parts}$$

$$= \frac{x^{n+1}}{n+1} \ln x - \frac{1}{n+1}\int x^n \, dx \qquad \text{Simplify}$$

$$= \frac{x^{n+1}}{n+1} \ln x - \frac{1}{n+1} \cdot \frac{x^{n+1}}{n+1} + C \qquad \text{Antiderivative}$$

$$= \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + C \qquad \text{Factor}$$

(b) Fill in the gap for this formula. That is, evaluate the integral when $n = -1$.

$$\int x^{-1} \cdot \ln x \, dx = \int \frac{1}{x} \cdot \ln x \, dx$$

$$u = \ln x \quad \Rightarrow \quad du = \frac{1}{x}\, dx \quad \Rightarrow \quad dx = x \, du$$

$$\int \frac{1}{x} \cdot \ln x \, dx = \int \frac{1}{x} \cdot u \cdot x \, du \qquad \text{Change variables}$$

$$= \int u \, du = \frac{u^2}{2} + C \qquad \text{Simplify; antiderivative}$$

$$= \frac{(\ln x)^2}{2} + C \qquad \text{Back to } x\text{'s}$$

4. Uncle Stanley found a differentiable function f with the property $f'(x) = -f(x)$ for all x.

(a) Show that for Uncle Stanley's function

$$I = \int f(x)\sin x\,dx = \frac{-\sin x - \cos x}{2} \cdot f(x) + C$$

$$u = f(x) \qquad dv = \sin x\,dx$$

$$du = f'(x)\,dx \qquad v = \int \sin x\,dx = -\cos x$$

$$I = f(x)\cdot(-\cos x) - \int (-\cos x)\,f'(x)\,dx \qquad \text{Integration by parts}$$

$$= -f(x)\cos x - \int f(x)\cos x\,dx \qquad \text{Property of } f$$

$$u = f(x) \qquad dv = \cos x\,dx$$

$$du = f'(x)\,dx \qquad v = \int \cos x\,dx = \sin x$$

$$I = -f(x)\cos x - \left[f(x)\cdot\sin x - \int \sin x\cdot f'(x)\,dx\right] \qquad \text{Integration by parts}$$

$$= -f(x)\cos x - f(x)\sin x - \int f(x)\sin x\,dx \qquad \text{Property of } f$$

$$2I = f(x)(-\sin x - \cos x)$$

$$\int f(x)\sin x\,dx = \frac{-\sin x - \cos x}{2}\cdot f(x) + C$$

(b) Evaluate $\int f(x)\cos x\,dx$ where f is Uncle Stanley's function.

Similar to part (a). Use integration by parts twice, and solve for the original integral.

(c) Explain the consequences of using integration by parts to evaluate $\int f(x)e^x\,dx$.

$$u = f(x) \qquad dv = e^x\,dx$$

$$du = f'(x)\,dx \qquad v = \int e^x\,dx = e^x$$

$$\int f(x)e^x\,dx = f(x)\cdot e^x - \int e^x \cdot f'(x)\,dx \qquad \text{Integration by parts}$$

$$= f(x)e^x + \int f(x)e^x\,dx \qquad \text{Property of } f$$

If we subtract $\int f(x)e^x\,dx$ from both sides, we get

$$\int 0\,dx = f(x)e^x$$

(d) Find a candidate for Uncle Stanley's function.

Since $\int 0\,dx$ is a constant, then $f(x)e^x = C$ for some constant C.

Therefore $f(x) = Ce^{-x}$.

And indeed, $f'(x) = -Ce^{-x} = -f(x)$