Monday Night Calculus

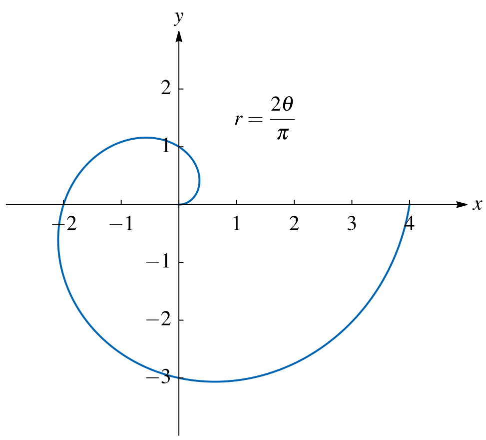
Polar Equations

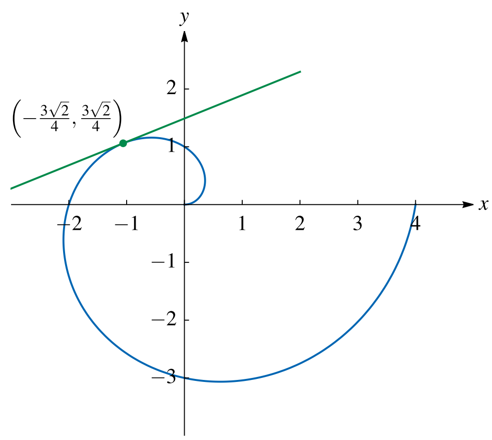
Exercises

1. Spiraling Under Control

Consider the curve C given by the polar equation $r = \dfrac{2\theta}{\pi}$ for $0 \le \theta \le 2\pi$.

(a) Sketch the graph of the curve C and find an equation of the tangent line to the curve at the point where $\theta = \dfrac{3\pi}{4}$.





For $\theta = \dfrac{3\pi}{4}$:

$$x = r\left(\frac{3\pi}{4}\right) \cdot \cos\frac{3\pi}{4} = \frac{3}{2} \cdot \left(-\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{2}}{4}$$

$$y = r\left(\frac{3\pi}{4}\right) \cdot \sin\frac{3\pi}{4} = \frac{3}{2} \cdot \left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{4}$$

Slope of the tangent line:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\theta\cos\theta + \sin\theta}{\cos\theta - \theta\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta = \frac{3\pi}{4}} = \frac{\frac{3\pi}{4}\cos\frac{3\pi}{4} + \sin\frac{3\pi}{4}}{\cos\frac{3\pi}{4} - \frac{3\pi}{4}\sin\frac{3\pi}{4}} = \frac{3\pi - 4}{3\pi + 4}$$

Equation of the tangent line: $y = \left(\dfrac{3\pi - 4}{3\pi + 4}\right)\left(x + \dfrac{3\sqrt{2}}{4}\right) + \dfrac{3\sqrt{2}}{4}$

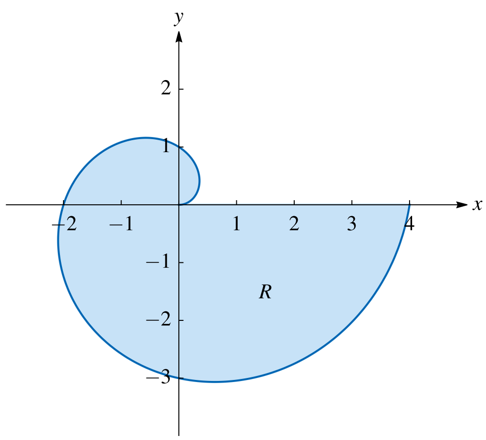
(b) Find the first value in the interval $0 \le \theta \le 2\pi$ for which the tangent line to the curve C is vertical.

$$\text{Solve } \frac{dx}{d\theta} = \frac{2}{\pi}(\cos\theta - \theta\sin\theta) = 0 \quad \Rightarrow \quad \theta = 0.860334$$

$$\text{Check: } \left.\frac{dy}{d\theta}\right|_{\theta=0.860334} 0.839801 \neq 0$$

The first value of θ in the interval $0 \le \theta \le 2\pi$ where the tangent line is vertical is $\theta = 0.860334$.

(c) The region R is bounded by the curve C and the line segment that connects the origin to the point $(x, y) = (4, 0)$. Find the area of the region R.



$$A = \frac{1}{2}\int_0^{2\pi} \left(\frac{2\theta}{\pi}\right)^2 d\theta = \frac{1}{2}\cdot\frac{4}{\pi^2}\int_0^{2\pi} \theta^2\, d\theta$$

$$= \frac{2}{\pi^2}\left[\frac{\theta^3}{3}\right]_0^{2\pi} = \frac{2}{\pi^2}\cdot\frac{8\pi^3}{3} = \frac{16\pi}{3}$$

(d) Find the length of the curve C.

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \cdots = \frac{2}{\pi}\int_0^{2\pi} \sqrt{1+\theta^2}\, d\theta$$

$$= \frac{2}{\pi}\cdot\frac{1}{2}\left[\theta\sqrt{1+\theta^2} + \theta^2 \ln\left|\theta + \sqrt{1+\theta^2}\right|\right]_0^{2\pi}$$

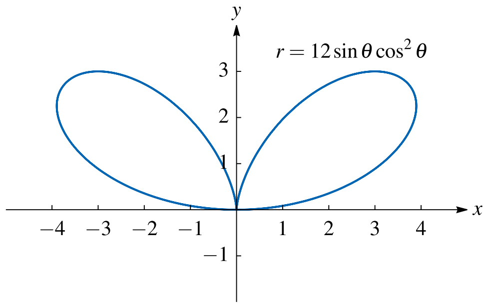
$$= \cdots = \frac{\ln\left(\sqrt{1+4\pi^2} + 2\pi\right) + 2\pi\sqrt{1+4\pi^2}}{\pi}$$

$$= 13.532$$

2. Rabbit Ears (Bifolium)

Consider the curve C defined by the polar equation $r(\theta) = 12\sin\theta\cos^2\theta$ for $0 \le \theta \le \pi$.

(a) Sketch the graph of the curve C. Find the polar coordinates (r, θ) of the point on the curve in the first quadrant that is farthest from the origin.



$$\frac{dr}{d\theta} = 12\left[\cos\theta\cdot\cos^2\theta + \sin\theta\cdot 2\cos\theta(-\sin\theta)\right]$$
$$= 12\cos\theta\left[\cos^2\theta - 2\sin^2\theta\right]$$

$\dfrac{dr}{d\theta} = 0 \;\Rightarrow\; \theta = 0.61548$ and $\dfrac{dr}{d\theta}$ changes sign from positive to negative there.

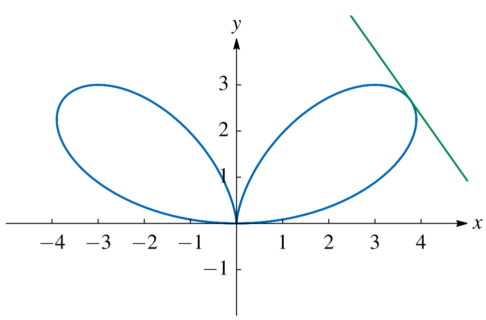
The polar coordinates of the point in the first quadrant farthest from the origin are $(4.6188, 0.61548)$

(b) Find an equation of the line tangent to the curve C at the point found in part (a).

At $\theta = 0.61548$:

$x = 3.77124, \quad y = 2.66667, \quad$ and $\quad \dfrac{dy}{dx} = -1.41421$

An equation of the tangent line: $y - 2.667 = -1.41421(x - 3.77124)$



(c) Find the total area enclosed by the curve C.

$$A = 2 \cdot \frac{1}{2} \int_0^{\pi/2} 144 \sin^2\theta \cos^4\theta \, d\theta$$

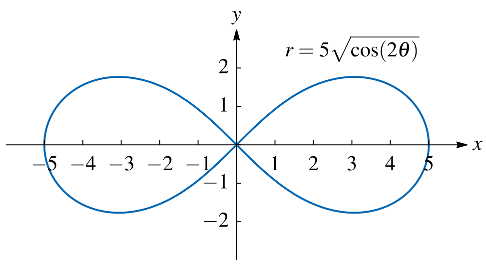
$$= 144 \int_0^{\pi/2} \left(\cos^4\theta - \cos^6\theta\right) \, d\theta$$

$$= \cdots = \frac{9\pi}{2}$$

3. **An Infinity Curve**

Consider the curve C defined by the polar equation $r = 5\sqrt{\cos 2\theta}$.

(a) Sketch the graph of the curve C.



(b) There are two horizontal lines tangent to the curve. Find these lines and the values for θ, $0 \le \theta \le 2\pi$, at which they occur.

$$\frac{dy}{d\theta} = 5 \cdot \frac{1}{2}(\cos 2\theta)^{-1/2} \sin\theta + 5\sqrt{\cos 2\theta}\cos\theta = \cdots = \frac{5\cos 3\theta}{\sqrt{\cos 2\theta}}$$

$$\frac{dy}{d\theta} = 0 \quad \Rightarrow \quad \theta = \frac{\pi}{6}, \ \frac{5\pi}{6}, \ \frac{7\pi}{6}, \ \frac{11\pi}{6}$$

$$\frac{dx}{d\theta} \neq 0 \ \text{ at each of these values.}$$

At $\theta = \dfrac{\pi}{6}$ and $\theta = \dfrac{5\pi}{6}$: $y = \dfrac{5\sqrt{2}}{4}$

At $\theta = \dfrac{7\pi}{6}$ and $\theta = \dfrac{11\pi}{6}$: $y = -\dfrac{5\sqrt{2}}{4}$

The two horizontal lines are: $y = \dfrac{5\sqrt{2}}{4}$ and $y = -\dfrac{5\sqrt{2}}{4}$

(c) Find $\displaystyle\lim_{\theta\to(\pi/4)^-}\frac{dr}{d\theta}$ or explain why it does not exist.

$$\lim_{\theta\to(\pi/4)^-}\frac{dr}{d\theta} = \lim_{\theta\to(\pi/4)^-} -\frac{5\sin 2\theta}{\sqrt{\cos 2\theta}} = -\infty$$

The numerator approaches -5 and the denominator approaches 0 through small positive values. Therefore the fraction decreases without bound, or approaches $-\infty$.

(d) Find $\displaystyle\lim_{\theta\to(\pi/4)^-}\frac{dy}{dx}$ or explain why it does not exist.

$$\lim_{\theta\to(\pi/4)^-}\frac{dy}{dx} = \lim_{\theta\to(\pi/4)^-} -\cot 3\theta = 1$$

(e) Find the total area enclosed by the curve C.
Hint: Carefully consider the domain of r.

The curve is traced out in three pieces for values of θ in three subintervals of $[0, 2\pi]$.

$0 \le \theta \le \dfrac{\pi}{4}$: top right quarter.

$\dfrac{3\pi}{4} \le \theta \le \dfrac{5\pi}{4}$: entire left loop.

$\dfrac{7\pi}{4} \le \theta \le 2\pi$: bottom right quarter.

By symmetry, we can find the entire area by taking twice the area of the left loop.

$$A = 2\cdot\frac{1}{2}\int_{3\pi/4}^{5\pi/4} 25\cos 2\theta\, d\theta = 25\cdot\frac{1}{2}\sin 2\theta\Big]_{3\pi/4}^{5\pi/4}$$

$$= \frac{25}{2}\left[\sin\frac{5\pi}{2} - \sin\frac{3\pi}{2}\right]$$

$$= \frac{25}{2}[1-(-1)] = 25$$