VCE Mathematical Methods Teacher Resource Book *for*

TI-Nspire™ CX II CAS graphing calculator

Teachers Teaching with Technology™
Professional Development from Texas Instruments

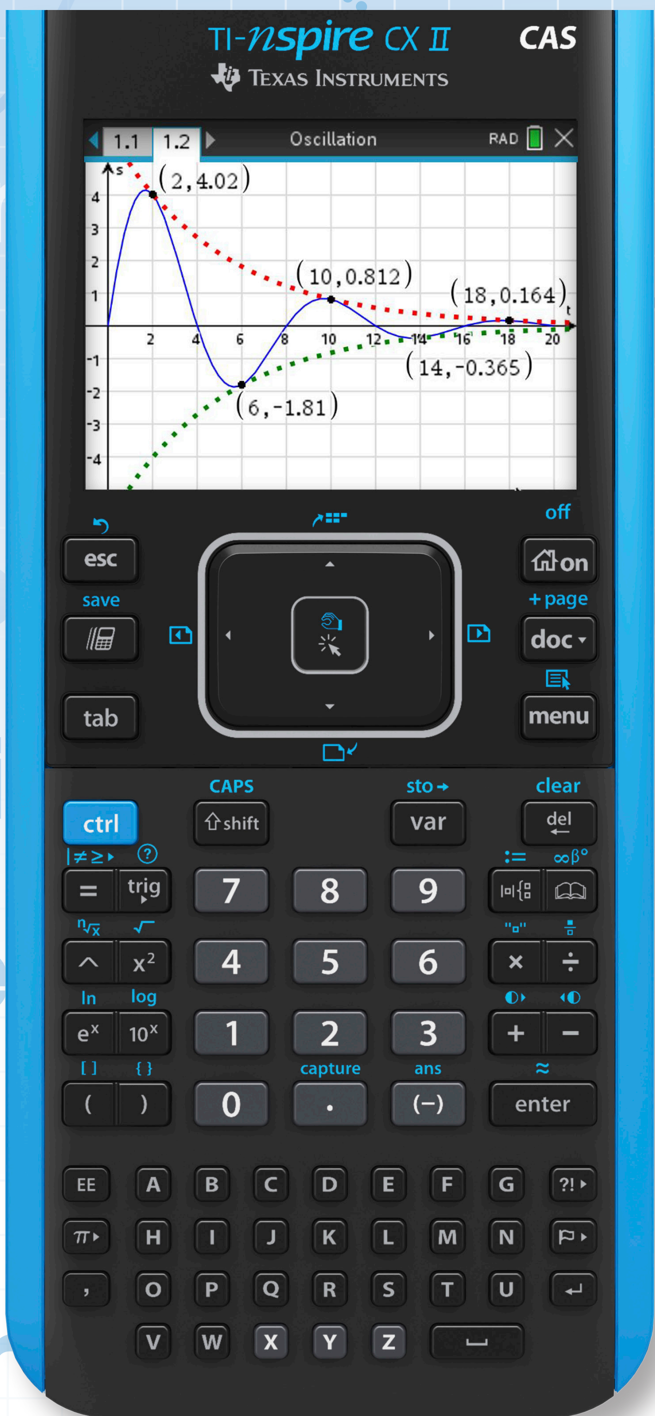
Authored by:
Peter Flynn
Frank Moya
David Tynan

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

$$\int x^n dx=\frac{1}{n+1}x^{n+1}+c,\ n\neq -1$$

$$\sigma^2=\int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx$$



Texas Instruments

Teachers Teaching with Technology™
Professional Development from Texas Instruments

The Teachers Teaching with Technology™ (T3™) Australia professional learning organization is comprised of some of the most creative and innovative mathematics and STEM teachers in the world. They are dynamic and passionate educators who share their knowledge and expertise with secondary teachers and students through professional development events and resource creation.

stationary points

composite functions

discrete random variables

properties of integrals

function modelling

statistical inference

sample proportion

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

$$\int x^n dx=\frac{1}{n+1}x^{n+1}+c,\ n\neq -1$$

$$\sigma^2=\int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx$$

Contents (Unit 1)

Contents (Unit 2)

Contents (Units 3&4)

Contents (Units 3&4)

Introduction

This publication, *VCE Mathematical Methods Teacher Resource Book for the TI-Nspire™ CX II CAS*, is intended to support senior secondary school mathematics teachers in Victoria as they seek to teach the *VCAA Mathematics Study Design 2023*.

Specifically, the publication highlights ways in which *TI-Nspire CAS* technology might be used to assist in the teaching, learning and assessment of *VCE Mathematical Methods Units 1 to 4*.

It is not a complete manual for using this technology, rather it tries to look at each syllabus dot point and make suggestions for possible classroom use.

It has been developed by experienced educators and reviewed by senior mathematics teachers from Victorian schools. We hope you find this to be a useful and supportive publication.

[**Note:** A digital version of this publication can be found at *https://education.ti.com/aus/VIC*].

Notes for teachers

To maximise the usefulness of *VCE Mathematical Methods Teacher Resource Book for the TI-Nspire™ CX II CAS* to teachers, the authors have provided the following explanatory notes.

- It is assumed that the user of this teacher resource book has a basic familiarity with navigating calculator documents and pages. Readers requiring an introduction to this are referred to tutorials at https://education.ti.com/aus and https://www.youtube.com/@TIAustralia.
- Throughout this publication, unless otherwise specified, the default calculator document settings have been used. The calculator user interface language has been set to *English (U.K.)*.
- For each example task, it is desirable to start a new calculator document ([ctrl] [N] or [on] [1]). Alternatively, insert a new problem ([doc▾] > **Insert** > **Problem**).
- When working with functions, use of the **assign** command (via '**:=**') has been privileged over the **define** command. While both commands essentially perform the same role, the **assign** command is a more natural command to use in the **Notes** and **Calculator** applications.
- Implied multiplication has been assumed when working with products such as $6x$. However, where it is necessary to use the multiplication key when entering the product *bx*, for example, the symbol '·' or '×' is used to denote multiplication.
- In some instances in this publication, space has been added to the syntax of commands to improve readability, even though in general spaces should **not** be used in calculator commands without a clear reason. For example, when entering a function, the authors may have expressed this as ***f*(*x*) := *a* + *bx*,** but on the calculator, it will appear as *f*(*x*):=*a*+*b*·*x*.
- There will be some variation in the formatting of commands and text to be entered, but the authors have attempted to use bold formatting when referring to commands to be entered or accessed via the calculator.
- For screenshots from the **Graphs** Application, grid and label settings will vary. Use [menu] commands (or [ctrl] [menu]) to modify these settings for an open document. The default settings (for all documents) for grids and labels can be edited by pressing [menu] and then select **Settings**.
- When catalogue commands are mentioned, pressing [📖] and then [1] will display catalog commands in alphabetical order. Pressing the first letter of the desired command will locate it more quickly.
- To make this publication as practical and concise as possible, mathematical problems considered have been restricted to those that can be attempted by teachers and students without using pre-prepared files. For more interactive digital resources aligned to *VCE Mathematical Methods Units 1 to 4*, go to https://education.ti.com/aus.

VCE Mathematical Methods Unit 1

1.1 Linear relations & coordinate geometry

1.1.1 Coordinate geometry

Analysing points, line segments and lines in the Calculator application

Question

For the two points $A(2,3)$ and $B(7,5)$, find:

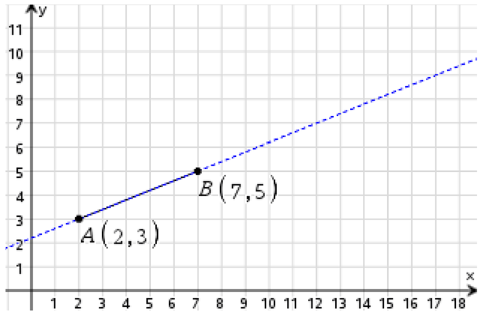
(a) the coordinates of the midpoint of AB.

(b) the gradient of AB.

(c) the length of AB.

(d) the equation of the line passing through A and B.

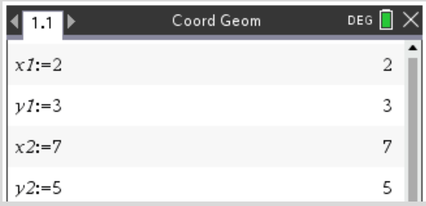
(e) the equation of the line perpendicular to AB that passes through B.



Solution

To begin, on a **Calculator** page, store the values of the coordinates as follows:

- Enter the command ***x*1:=2**.
- Enter the command ***y*1:=3**.
- Enter the command ***x*2:=7**.
- Enter the command ***y*2:=5**.

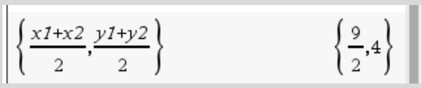


Note: The assign symbol ":=" can be found via ctrl [templates key].

(a) The midpoint of a line segment connecting any two points (x_1, y_1) and (x_2, y_2) has coordinates $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

To find the coordinates of the midpoint, on a **Calculator** page:

- Enter the command $\left\{\frac{\mathbf{x1}+\mathbf{x2}}{\mathbf{2}}, \frac{\mathbf{y1}+\mathbf{y2}}{\mathbf{2}}\right\}$.



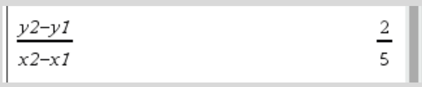
Note: The "{}" can be found via ctrl).

Answer: The midpoint has coordinates $\left(\frac{9}{2}, 4\right)$.

(b) The gradient of a line segment connecting any two points (x_1, y_1) and (x_2, y_2) is gradient $= \frac{y_2 - y_1}{x_2 - x_1}$.

To find m, the gradient of AB, on the **Calculator** page:

- Enter the command $\frac{\mathbf{y_2 - y_1}}{\mathbf{x_2 - x_1}}$



Answer: The gradient is $m = \frac{2}{5}$.

... continued

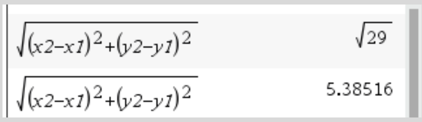
Solution (continued)

(c) The length of a line segment connecting any two points (x_1, y_1) and (x_2, y_2) is given by the distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

To find the value of d:

- Enter the command $\sqrt{(\boldsymbol{x2 - x1})^2 + (\boldsymbol{y2 - y1})^2}$



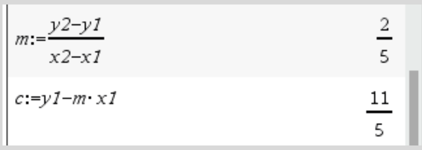
Answer: $d = \sqrt{29}$. Press ctrl enter to display the approximate answer $d \approx 5.38516 \approx 5.4$ units.

(d) For a line with equation $y = mx + c$, for any two points (x_1, y_1) and (x_2, y_2), the values of the slope (m) and the y-intercept (c) can be calculated as follows:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \text{ and } c = y_1 - mx_1$$

To find the values of m and c:

- Enter the command $\boldsymbol{m := \frac{y2 - y1}{x2 - x1}}$
- Enter the command $\boldsymbol{c := y1 - m \times x1}$

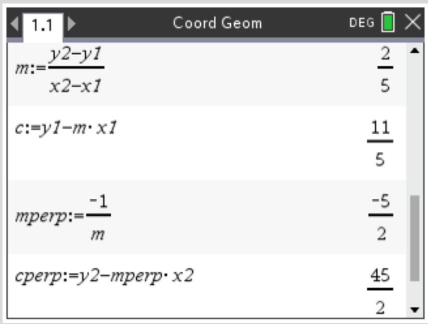


Answer: The equation of the line passing through A and B is $y = \frac{2}{5}x + \frac{11}{5}$.

(e) The line perpendicular to the line segment AB has a gradient which is the negative reciprocal of the gradient m.

The gradient and y-intercept of the perpendicular line passing through $B(x2, y2)$ can be found as follows:

- Enter the command $\boldsymbol{mperp := \frac{-1}{m}}$
- Enter the command $\boldsymbol{cperp := y2 - mperp \times x2}$



Answer: The equation of the line perpendicular to the line segment AB passing through B is $y = \frac{-5}{2}x + \frac{45}{2}$.

Analysing points, line segments and lines in the Notes application

A **Notes** page can be constructed to calculate, for two given points (x_1, y_1) and (x_2, y_2), the midpoint, gradient and length of a line segment, as well as the equation of the line passing through the two points. For demonstration purposes, the points $A(2,3)$ and $B(7,5)$ will be used.

Question

For the two points A (2,3) and B (7,5), find:

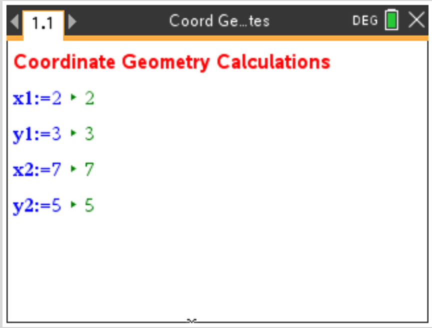
- the coordinates of the midpoint of line segment AB.
- the gradient of the line segment AB.
- the length of the line segment AB.
- the equation of the line passing through A and B.

Solution

To set up a template to answer the above (and similar questions), on a **Notes** page:

- Enter the template title text '**Coordinate Geometry Calculations**' as shown in the screenshot.

***Note**: While not essential, the title text has been formatted to **bold** and **red**. This can be done via* menu *> **Format**.*

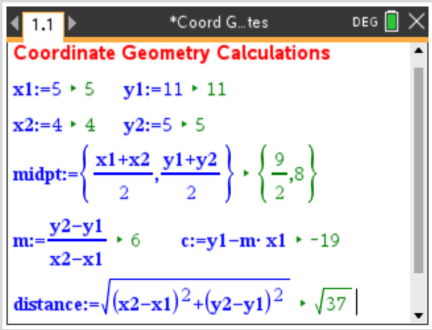


- Press menu > **Insert** > **Maths Box** (or press ctrl M) and enter the command ***x*1:=2** (then press enter).
- Repeat the last step to enter the following (shown right): ***y*1:=3** ***x*2:=7** ***y*2:=5**.
- For the midpoint, in a **Maths Box**, enter the command:

$$midpt := \left\{\frac{x1+x2}{2}, \frac{y1+y2}{2}\right\}.$$

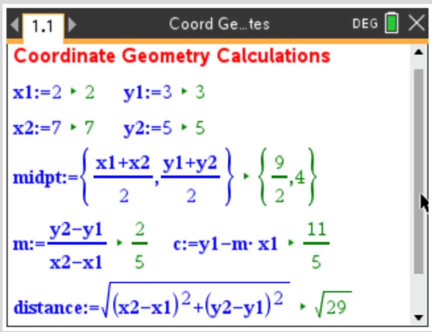
- For the gradient of AB, in a **Maths Box**, enter the command $m := \dfrac{y_2 - y_1}{x_2 - x_1}$.
- For the y-intercept of a line passing through AB, in a **Maths Box**, enter the command $c := y1 - m \times x1$.
- To find the length of the line segment AB, in a **Maths Box**, enter the command:

$$\mathbf{distance:} = \sqrt{(x2-x1)^2 + (y2-y1)^2}.$$



Answers*:* The answers match those given in the previous example. Note that recalculation will occur if any of the coordinate values are changed.

***Note**: The entries/objects on a **Notes** page can be rearranged in ways like working with word processor software. As an example, note that the coordinate values have been moved so that each coordinate pair has its own line. The* ␣ *and* del *keys are helpful for positioning Maths boxes.*



1.1.2 Graphs of linear relations

Graphing lines

In this section, the plotting of single and multiple lines is demonstrated. It is also possible to plot lines for equations not expressed in the functional form $y = ax + b$, using relation graphing methods.

Note: *There are many options for changing the line style, colour and labels – these will be shown in later examples in this section.*

Question

Plot the graphs of the following linear equations:

(a) $y = 2x - 1$ **(b)** $x = 2$ **(c)** $3x + y - 5 = 0$

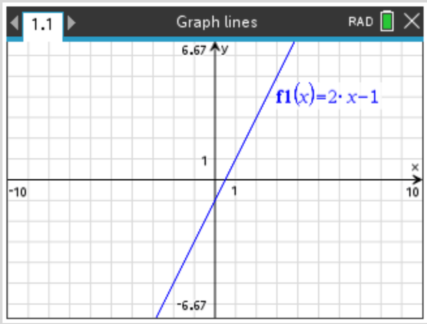
Solution

(a) To plot a line with equation $y = 2x - 1$, on a **Graphs** page:

- Enter the rule $\boldsymbol{f1(x) = 2x - 1}$.

This will plot the line with the above equation using the current window settings.

Note: *The lined grid is displayed by pressing* [menu] **> *View* > *Grid* > *Lined Grid***.

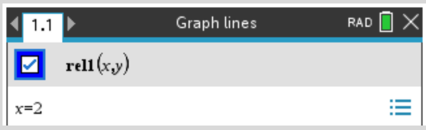


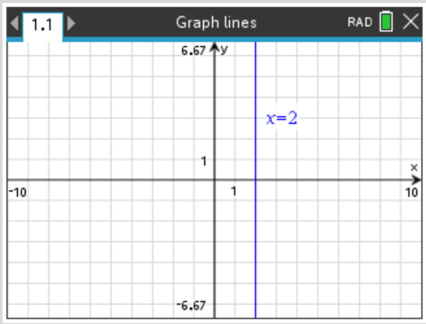
(b) To plot the line with equation $x = 2$ (which is not expressed as a function), on the same **Graphs** page:

- Press [menu] **> Graph Entry/Edit > Relation**.
- For *rel1*(x,y), enter $\boldsymbol{x = 2}$**.**

This will plot the vertical line with equation $x = 2$.

Note: *The equation $x = 2$ is not a function, and so it is necessary to use relation graphing methods, not function graphing methods.*



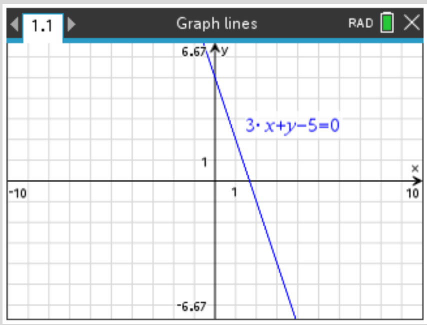


(c) To plot a line for $3x + y - 5 = 0$ (which is not expressed as a function), on the same **Graphs** page:

- Press [ctrl] [G] (or [tab]) and then ▲ to view $x = 2$, the previous rule for *rel1*(x,y).
- Delete $\boldsymbol{x = 2}$, and enter the $\boldsymbol{3x + y - 5 = 0}$.

This will plot the equation on a Cartesian plane using the current window settings.

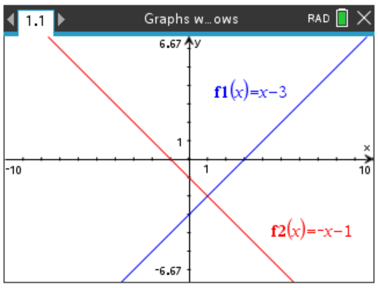
Note: *To return to function graphing mode, press* [menu] > ***Graph Entry/Edit* > *Function***.



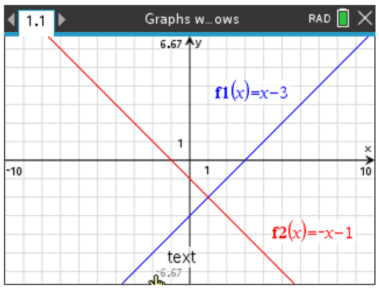
Modifying the graph and graph window settings

The graphing window used in the **Graphs** application has many settings that can be modified to suit the intended purpose. Below are three views of the same two lines with different settings applied.

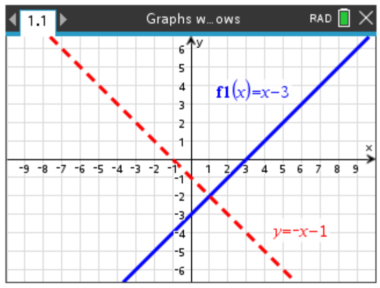
View 1



View 2



View 3



There are several settings or attributes that have been changed to create the different views.

- **Graphs Settings**: Press [menu] > **Settings …**
- **Graph attributes**: hover cursor over a graph, press [ctrl] [menu] then select **Attributes**.
- **Axes attributes**: hover cursor over an axis, press [ctrl] [menu] then select **Attributes**.

Question

For the lines with equations $y = x - 3$ and $y = -x - 1$, modify the graph window to create:

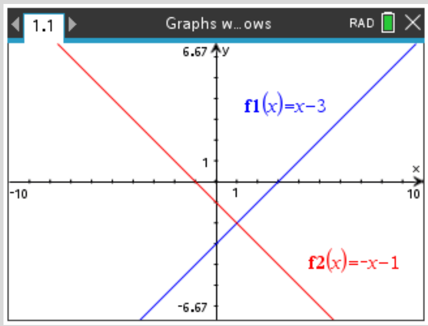
(a) *View 1* **(b)** *View 2* **(c)** *View 3*

Solution

(a) To create *View 1*, on a **Graphs** page:

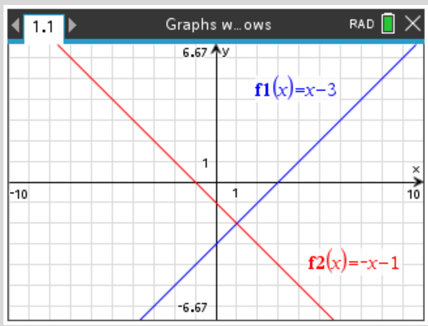
- Press [ctrl] [G] and enter the rule $f1(x) = x - 3$.
- Press [ctrl] [G] and enter the rule $f2(x) = -x - 1$.

Assuming the default settings, this will plot the two lines in the 'standard' viewing window which has the dimensions [–10,10] by [–6.67, 6.67], with tick marks on each axis at every 1 unit (and the same scale on each axis). The default settings are for no grid, and for the end values of the viewing window to be displayed.



(b) To create *View 2*, on the same **Graphs** page:

- Press [menu] > **Settings …** and observe the following options for the default **Graphs** settings:
  - **Grid**: to set whether no grid, lined grid or dotted is to be displayed.
  - **Automatically hide plot labels**: If box is checked, plot labels will only be displayed if the plot is selected (clicked).
- For **Grid**, select **Lined Grid**.
- Click **OK** to save these **Graphs** settings and observe the changes to the **Graphs** display.



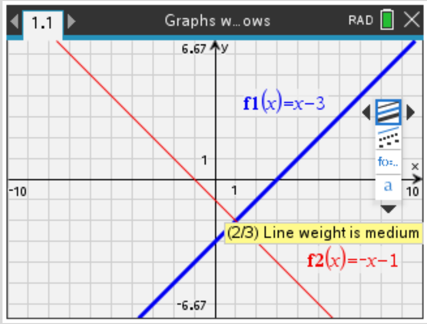
***Note**: The grid options can also be changed by pressing [menu] > **View** > **Grid** and selecting the required grid setting.*

... continued

Solution (continued)

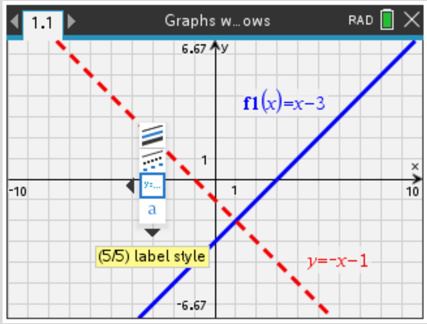
(c) To create *View 3*, on the same Graphs page:

Change the attributes of the graph of $\boldsymbol{f1(x) = x - 3}$ as follows:



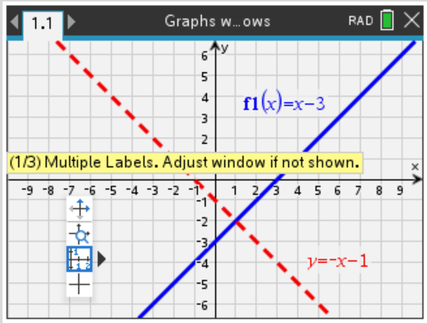
- Hover over the graph of $\boldsymbol{f1(x) = x - 3}$ and press ctrl menu.
- From the pop-up menu, select **Attributes**.
- In the first row of the pop-up menu, use ◀ or ▶ to set the **Line weight** attribute to **Medium**.
- Press enter to apply these attributes.

Change the attributes of the graph of $\boldsymbol{f2(x) = -x - 1}$ as follows:



- Hover over the graph of $\boldsymbol{f2(x) = -x - 1}$ and press ctrl menu.
- From the pop-up menu, select **Attributes** (see right).
- In the first row of the pop-up menu, use ◀ or ▶ to set the **Line weight** attribute to **Medium**.
- Use ▼ or ▲ to move to the second row of the pop-up menu, use ◀ or ▶ to set the **Line style** to **dashed**.
- Use ▼ or ▲ to move to the third row of the pop-up menu, set the **Label style** attribute to '**y=…**'
- Press enter to apply these attributes.

Change the attributes of the axes as follows:

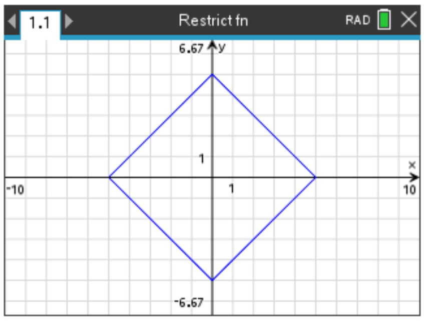


- Hover over the either of the axes, and press ctrl menu.
- From the pop-up menu, select **Attributes** (see right).
- Use ▼ or ▲ to move to the third row of the pop-up menu, set the **Tick Labels** attribute to **Multiple Labels**.
- Press enter to apply these attributes.

Graphing lines with domain restrictions using functional form

It is possible to plot lines with restrictions on the domain for linear equations expressed in the form $y = ax + b$ (i.e. function form). For instance, the entry $\boldsymbol{f1(x) = 2x - 1 \mid x > 0}$ will plot the line $y = 2x - 1$ for positive values of x only.

Note*: The symbol '|' is used to specify that a restriction or condition is to be imposed. This symbol, and the inequality symbols can be accessed via* ctrl =.



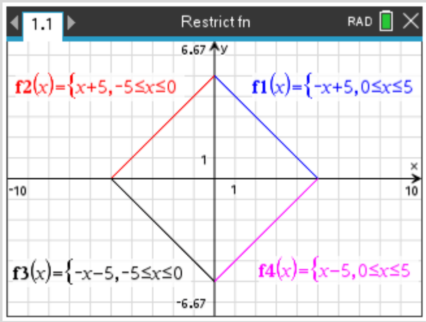
Question

Create the following shape with four line segments using suitable domain restrictions.

Solution

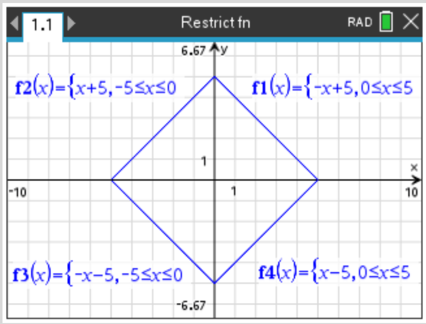
To plot the four line segments, on a **Graphs** page:

- For the 1st quadrant line segment, enter the rule $\boldsymbol{f1(x) = -x + 5 \mid 0 \le x \le 5}$.
- For the 2nd quadrant line segment, enter the rule $\boldsymbol{f2(x) = x + 5 \mid -5 \le x \le 0}$.
- For the 3rd quadrant line segment, enter the rule $\boldsymbol{f3(x) = -x - 5 \mid -5 \le x \le 0}$.
- For the 4th quadrant line segment, enter the rule $\boldsymbol{f4(x) = x - 5 \mid 0 \le x \le 5}$.



This will plot the four line segments, each with a different colour. To set all of the line segments to be the same colour (for example all to be coloured blue):

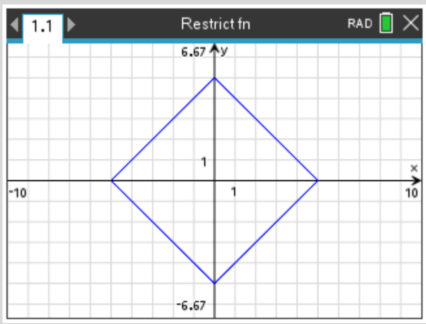
- Hover the cursor over the graph of ***f*2**(x) and then press ctrl menu
- Select **Colour** > **Line Colour**, then select the blue colour.
- Repeat these steps for graphs defined for ***f*3(*x*)** and ***f*4(*x*)**.



Note*: Once entered, the rules in* ***f1*** *to* ***f4*** *are automatically altered to use an opening brace, in the conventional piecewise format (e.g.* $\boldsymbol{f1(x) = \{-x + 5, 0 \le x \le 5}$*).*

To prevent the labels from being displayed,

- Press menu > **Settings** and click the option for **Automatically hide plot labels**.
- Click OK to set this option and display the line segments without the labels visible.



Note*: With the option checked for* ***Automatically hide plot labels****, the labels can still be shown if the cursor is hovered over each graph, or if the graph is clicked directly.*

Graphing lines with domain and/or range restrictions using relational form

Using the relation graphing feature, it is possible to plot lines with restrictions on the domain and/or range. To specify range restrictions, or a combination of domain and range restrictions, the graph type must be changed to **Relation** type, as the following example illustrates.

Question

Plot the graphs for each the following linear relations, including the stated restrictions:

- $y = 2x - 1,$ for $x > 1$ and $y \le 4$
- $x = -5,$ for $-3 \le y \le 3$
- $x + y = 3,$ for $y < -2.$

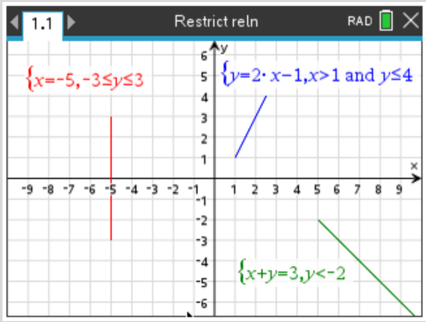
Solution

***Note**: The symbol '|' is used to specify that a restriction or condition is to be imposed. This symbol, and the inequality symbols can be accessed via* [ctrl] [=].

To plot a line segment as defined above, on a **Graphs** page:

- Press [menu] > **Graph Entry/Edit** > **Relation**.
- For **rel1**, enter the equation $\boldsymbol{y = 2x - 1 \,|\, x > 1 \text{ and } y \le 4}$.
- For **rel2**, press [ctrl] [G] (or [tab]), and then enter the equation $\boldsymbol{x = -5 \,|\, -3 \le y \le 3}$.
- For **rel3**, press [ctrl] [G] (or [tab]), and then enter the equation $\boldsymbol{x + y = 3 \,|\, y < -2}$.

This will plot the equations with the associated domain and/or range restrictions on a Cartesian plane using the current window settings.



Graphing multiple lines using list syntax

It is possible to plot several lines in an efficient way by defining a set of lines in the same statement using the braces. For example, the definition $f1(x) = \{-2, -1, -0.5, 0.5, 1, 2\} \times x$ will produce a set of six lines with gradients given by the elements in the bracketed list.

Question 1

Use the lists syntax to graph $y = kx,$ where $k \in \{-2, -1, -0.5, 0.5, 1, 2\}$ on the Cartesian plane.

Solution

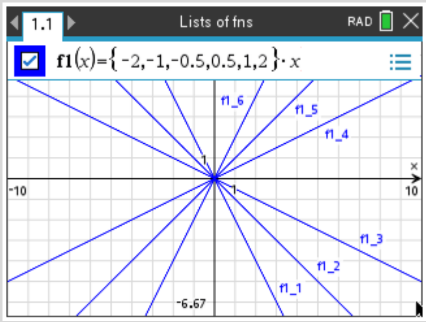
To plot this set of lines, on a **Graphs** page:

- Enter the rule $\boldsymbol{f1(x) = \{-2, -1, -0.5, 0.5, 1, 2\} \times x}$.

This will plot six lines with equations:

$$y = -2x,\ y = -x,\ y = -0.5x,\ y = 0.5x,\ y = x,\ y = 2x.$$

***Note**: Alternatively, if the six listed **k** values had been stored on a **Calculator** page as* $\boldsymbol{k := \{-2, -1, -0.5, 0.5, 1, 2\}}$*, the set of lines could be defined as* $\boldsymbol{f1(x) = k \times x}$.



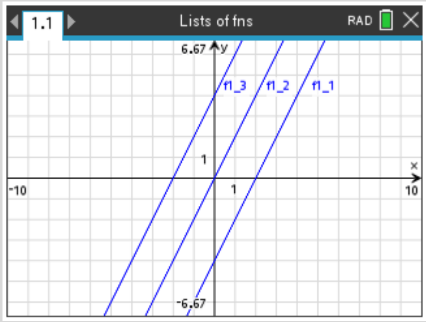
Question 2

Use the lists syntax to graph $y = 2x - 4$, $y = 2x$ and $y = 2x + 4$ on the Cartesian plane, and modify the labels so that the above equations are displayed next to the relevant graph.

Solution

To plot these set of lines, on a **Graphs** page:

- Press [ctrl] [G] and then ▲ to view the rule for $f1(x)$.
- Delete any previous rule defined in $f1(x)$.
- Enter the rule $f1(x) = 2x + \{-4, 0, 4\}$.

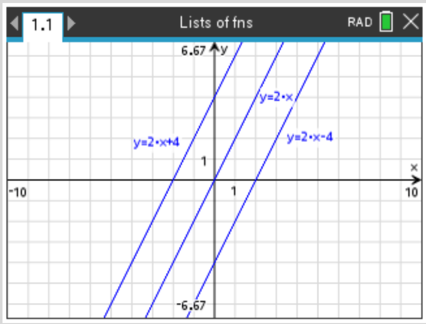


This will plot 3 lines with the following equations (subscripts are used for the graph labels:
$f1\_1 = 2x - 4$, $f1\_2 = 2x$ and $f1\_3 = 2x + 4$.

***Note**: The labels next to the lines match the order of the elements in the list {–4,0,4}. The type of labels can be altered by hovering over each line, pressing [ctrl] [menu] and then selecting **Attributes**.*

Change the label style of the graph of ***f1_1*** to display more helpful labels as follows:

- Hover over the graph of ***f1_1*** and press [ctrl] [menu].
- From the pop-up menu, select **Attributes** (see right).
- Press ▼ to move to the third row of the pop-up menu, then press the ▶ key to set the **Label style** attribute to '**y=…**'
- Press [enter] to apply these attributes.
- Repeat this process for ***f*1_2** and ***f*1_3**.



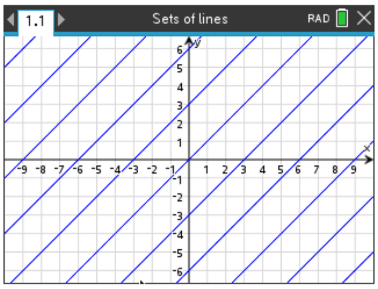
Graphing multiple lines with the sequence command

It is possible to plot several lines in an efficient way by defining a set of lines in the same statement using the **sequence** command. The **sequence** command has the basic syntax **seq(expression, variable, start, finish, step size)**. As an example, the command **seq(2*n*+1,*n*,0,3,1)** will produce the set of numbers $\{1,3,5,7\}$.

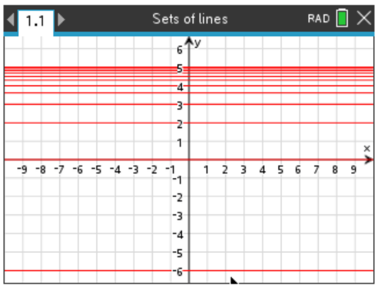
Question

Use the sequence command to define the following sets of lines on the Cartesian plane.

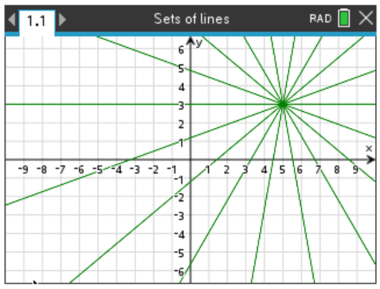
(a)



(b)



(c)



Solution

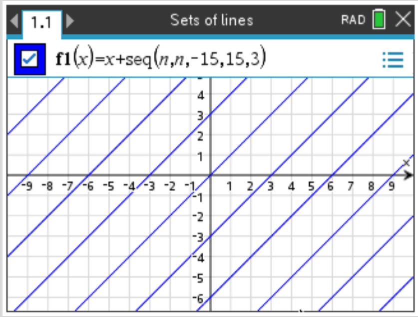
(a) To plot this set of lines, on a **Graphs** page:

- Enter the rule $\boldsymbol{f1(x) = x + \text{seq}(n, n, -15, 15, 3)}$.

This will plot a sequence of lines with equations:

$y = x - 15$, $y = x - 12$, $y = x - 9$, …, $y = x + 12$, $y = x + 15$.

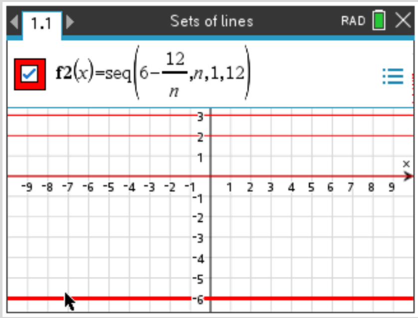
***Note**: This set of graphs could also be defined as:*
$\boldsymbol{f1(x) = x + \text{seq}(3n, n, -5, 5, 1)}$, *or alternatively as*
$\boldsymbol{f1(x) = x + \{-15, -12, -9, -6, -3, 0, 3, 6, 9, 12, 15\}}$.



(b) To plot this set of lines, on a **Graphs** page:

- Press ctrl G and then ▲ to view the rule for ***f*1(*x*)**.
- Click the checkbox to hide graphs defined in ***f*1(*x*)**.
- Press ▼ and enter the rule $\boldsymbol{f2(x) = \text{seq}\left(6 - \frac{12}{n}, n, 1, 12\right)}$.

***Note**: If the final parameter in the sequence command for step size is omitted, a default step size of 1 is used.*

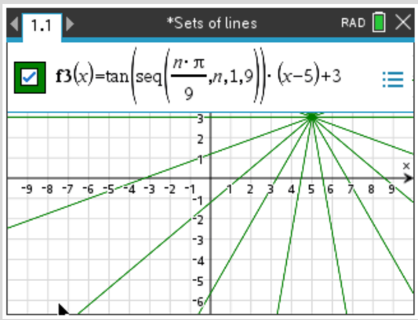


(c) To plot this set of lines, on a **Graphs** page:

- Press ctrl G and then ▲ to view the rule for $\boldsymbol{f2(x)}$.
- Click the checkbox to hide graphs defined in $\boldsymbol{f2(x)}$.
- Press ▼ and enter the rule

$$\boldsymbol{f3(x) = \tan\left(\text{seq}\left(\frac{n\pi}{9}, n, 1, 9\right)\right) \times (x - 5) + 3}.$$

This will produce a set of lines that pass through the point $(5,3)$ with gradients $m = \left\{\tan\frac{\pi}{9}, \tan\frac{2\pi}{9}, \ldots, \tan\frac{9\pi}{9}\right\}$.



1.1.3 Linear equations

Substitution and solving with linear equations

Question

Consider the equation $v = u + at$. Use substitution and solving commands to answer the following.

(a) Find v if $u = 20$, $a = 9.8$ and $t = 1.5$.

(b) Find u if $v = 112$, $a = 9.8$ and $t = 3$.

(c) Transpose the original equation so that t is the subject (using Solve command)

(d) Transpose the original equation so that t is the subject (using a sequence of operations)

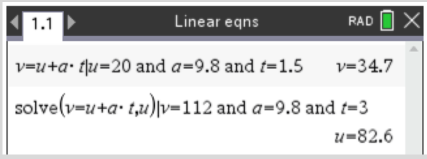
Solution

Notes:

(1) In the following, the vertical line symbol '|' is used to denote substitution or 'when' or 'given that'. This symbol can be accessed via [ctrl] [=].

(2) When entering the term 'at' in the equation, ensure that the multiplication symbol is added between the a and the t.

(3) The word 'and' and the 'space' character ([␣]) *can just be entered from the alphabetic keys at the bottom of the calculator. Alternatively, 'and' can be found via* [📖] [1] [A].



To complete these calculations, on a **Calculator** page:

(a) Substitution

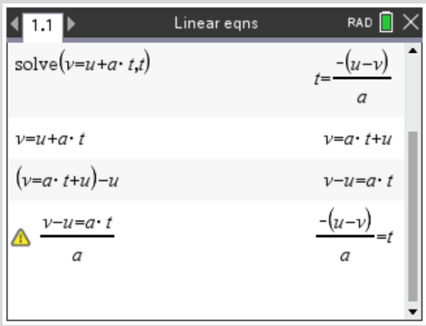
- Enter $\boldsymbol{v = u + at \mid u = 20}$ **and** $\boldsymbol{a = 9.8}$ **and** $\boldsymbol{t = 1.5}$.

Answer: $v = 34.7$.

(b) Solving

- Press [menu] > **Algebra** > **Solve** and then enter as follows: **solve(**$\boldsymbol{v = u + at, u}$**)** $\boldsymbol{\mid v = 112}$ **and** $\boldsymbol{a = 9.8}$ **and** $\boldsymbol{t = 3}$.

Answer: $u = 82.6$.



(c) Transposition (via Solve)

- Press [menu] > **Algebra** > **Solve** and then enter as follows: **solve(**$\boldsymbol{v = u + at, t}$**)**.

(d) Transposition (via a sequence of operations)

- Enter $\boldsymbol{v = u + at}$.
- To subtract u from both sides of the previous equation, press [−] [U] [enter].
- To divide both sides of the previous equation by a, press [÷] [A] [enter].

Answer to (c) and (d): $t = \dfrac{-(u - v)}{a}$.

1.1.4 Simultaneous linear equations

Solving simultaneous linear equations by a sequence of arithmetic operations

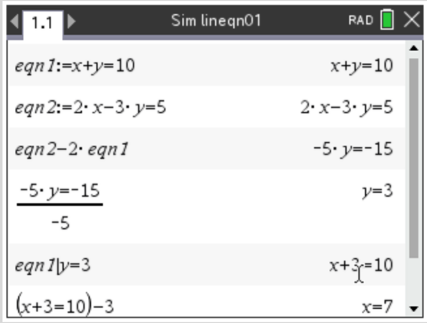
Question

Solve the equations $x + y = 10$ and $2x - 3y = 5$ simultaneously by the elimination method.

Solution

The pair of equations can be solved simultaneously by a sequence of arithmetic operations. On a **Calculator** page:

- To store the first equation, enter $\boldsymbol{eqn1 := x + y = 10}$.
- To store the second equation, enter $\boldsymbol{eqn2 := 2x - 3y = 5}$.
- To eliminate the variable x, enter $\boldsymbol{eqn2 - 2eqn1}$.
- To solve for y, press [÷] [(−)] [5] [enter].
- To solve for x, enter $\boldsymbol{eqn1 \mid y = 3}$ and then press [−] [3] [enter] to subtract 3 from the resulting equation.



Answer: The solution is $x = 7$ and $y = 3$.

***Note**: The assign symbol ":=" can be found via [ctrl] [▫◦{▫]. The vertical line symbol can be found via [ctrl] [=].*

Solving simultaneous linear equations using solving commands

Question

Solve the equations $x + y = 10$ and $2x - 3y = 5$ simultaneously using CAS solving commands.

Solution

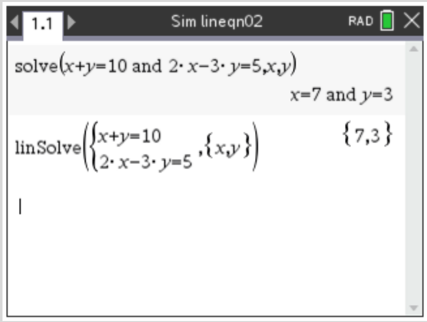
The pair of equations can be solved simultaneously by using CAS solving commands. To do this, on a **Calculator** page:

Method 1

- Enter $\mathbf{solve}(\boldsymbol{x + y = 10}$ **and** $\boldsymbol{2x - 3y = 5, x, y})$.

Method 2

- Press [menu] **> Algebra > Solve System of Equations >Solve System of Linear Equations …**
- In the dialog box that follows
  - For **Number of equations**, enter **2**.
  - For **Variables**, enter ***x*,*y***.
- Enter the two equations into the template and then press [enter].



Answer: The solution is $x = 7$ and $y = 3$.

***Note**: Method 2 expresses the solution in set notation, and this method may be more practical if there are more than 2 equations or variables.*

Generalising with simultaneous equations

Question

A teacher is trying to generate some student worksheets involving simultaneous linear equation solving. She wants to find pairs of linear equations that have integer solutions. Specifically, she is trying to find integer values of a and b that will give integer solutions to the following pair of linear equations.

$$x + y = 10$$
$$ax + by = 20$$

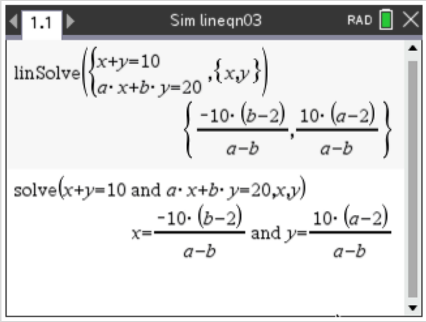
(a) Find a general solution for x and y in the above pair of equations in terms of a and b.

(b) Investigate some integer values of a and b for which there will be integer solutions to the above pair of equations.

Solution

(a) The pair of equations can be solved simultaneously by using CAS solving commands. On a **Calculator** page:

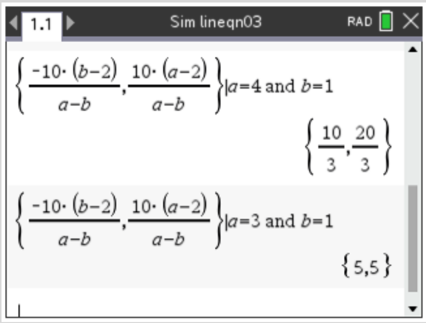
- Press menu > **Algebra > Solve System of Equations >Solve System of Linear Equations …**
- In the dialog box that follows:
  - For **Number of equations**, enter **2**.
  - For **Variables**, enter ***x*,*y***.
- Enter the two above equations into the template and then press enter.



Answer: The solution is $x = \dfrac{-10(b-2)}{a-b}$ and $y = \dfrac{10(a-2)}{a-b}$.

***Note**: When entering the equation ax + by = 20, ensure that you press the multiplication key between a and x, and between b and y.*

(b) Students will benefit from experimenting with values of a and b and observing whether these values result in integer solutions. As an example, $a = 4$ and $b = 1$ will not result in integer solutions, but $a = 3$ and $b = 1$ will do so (see some example substitutions right).

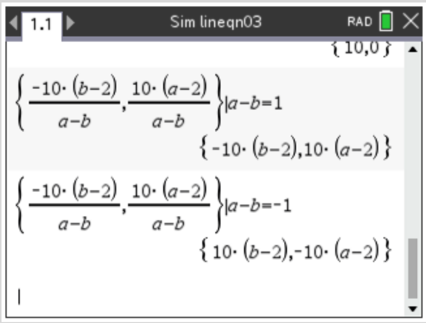


Answer: Integer solutions will occur if

$a - b = \pm 1, \pm 2, \pm 5, \pm 10$

$a = 2$ or $b = 2$

Other integer solutions are possible.



Working with general solutions of a pair of simultaneous equations

Question

Observe the following pair of equations.

$$x + y = 10 \quad \ldots[1]$$
$$2x + 2y = 20 \quad \ldots[2]$$

Note also that there is only one unique equation present, since equation [2] = 2 × equation [1].

Find a general solution for the above pair of equations using an extra parameter.

***Note**: It is preferable that students construct the general solution by hand and head for an example such as this. The solution below is used to illustrate how the CAS handles a general solution case.*

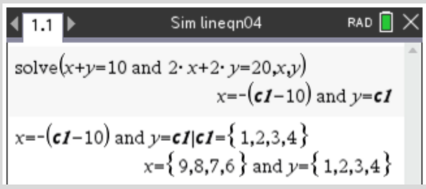
Solution

(a) The pair of equations can be solved simultaneously by using CAS solving commands. On a **Calculator** page:

- Enter **solve(**$\boldsymbol{x + y = 10}$ **and** $\boldsymbol{2x + 2y = 20, x, y}$**)**

Answer: The general solution is $x = -(\mathbf{c1} - 10)$ and $y = \mathbf{c1}$, where **c1** is any real number (The *TI-Nspire CAS* uses the letter c to denote a parameter which is a real number).

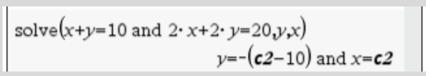
The general solution can also be expressed as $x = 10 - k, y = k,$ where $k \in R$.



(b) The general solution can be used to give some possible solutions by substituting values into ***c1***. To do this, copy and paste the bolded ***c1*** into an expression where substitution is used (see example above right).

***Notes**: (1) Typing 'c1' will not work for the purposes of substitution – the CAS treats 'c1' and '**c1**' as different objects.*

(2) It is possible to create a general solution starting from y = k by reversing the order of x and y in the solve command (see right). This leads to the alternatively expressed general solution of the form $x = k, y = 10 - k,$ where $k \in R$.



Modelling with simultaneous linear equations

Question

OZ-Electrics is a newly launched electric bike hirer that will offer competitive cost rates to attract customers from its competitors. Customers can choose from three different daily cost schemes.

- *OZ-Electrics Scheme A - $10 rental per day and a distance usage rate of $1.20 per km.*
- *OZ-Electrics Scheme B - $20 rental per day and a distance usage rate of $0.80 per km.*
- *OZ-Electrics Scheme C - $35 rental per day and a distance usage rate of $0.40 per km.*

Note that if only part of a kilometre is used, only that part of a kilometre is charged to the user. Let $A(x)$, $B(x)$, and $C(x)$ be the daily cost functions in dollars for each of the schemes where x represents the distance the user rides in kilometres on that day.

(a) Write down rules for $A(x)$, $B(x)$, and $C(x)$. Use these rules to fill the missing values in the table.

Distance travelled	Cost for Scheme *A*	Cost for Scheme *B*	Cost for Scheme *C*
20 km	$34.00	$36.00	$43.00
30 km			
40 km			
50 km			

(b) Enter and plot each of the three cost functions on the same set of axes. Select a viewing window to show all three lines and their points of intersection.

(c) Find the coordinates of the points at which the graphs intersect. What do these points represent?

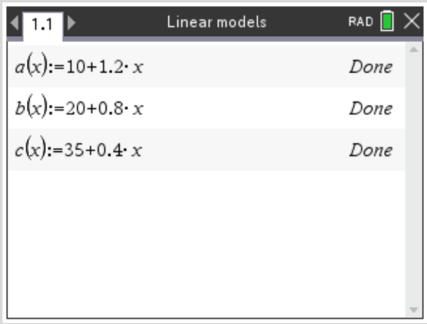
(d) Summarise which scheme will be best for customers, based upon the number of kilometres they expect to travel in a day.

Solution

(a) To construct rules for each scheme, on a **Calculator** page:

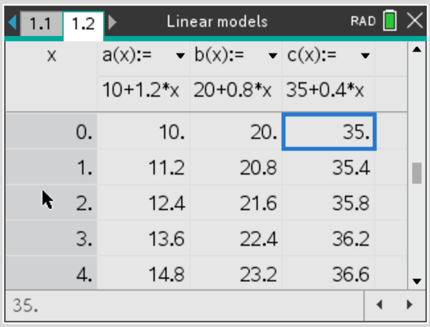
- For Scheme A, enter the rule $\boldsymbol{a(x) := 10 + 1.2x}$.
- For Scheme B, enter the rule $\boldsymbol{b(x) := 20 + 0.8x}$.
- For Scheme C, enter the rule $\boldsymbol{c(x) := 35 + 0.4x}$.

Note: The assign symbol ":=" can be found via ctrl [templates key].



(b) To construct a table to evaluate the rules for each scheme, add a **Lists & Spreadsheet** page, and then:

- Press ctrl T to change the view to **Table** mode.
- In the pop-up list that follows, select ***a*** to display a table of values for ***a*(*x*)** for the listed ***x*** values (Scheme A costs).
- Click at the top right of the next empty column to view a pop-up list, then select ***b*** to display a table of values for ***b*(*x*)** for the listed ***x*** values (Scheme B costs).

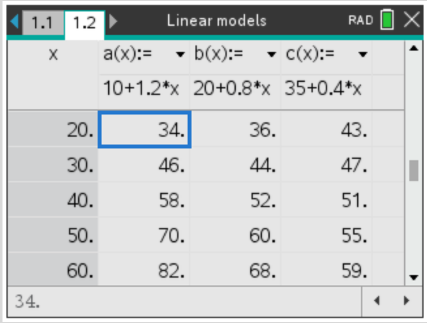


... continued

Solution (continued)

- Click at the top right of the next empty column to view a pop-up list, then select ***c*** to display a table of values for ***c*(*x*)** for the listed *x* values (Scheme *C* costs).
- To change the *x* values so that they start at 20 km and increment by 10 km, press [menu] > **Table > Edit Table Settings …**
- In the dialog box that follows, enter the following values: **Table Start = 20** and **Table Step = 10.**

Now the missing values can be filled in from this table.

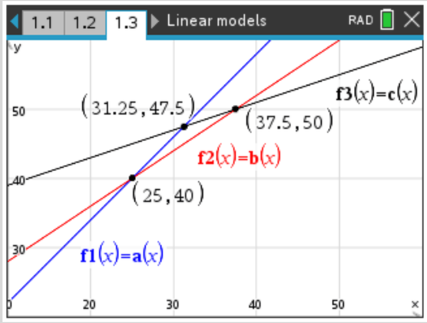


(c) To construct the lines for the costs of each scheme, add a Graphs page and then:

- Enter $f1(x) = a(x)$.
- Press [ctrl] [G] and enter $f2(x) = b(x)$.
- Press [ctrl] [G] and enter $f3(x) = c(x)$.

Observing the table values, a better view of the lines could be made by editing the **Window Settings** as follows:

- Press [menu] > **Window/Zoom > Window Settings**.
  Adjust the window settings as shown.
  XMin = 10 Xmax = 60 XScale = 10
  YMin = 0 YMax = 60 YScale = 10

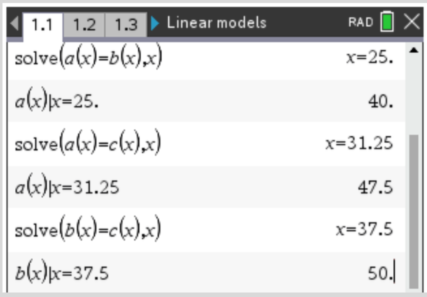


Note: The axes tick marks and grid settings can be modified using methods demonstrated in Section 1.1.3. The display precision of the coordinates of the points of intersection can be changed by pressing [menu] > ***Settings*** *and then modify the* ***Display Digits*** *value.*

The points of intersection are found graphically as follows:

- Press [menu] > **Analyse Graph** > **Intersection**.
- Click the graphs for *a*(*x*) and *b*(*x*), then click to the left (for lower bound) and then click to the right (for upper bound) of that intersection point.
- Repeat this procedure to find the coordinates of the other two intersection points.

Now the coordinates of the points of intersection are displayed as (25, 40), (31.25,47.5) and (37.5, 50). These coordinate values can be confirmed by algebraic solving, or by using the solve command in the **Calculator** application.



(d) If it is expected that the daily ride will be less than 25 km, Scheme *A* is cheapest option. Scheme *B* is cheapest for daily rides between 25 and 37.55 km. Scheme C becomes the cheapest for daily rides of greater than 37.5 minutes.

1.2 Introduction to functions and their inverses

1.2.1 Representation of a function by rule, graph and table

Playing the 'Guess the rule' game

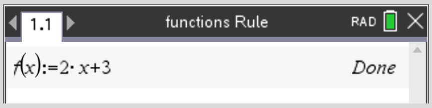
Question

Play the 'Guess the rule' game with a partner using the method described below.

Solution

To define a function rule for your partner to figure out, on a **Calculator** page:

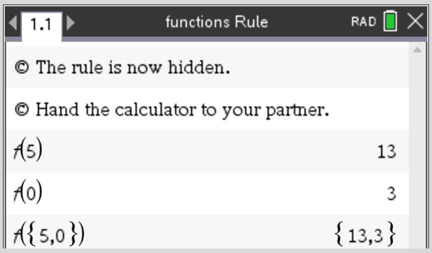
- Enter $\boldsymbol{f(x):=2x+3}$, pressing [ctrl] [:=] for **assign** symbol.



This defines a rule that will double any input value and then add 3 to the result.

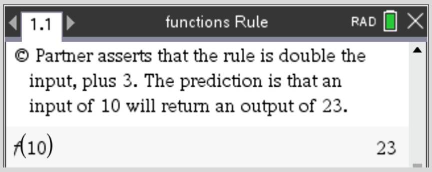
To clear the screen so that the rule is no longer visible:

- Press [menu] > **Actions** > **Clear History**.



To allow your partner to deduce the function rule:

- Pass the calculator to your partner and ask them to figure out the hidden rule by testing various input values and observing the resulting output values.
- This can be achieved by entering such things as $\boldsymbol{f(5)}$ etc.
- When your partner has tested enough input values (two tests should be sufficient if you tell them that the rule is linear), they state what they believe the rule to be and then test their assertion by predicting the output for their next input value.



Note*: To make the challenge achievable, it is best to first agree on some guidelines. E.g. The function will be linear.*

Determining the rule for a linear or quadratic function from a table of values

Question

Two functions are represented by the tables below.

(i)

x	-2	-1	0	1	2	3
$f(x)$	5	$\frac{7}{2}$	2	$\frac{1}{2}$	-1	$-\frac{5}{2}$

(ii)

x	-2	-1	0	1	2	3
$f(x)$	-2	$-\frac{7}{2}$	-4	$-\frac{7}{2}$	-2	$\frac{1}{2}$

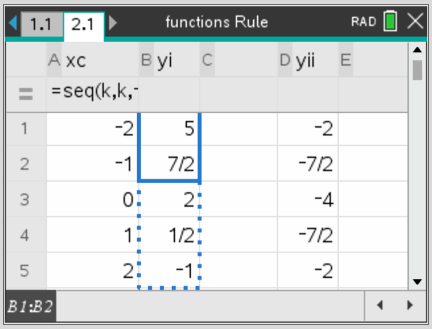
(a) Use the method of common differences to determine whether a function is linear or quadratic.

(b) Verify the function rule by fitting a linear or quadratic graph to a scatter plot for each table.

Note*: Although not required, exploring common differences is a simple and insightful way to determine or confirm the degree of a polynomial from numerical data.*

Solution

(a) To enter the table values, on a **Lists & Spreadsheet** page:

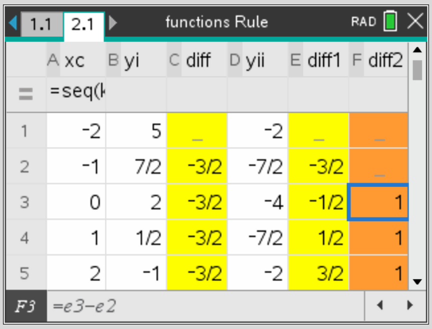


- In the heading row, enter the titles for column A, ***xc***, column B, ***yi***, and column D, ***yii***, as shown.
- In the column A formula cell, enter $=\mathbf{seq}(k,k,-2,3)$ by pressing [book] [1] [S] to select **seq**(Expr, Var, Low, High).
- Enter the number **5** in cell B1 and the number **7/2** in B2.

To test whether the rule for table (i) is linear:

- Navigate to cell B2, press [shift] ▲ then [ctrl] [menu] > **Fill**.
- Press ▼ down to cell B6, then press [enter].

The column B values are auto-filled correctly by following the linear pattern 5, 7/2, …, indicating that the function is linear.



- In column D, enter the *f*(*x*) values for table (ii).

To determine the common differences for table (i):

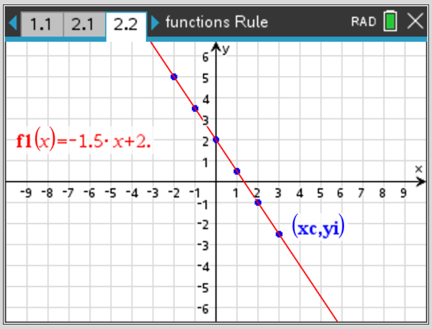
- In cell C2, enter the formula =***b2*** – ***b1***.
- Navigate to cell C2 and press [ctrl] [menu] > **Fill**. Press ▼ down to cell B6, then press [enter].

To determine the common differences for table (ii):

- In cell E2, enter the formula =***d2*** – ***d1***.
- Fill down the formula to cell E6, as described above.
- In cell F3, enter the formula =***e3*** – ***e2***.
- Fill down the formula to cell F6, as described above.

Answer: Table (i). The difference between consecutive values of *f*(*x*) is – 3/2, indicating linear change by constant addition. Table (ii). Constant second difference. A quadratic function.

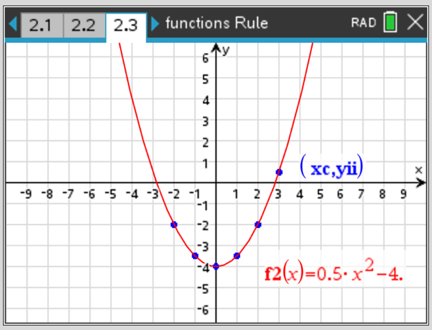
(b) To create scatter plots from the tables, on a **Graphs** page:



- Press [menu] > **Graph Entry/Edit** > **Scatter Plot**.
- For **s1**, enter $x \leftarrow \boldsymbol{xc}$ and $y \leftarrow \boldsymbol{yi}$.
- On a separate **Graphs** page, enter $x \leftarrow \boldsymbol{xc}$ and $y \leftarrow \boldsymbol{yii}$.
- Press [menu] > **Settings**. Select **Float 3** and **Lined Grid**.

To fit a linear graph to the first scatter plot:

- Press [menu] > **Graph Entry/Edit** > **Function**.
- Enter $\boldsymbol{f1}(\boldsymbol{x}) = -\boldsymbol{x}$. Translate or change the gradient of the line by hovering over the line until ✥ or ⟲ appear.
- Press [ctrl] [grab] to grab and move the line, [esc] to release.



To fit a quadratic graph to the second scatter plot:

- Repeat but use $\boldsymbol{f2}(\boldsymbol{x}) = \boldsymbol{x}^2$ as the starting graph.

Answer: The graphs that fit the scatter plots are
(i) $f(x) = 1.5x + 2$ and **(ii)** $f(x) = 0.5x^2 - 4$.

1.2.2 Domain and range of a function

Graphing a function with a restricted domain

Question

Graph the following functions and state the domain, codomain and range in each case.

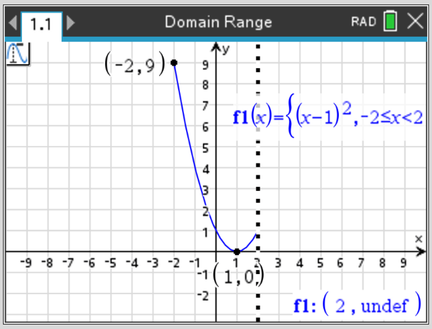
(a) $f:[-2,2)\to \mathrm{R}, f(x)=(x-1)^2$

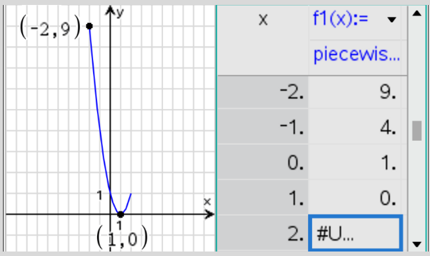
(b) $g:D\to R, g(x)=\dfrac{1}{x}+1$, where D is the maximum domain within the interval $-2<x\le 2$.

Solution

(a) To graph *f*, on a **Graphs** page:

- Enter $\boldsymbol{f1(x)=(x-1)^2 \mid -2\le x<2}$ by pressing [ctrl] [=] to select the 'given', |, and inequality, $\le, <$, symbols.
- Press [menu] > **Window/Zoom** > **Window Settings**.
  Adjust the window settings as shown.
  XMin = –10 Xmax = 10 XScale = 1
  YMin = –3 YMax = 10 YScale = 1
- Hover over an axis, press [ctrl] [menu] > **Attributes** and select **Multiple Labels**.
- Press [menu] > **Trace** > **Graph Trace**, then press the ◀ or ▶ keys to view the coordinates of points on the graph.
- To 'jump' to a particular point, enter the *x*-value, e.g. 'jump' to $(1, f(1))$ by pressing [1] [enter].
- Press [enter] again to pin the coordinates of the point.
- Press [esc] to exit the **Trace** tool.

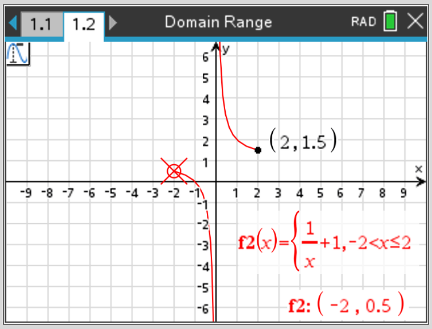


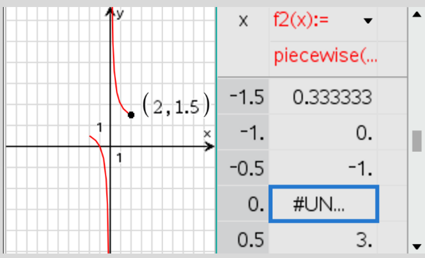


Answer: The domain is given as $[-2,2)=\{x:-2\le x<2\}$, range is $[0,9]=\{y:0\le y\le 9\}$. The codomain is $R=(-\infty,\infty)$ which is the target set of *potential* output values of *f*.

(b) To graph *g*, on a **Graphs** page:

- Enter $\boldsymbol{f2(x)=\dfrac{1}{x}+1 \mid -2<x\le 2}$.
- Press [menu] > **Trace** > **Graph Trace**, to explore the endpoints of the graph.
- With the **Trace** tool active, key in – **1.999** and press [enter] to find the open endpoint at *x* = – 2.
- Press [ctrl] [T] to toggle a table of values.
- Press [menu] > **Table** > **Edit Table Settings** to customise the table.





Answer: The graph shows the:
domain is $(-2,0)\cup(0,2]=(-2,2]\setminus\{0\}$;
range is $y\in(-\infty,0.5)\cup[2,\infty)$. The codomain is $R=(-\infty,\infty)$.

Exploring the maximal or implied domain of a function

Question

Determine the maximal domain and range of the functions with the following rules.

(a) $f(x)=2\sqrt{x+5}-3$ **(b)** $g(x)=2-\sqrt{16-(x-1)^2}$ **(c)** $h(x)=4-\sqrt{x^2+2x}$

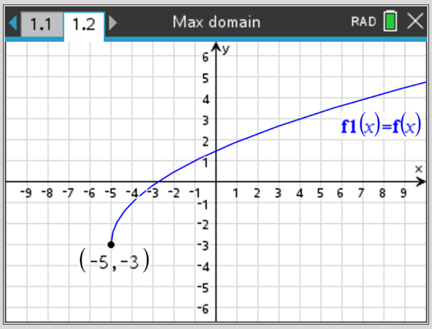
Solution

(a) To find the maximal domain of *f*, on a **Calculator** page:

- Enter $\boldsymbol{f(x):=2\sqrt{x+5}-2}$, pressing [ctrl] [template] for **:=**.
- Press [book] [1] [D] and select **domain**.
- Enter **domain(*f*(*x*), *x*)**. The syntax is **domain**(Expr, Var).

To explore the range of *f*, add a **Graphs** page, then:

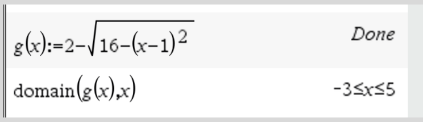
- Enter ***f*1(*x*) = *f*(*x*)**.
- Hover over an axis, press [ctrl] [menu] > **Attributes** and select **Multiple Labels**.
- Press [menu] > **Trace** > **Graph Trace** to explore the endpoints of the graph, as shown in the previous problem.



Answer: Domain: $[-5,\infty)=\{x:-5\le x<\infty\}$. Range: $[-3,\infty)$.

(b) To find the maximal domain of *g*, on a **Calculator** page:

- Enter $\boldsymbol{g(x):=2-\sqrt{16-(x-1)^2}}$.
- Enter **domain(*g*(*x*), *x*)**.

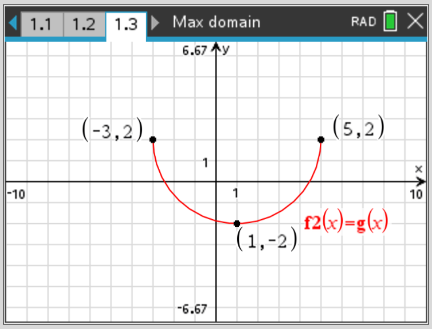


To explore the range of *g*, add a **Graphs** page, then:

- Enter ***f*2(*x*) = *g*(*x*)**, then use the **Trace** tool to explore the endpoints of the graph, as in previous problems.

To find the coordinates of the minimum point of the graph:

- Press [menu] > **Analyse Graph** > **Minimum**. Move the cursor to the left of the minimum for a lower bound and press [enter], then to the right and press [enter].



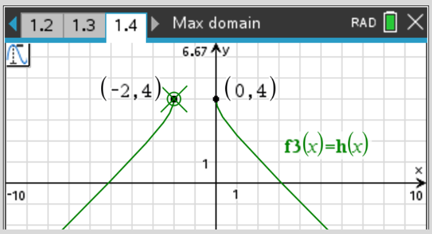
Answer: Domain: $[-3,5]=\{x:-3\le x\le 5\}$. Range: $[-2,\ 2]$.

(c) To find the maximal domain of *h*, on a **Calculator** page:

- Enter $\boldsymbol{h(x):=4-\sqrt{x^2+2x}}$, then enter **domain(*h*(*x*), *x*)**.

To explore the range of *h*, add a **Graphs** page, then:

- Enter ***f*3(*x*) =*h*(*x*)**. Use the **Trace** tool to explore the endpoints of the graph, as in previous problems.



Answer: Domain:
$(-\infty,-2]\cup[0,\infty)=\{x:x\le -2\}\cup\{x:x\ge 0\}$. Range: $(-\infty,4]$.

1.2.3 The inverse of a function

Understanding inverse of a function through a pointwise approach

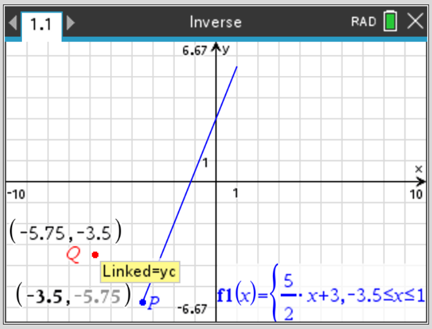
Question

Let $P(x_c, y_c)$ be a point on the graph of a function *f*. Explore the locus of a point *Q* with coordinates (y_c, x_c) and find the equation of a graph containing the locus of *Q*. Consider the cases where *f* is:

(a) $f:[-3.5,1]\to R,\ f(x)=\frac{5}{2}x+3$ **(b)** $f:[-3.5,1]\to R,\ f(x)=\frac{3}{2}(x+1)^2-4$

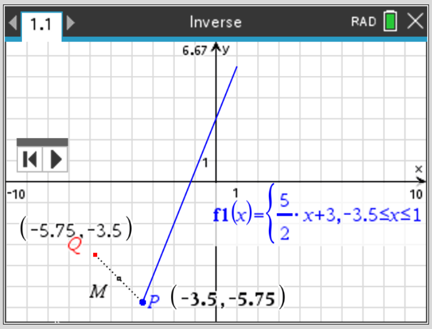
Solution

(a) To graph $f(x)=\frac{5}{2}x+3, x\in[-3.5,1]$, on a **Graphs** page:



- Enter $\boldsymbol{f1(x)=\frac{5}{2}x+3\,|\,-3.5\le x\le 1}$ by pressing [ctrl] [=] to select the 'given', |, and inequality, ≤ symbol.
- Press [menu] > **Trace** > **Graph Trace**. Key in **–3.5** and press [enter] [enter]. This places a point at an endpoint.
- Label this endpoint by hovering over it, pressing [ctrl] [menu] > **Label** and entering the label, ***P***.

To set up the locus of the point *Q* with coordinates (y_c, x_c):

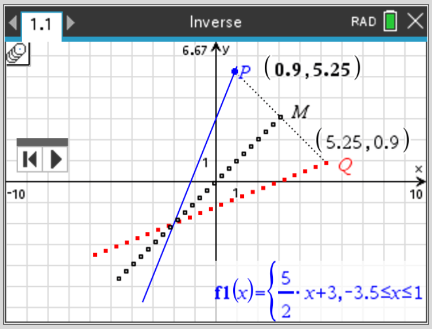


- Hover over the *x*-coordinate of *P*, press [ctrl] [menu] > **Store** and enter ***xc***:**= –3.5**.
- Similarly, store the *y*-coordinate of *P*, entering ***yc***:**= –5.75**.
- Press [P] > **Point by Coordinates** and enter $(\boldsymbol{yc, xc})$.
- Label this point ***Q*** by pressing [ctrl] [menu] > **Label**.

To set up the midpoint, *M*, of line segment *PQ*:

- Press [menu] > **Geometry** > **Points & Lines** > **Segment** then click points *P* and *Q*.
- Press [menu] > **Geometry** > **Constructions** > **Midpoint**, click segment *PQ* then [esc] to exit the tool.
- Label this point ***M*** by pressing [ctrl] [menu] > **Label**.

To animate point *P* with animation control buttons:



- Hover over point *P*, press [ctrl] [menu] > **Attributes**, then press ▼ [1] [enter] [enter]. This sets a unidirectional animation speed of 1 (on a scale of 0 to 9).
- Use the control buttons to start/pause/reset the animation.

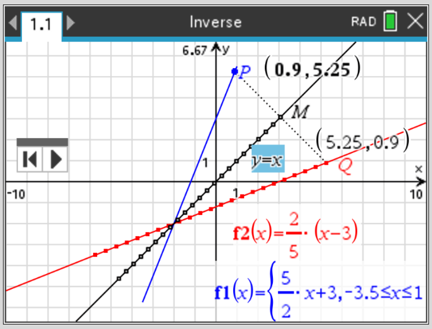
... continued

Solution (continued)

To obtain a trace of the locus of points Q and M:

- To multi-select points Q and M, click point ***Q*** then hover over point M and press ctrl menu > **Geometry Trace**. Start the animation of P to create a trace of points Q and M.

To graph functions that fit the traces of points Q and M:

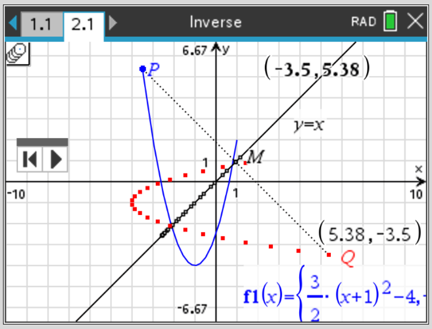


- Press ctrl G. Key in $\boldsymbol{f2(x)=\frac{2}{5}(x-3)}$, then press ▼ del. Select the '$\boldsymbol{y=}$' option and enter $\boldsymbol{y=x}$.

Answer: The graphs with equations $y=\frac{2}{5}(x-3)$ and $y=x$ fit the traces of Q and M. The graph of the inverse of f, $y=f^{-1}(x)$, is a reflection of $y=f(x)$ in the line $y=x$.

***Note**: To erase the geometry trace, select **Erase Geometry Trace** from the context menu (via ctrl menu) or by pressing menu > **Trace** > **Erase Geometry Trace**.*

(b) To repeat the above with $f(x)=\frac{3}{2}(x+1)^2-4$:

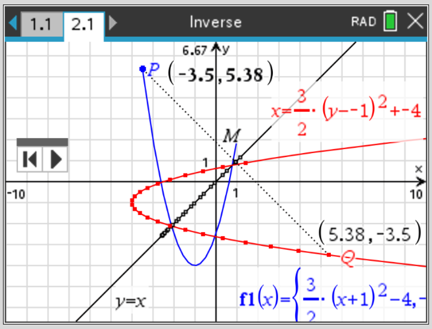


- Make a clone of the problem. Press ctrl ▲, navigate to the heading '**Problem 1**', pressing ctrl C then ctrl V.
- On page **2.1**, delete the rule for graph $\boldsymbol{f2}$ and edit $\boldsymbol{f1}$ to $\boldsymbol{f1(x)=\frac{3}{2}(x+1)^2-4\,|\,-3.5\le x\le 1}$.

To obtain a trace of the locus of points Q and M:

- To multi-select points Q and M, click point Q then hover over point M and press ctrl menu > **Geometry Trace**. Start the animation of P to create a trace of points Q and M.

To find the equation of a graph that contains the locus of Q:



- Press menu > **Graph Entry/Edit** > **Equation Templates** > **Parabola** > **Vertex form** $\boldsymbol{x=a\cdot(y-k)^2+h}$. Use the template to enter $\boldsymbol{x=\frac{3}{2}\cdot(y-(-1))^2+(-4)}$.

***Note**: Alternatively, enter the equation as a **Relation**.*

Answer: The inverse of f is not a functional relation because f is not a one-to-one function. The graph of the parabola with equation $x=\frac{3}{2}(y+1)^2-4$ contains the locus of point Q.

Texas Instruments

Determining the rule and domain of the inverse of a one-to-one function

Question

Determine the rule and domain of the inverse function for each of the following functions. Show the graphs of the original function and its inverse function on the same set of axes.

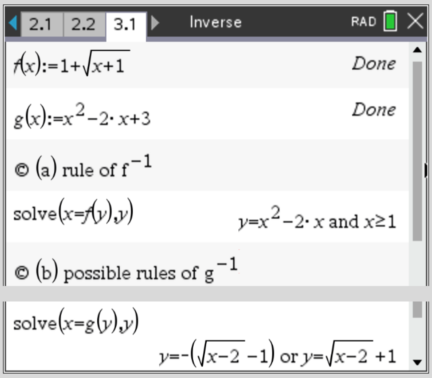
(a) $f: D \to R,\ f(x) = 1 + \sqrt{x+1}$, where D is the maximal domain of *f*.

(b) $g: [1,3] \to R,\ g(x) = x^2 - 2x + 3$.

Note*: Inserting a* ***New Problem*** *(via* doc▾ ***> Insert > Problem*** *) clears any existing definitions for functions and variables, and allows the user to combine multiple problems in the same document.*

Solution

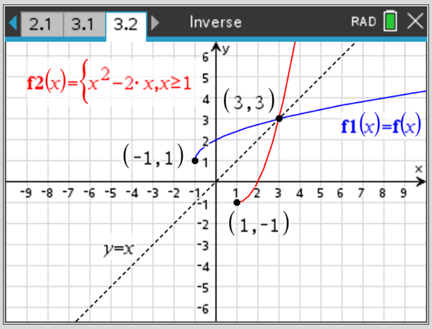
To find possible rules of f^{-1} and g^{-1}, press doc▾ > **Insert** > **Problem**, add a **Calculator** page and then:



- Enter $\boldsymbol{f(x) := 1 + \sqrt{x+1}}$ and $\boldsymbol{g(x) := x^2 - 2x + 3}$, pressing ctrl ⊡ := for the **assign** symbol.
- Press menu > **Algebra** > **Solve**. Enter $\mathbf{solve}(\boldsymbol{x = f(y), y})$.
- Similarly, enter $\mathbf{solve}(\boldsymbol{x = g(y), y})$.

Answer: The rules of the inverse functions are:
(a) $f^{-1}(x) = x^2 - 2x, x \geq 1$. Hence $\text{dom} f^{-1} = \text{ran} f = [1, \infty)$.
(b) Either $g^{-1}(x) = 1 - \sqrt{x-2}$ or $g^{-1}(x) = 1 + \sqrt{x-2}$, depending on the domain of *g*. This is explored below.

(a) To graph the functions *f* and f^{-1}, and determine their domain and range, on a **Graphs** page:



- Enter $\boldsymbol{f1(x) = f(x)}$ and $\boldsymbol{f2(x) = x^2 - 2x \mid x \geq 1}$
- Press menu > **Graph Entry/Edit** > **Relation**. Enter $\boldsymbol{y = x}$.
- Confirm the coordinates of the endpoints by pressing menu > **Trace** > **Graph Trace.**
- Press (−) 1 enter enter to 'jump' to and label the endpoint of $\boldsymbol{f1}$ at $x = -1$.
- Press ▲ to move to $\boldsymbol{f2}$ and press 1 enter enter to 'jump' to and label the endpoint of $\boldsymbol{f2}$ at $x = 1$.

Note*: When moving between graphs in* ***Trace*** *mode, it may be necessary to use the ◀ or ▶ keys before using the ▲ or ▼ keys. This is often the case if domains are different.*

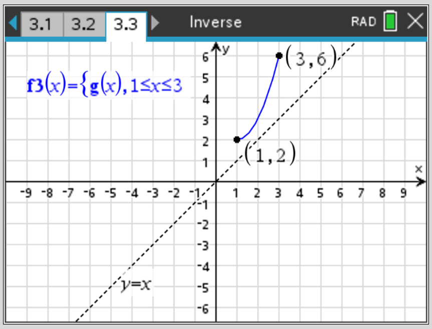
- Press menu > **Geometry** > **Points & Lines** > **Intersection Point(s)**. Click any pair of graphs, then press esc to exit.

Answer: $\text{dom} f = \text{ran} f^{-1} = [-1, \infty)$, $\text{ran} f = \text{dom} f^{-1} = [1, \infty)$, $f^{-1}: [1, \infty) \to R, f^{-1}(x) = x^2 - 2x$. Intersection at $(3,3)$.

... continued

Solution (continued)

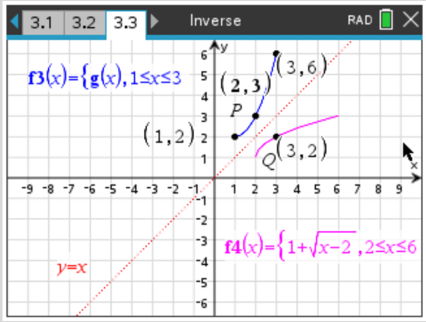
(b) To graph the functions g and g^{-1}, on a **Graphs** page:



- Enter $\boldsymbol{f3(x)=g(x)\,|\,1\le x\le 3}$.
- Use the **Trace** tool, as described above, to confirm the coordinates of the endpoints of $\boldsymbol{f3}$ are (1, 2) and (3,6).

Answer: $\text{ran}\, g=\text{dom}\, g^{-1}=[2,6]$

To confirm the rule and domain of g^{-1}:



- Enter $\boldsymbol{f4(x)=1+\sqrt{x-2}\,|\,2\le x\le 6}$.
- Using the method described in the previous problem, press [P] > **Point** and place a point ***P*** on graph $\boldsymbol{f3}$.
- Store the coordinates of *P* as (***xc***, ***yc***).
- Press [P] > **Point by Coordinates** and enter $(\boldsymbol{yc},\boldsymbol{xc})$.

  Move point *P* and observe point (yc,xc) move along $\boldsymbol{f4}$.

Answer: $g^{-1}:[2,6]\rightarrow R,\ g^{-1}(x)=1+\sqrt{x-2}$

1.3 Power and polynomial functions

1.3.1 Power functions

Investigating the graphs of power functions with integer powers

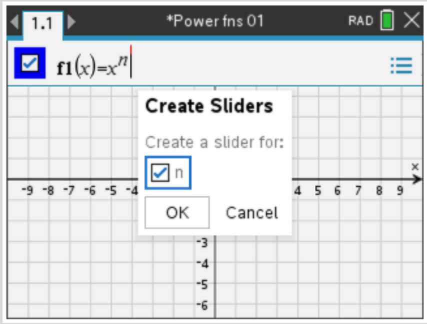
Question

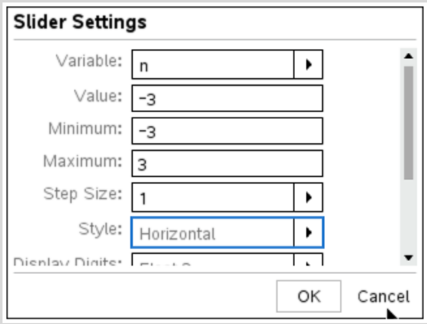
A power function can be expressed in the form $f(x)=x^n$, for $n \in Q$. In this example, we will consider only integer values of n. Construct a slider to display graphs of $y = f(x)$ for integer values from $n = -3$ to 3.

Solution

To plot these graphs, on a **Graphs** page:

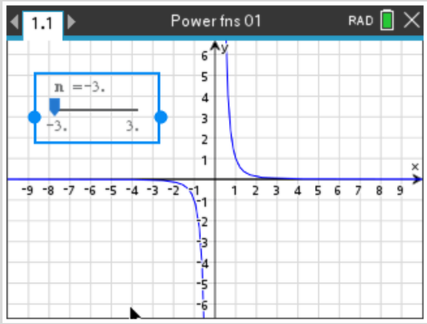
- Enter the rule $\boldsymbol{f1(x)=x^n}$.
- Click **OK** to create a slider for the power n.
- Press [enter] to locate the slider on the page
- Hover the cursor over the slider and press [ctrl] [menu] then select **Settings …**
- In the **Slider Settings** dialog box that follows, enter the following values:
  Value = –3
  Minimum = –3
  Maximum = 3
  Step Size = 1
- Click **OK** to save these slider settings and return to the graph page.





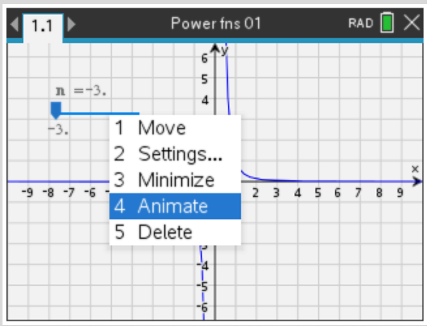
To use the slider, click on it (it will be coloured blue when selected), and then click the arrow keys to change the value of n within the **Slider Settings** constraints.

The example shown right shows the graph of a power function with $n = -3$. $\left(\text{If } n=-3,\ f1(x)=x^{-3}=\dfrac{1}{x^3}\right)$.



It is also possible to animate the effect of changing the n value by:

- hovering over the slider and press [ctrl] [menu].
- Select **Animate** from the pop-up menu to cycle the graphs through the values of n entered in the **Slider Settings.**



The animation can be stopped by hovering over the slider and press [ctrl] [menu], and then select **Stop Animate** from the pop-up menu.

***Note**: Values of the parameter n can also be entered directly by clicking on the current value, and editing it as required.*

Investigating the graphs of power functions with rational powers

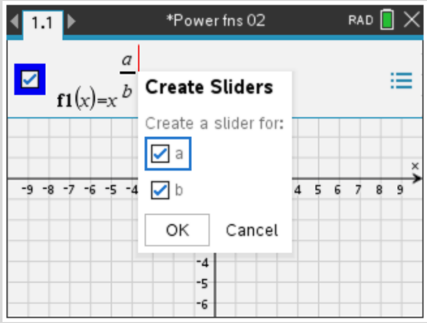
Question

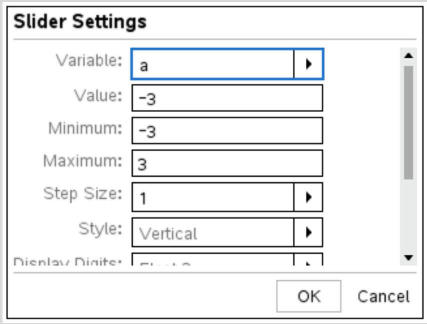
A power function can be expressed in the form $f(x)=x^{\frac{a}{b}}$, where $a,b\in Z$. Consider a restricted set of values for *a* and *b* such that $a,b\in\{-3,-2,-1,1,2,3\}$. Explore the behaviour of graphs for a range of odd and even values of *a* and *b*.

Solution

To plot these graphs, on a **Graphs** page:

- Enter the rule $\boldsymbol{f1(x)=x^{\frac{a}{b}}}$.
- Click **OK** to create sliders for the parameter *a* and *b*.
- Press [enter] to locate the sliders on the page.
- Hover the cursor over the slider for *a* and press [ctrl] [menu] then select **Settings …**
- In the **Slider Settings** dialog box that follows, enter the following values:
  Value = –3 Minimum = –3 Maximum = 3
  Step Size = 1 Style = Vertical
- Scroll down and check the **Minimised** box.
- Click **OK** to save these slider settings and return to the graph page.
- Repeat the above steps to enter the same slider settings for the parameter *b*.

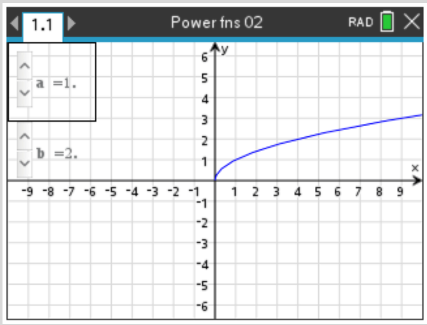




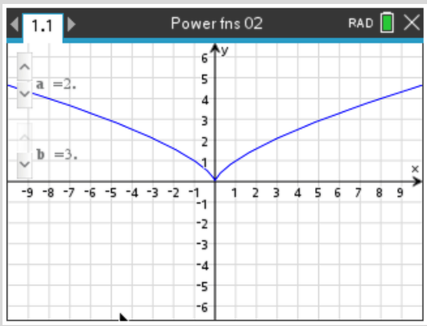
***Note**: To move the sliders into the second quadrant of the viewing window, hover over each slider and press [ctrl] [menu], then select **Move**.*

To use each slider, click on the slider arrow – this will increment or decrement the value of the parameter by one. Two example graphs are shown right.

The first example shown right shows the graph of a power function with *a* = 1 and *b* = 2. $\left(f1(x)=x^{\frac{1}{2}}\right)$.



The second example shown right shows the graph of a power function with *a* = 2 and *b* = 3. $\left(f1(x)=x^{\frac{2}{3}}\right)$.



***Note**: The exploration of the impact of the various values of parameters a and b prompt excellent possibilities for class discussion, and might consider the effect of various combinations of even/odd values of a and b, as well whether a is greater, less or equal to b.*

Exploring the concavity of power function graphs

Question

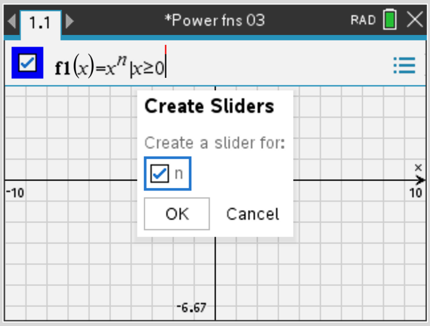
A power function can be expressed in the form $f(x) = x^n$, where $n \in Q$. Consider a further restricted set of values for n such that it can vary between 0.1 and 2, in steps of 0.1 (i.e. $n \in \{0.1, 0.2, 0.3, \ldots, 1.9, 2\}$). Consider the behaviour of this set of graphs in the first quadrant only. Comment on how the concavity of the graphs of these power functions in the first quadrant varies with changes in the value of n.

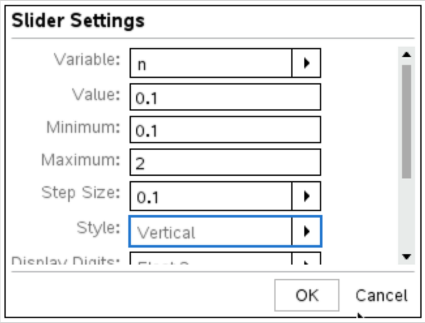
Solution

To plot these graphs, on a **Graphs** page:

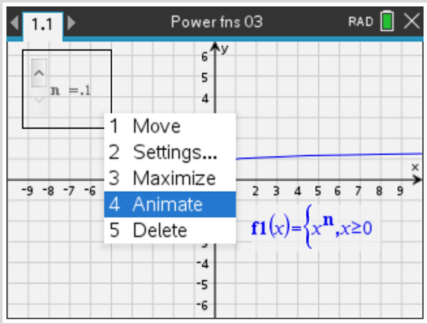
- Enter the rule $\boldsymbol{f1(x) = x^n \mid x \ge 0}$.
- Click **OK** to create a slider for the parameter n.
- Press [enter] to locate the slider on the page.
- Hover the cursor over the slider for n and press [ctrl] [menu] then select **Settings …**
- In the **Slider Settings** dialog box that follows, enter the following values:
  Value = 0.1 Minimum = 0.1 Maximum = 2
  Step Size = 0.1 Style = Vertical
- Scroll down and check the **Minimised** box.
- Click **OK** to save these slider settings and return to the graph page.

Note: *To move the slider into the second quadrant of the viewing window, hover over the slider and press* [ctrl] [menu], *then select* ***Move***.

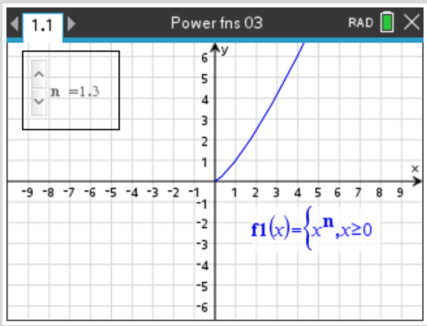




To animate the graphs using slider, hover the cursor over the slider for n and press [ctrl] [menu] then select **Animate**. This will start the display of a sequence of graphs using the range of values of a specified in the **Slider Settings**.



When you are ready, hover the cursor over the slider for n and then press [ctrl] [menu] then select **Stop Animate**.



Answer: Focussing on the behaviour of the power function graphs in the first quadrant (i.e. $x \ge 0$), the curvature changes can be summarised as follows:

For $0 < n < 1$, the graph is concave downwards.

For $n = 1$, the graph is straight.

For $n > 1$, the graph is concave upwards.

Note: *Instead of animating the change in the curvature, clicking the arrows inside the slider in turn will create a similar effect, with the benefit of greater control of the speed of the parameter changes.*

1.3.2 Transforming power functions

Constructing a graphing template to investigate transformations

A graphing template can be constructed which permits easier visualisation of the impact of transformation parameters.

***Notes**: (1) For ease of demonstrating in classrooms, this graphing template is best constructed and viewed using the TI-Nspire CAS Teacher Software, using **Computer Document Preview** mode.*
(2) The axes tick marks and grid settings can be modified using methods shown in Section 1.1.3.

Question

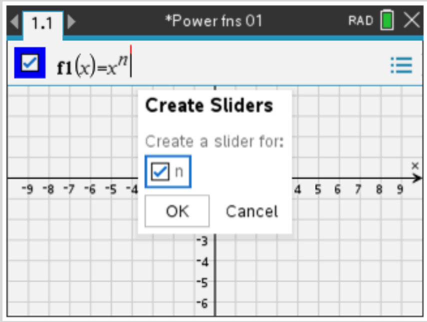
Let a set of power functions be defined as $f(x)=x^n, n\in\left\{-2,-1,\frac{1}{3},\frac{1}{2},1,2,3,4\right\}$. Construct a graphing template for viewing any graph of the form $y=a(x+b)^n+c$, where $a,b,c\in R$, and $a\neq 0$.

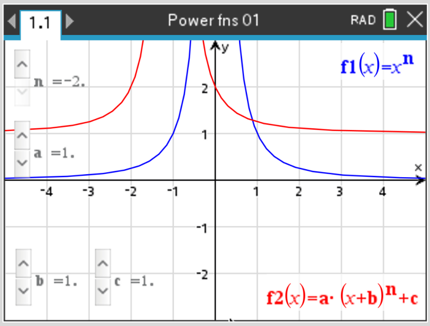
Solution

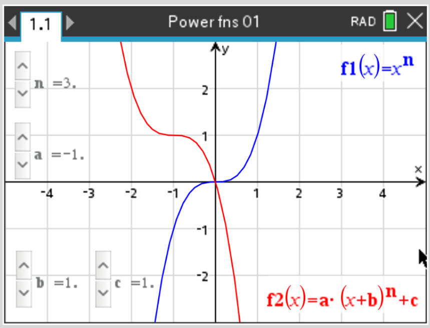
To plot these graphs, on a **Graphs** page:

- Enter the rule $f1(x)=x^n$.
- Click **OK** to create a slider for the power n.
- Press enter to locate the slider on the page
- Hover the cursor over the slider and press ctrl menu then select **Settings …**
- In the **Slider Settings** dialog box that follows, enter the following values:
  Value = 1 Minimum = –2 Maximum = 4
  Step Size = 1 Style = Vertical
- Scroll down and check the Minimised box.
- Click **OK** to save these slider settings and return to the graph page. Position the slider on the top left of the page.
- Press menu > **Window/Zoom** > **Window Settings**.
  Adjust the window settings as shown.
  XMin = –5 Xmax = 5 XScale = 1
  YMin = –3 YMax = 3 YScale = 1
- Press ctrl G and enter the rule $f2(x)=a\times(x+b)^n+c$.
- In the dialog box that follows, press enter to create sliders for a, b, and c.
- Hover over the slider for a, press ctrl menu and select **Settings**. Change the style to Vertical and Minimised.
- Repeat the previous step for b and c.
- Hover the cursor over the slider for a and press ctrl menu then select **Move**. Move it to the bottom right of the page.
- Repeat the previous step for b and c.

***Note**: The values of the transformation parameters **n**, **a**, **b**, and **c** can also be entered directly by clicking on the current value and editing it as required.*







Comparing dilations from the y-axis and from the x-axis

Students often have difficulty with the concept of dilation from the y-axis as it relates to function notation. For instance, understanding that the graph $y = f(nx)$ is a dilation of the graph of $y = f(x)$ by a factor of $1/n$ from the y-axis. To help students, the examples below provide an illustration of how the coordinates of a point on a graph are altered by dilations from the y-axis or from the x-axis.

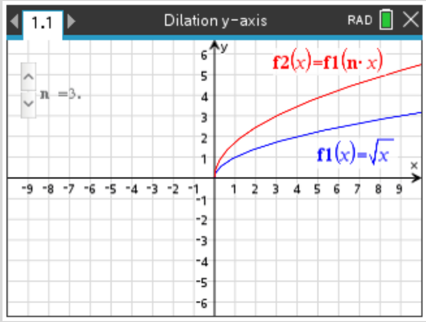
Question

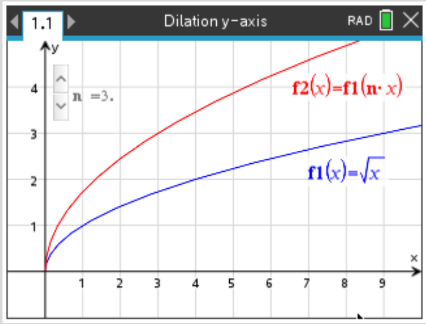
(a) Let $f(x) = \sqrt{x}$. Construct a graph of $y = f(x)$ and $y = f(nx)$ on the same axes to illustrate how the points (4,2) and (9,3) on the graph of $y = f(x)$ are transformed by the function $f(nx)$.

(b) Let $f(x) = \sqrt{x}$. Construct a graph of $y = f(x)$ and $y = af(x)$ on the same axes to illustrate how the points (4,2) and (9,3) on the graph of $y = f(x)$ are transformed by the function $af(x)$.

Solution

(a) To plot these graphs, on a **Graphs** page:



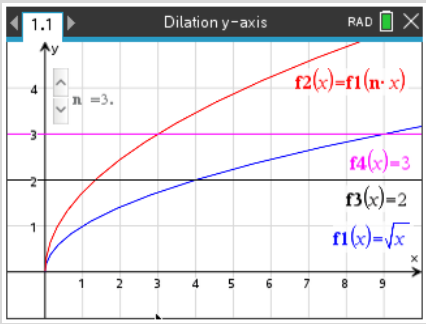


- Enter the rule $\boldsymbol{f1(x) = \sqrt{x}}$.
- Enter the rule $\boldsymbol{f2(x) = f1(nx)}$.
- Click **OK** to create a slider for the parameter n.
- Press [enter] to locate the slider on the page.
- Hover the cursor over the slider and press [ctrl] [menu] then select **Settings …**
- In the **Slider Settings** dialog box that follows, enter the following values:
  Value = 1 Minimum = 1 Maximum = 4
  Step Size = 1 Style = Vertical
- Click the **Minimised** checkbox and then click **OK** to save these slider settings and return to the graph page.

Click on the slider arrows to set $n = 3$. Now the two graphs displayed are for $\boldsymbol{f1(x) = \sqrt{x}}$ and $\boldsymbol{f2(x) = f1(3x) = \sqrt{3x}}$. To construct a more suitable window:

- Press [menu] > **Window/Zoom** > **Window Settings**.
  Adjust the window settings as shown.
  XMin = –1 Xmax = 10 XScale = 1
  YMin = –1 YMax = 5 YScale = 1

Construct two horizontal line graphs as follows:



- Enter the rule $\boldsymbol{f3(x) = 2}$.
- Enter the rule $\boldsymbol{f4(x) = 3}$.

This will plot two horizontal lines at $y = 2$ and $y = 3$.

… continued

Texas Instruments

Solution (continued)

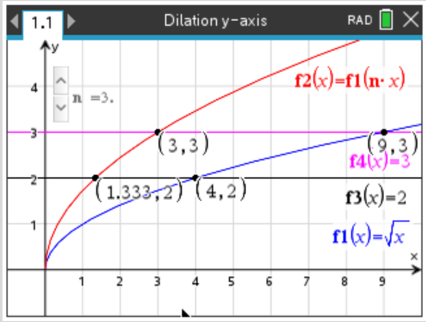
Now display coordinates on the graphs at the intersection points with the lines as follows:

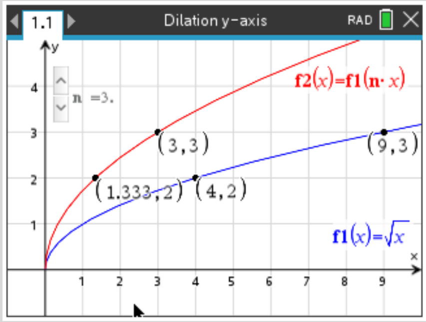
- Press [menu] > **Geometry >Points & Lines > Intersection points**.
- Click on the graph of ***f*1(*x*)** and then on the graph of ***f*3(*x*).**
- Click on the graph of ***f*1(*x*)** and then on the graph of ***f*4(*x*).**
- Click on the graph of ***f*2(*x*)** and then on the graph of ***f*3(*x*).**
- Click on the graph of ***f*2(*x*)** and then on the graph of ***f*4(*x*).**
- Press [esc] to escape the **Intersection point(s)** tool.

To hide the lines:

- Hover over the line graph of ***f*3(*x*)** and press [ctrl] [menu], then select **Hide** from the pop-up menu.
- Hover over the line graph of ***f*4(*x*)** and press [ctrl] [menu], then select **Hide** from the pop-up menu.

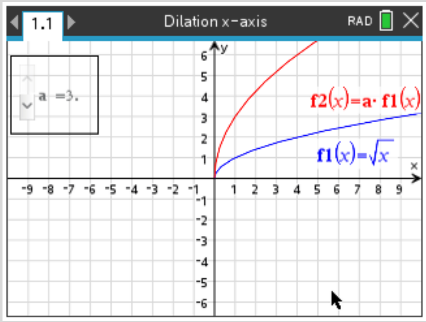
Click the slider and set $n = 1$. Then observe how the points (4, 2) and (9,3) are affected as the slider value of n is changed.





(b) To plot these graphs, create a new document (use [ctrl] [N]), and then on a **Graphs** page:

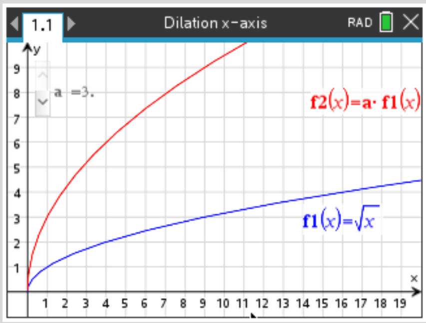
- Enter the rule $f1(x) = \sqrt{x}$.
- Enter the rule $f2(x) = af1(x)$.
- Click **OK** to create a slider for the parameter a.
- Press [enter] to locate the slider on the page.
- Hover the cursor over the slider and press [ctrl] [menu] then select **Settings …**
- In the **Slider Settings** dialog box that follows, enter the following values:
  Value = 1   Minimum = 1   Maximum = 3
  Step Size = 1   Style = Vertical
- Click the **Minimised** checkbox and then click **OK** to save these slider settings and return to the graph page.



Click on the slider arrows to set $a = 3$. Now the two graphs displayed are for $f1(x) = \sqrt{x}$ and $f2(x) = f1(3x) = \sqrt{3x}$.

To construct a more suitable window:

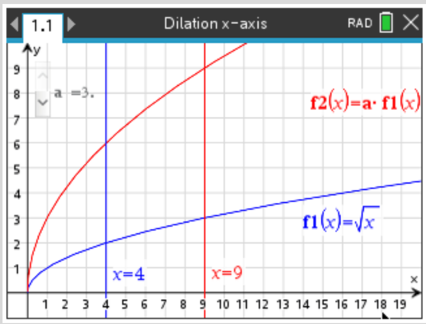
- Press [menu] > **Window/Zoom > Window Settings**.
  Adjust the window settings as shown.
  XMin = –1   Xmax = 20   XScale = 1
  YMin = –1   YMax = 10   YScale = 1



To construct two vertical line graphs, press [menu] > **Graph Entry/Edit > Relation**, and then:

- For **rel1(*x*,*y*)**, enter the rule ***x* = 4**.
- For **rel2(*x*,*y*)**, enter the rule ***x* = 9**.

This will plot two vertical lines at $x = 4$ and $x = 9$.

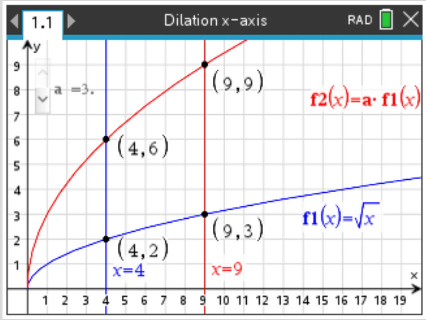


… continued

Solution (continued)

Now place some coordinates on the graphs at the intersection points with the vertical lines as follows:

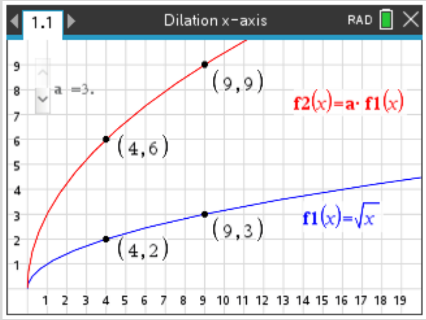
- Press [menu] > **Geometry >Points & Lines > Intersection points**.
- Click on the graph of ***f*1(*x*)** and then on the graph of ***x* = 4.**
- Click on the graph of ***f*1(*x*)** and then on the graph of ***x* = 9.**
- Click on the graph of ***f*2(*x*)** and then on the graph of ***x* = 4.**
- Click on the graph of ***f*2(*x*)** and then on the graph of ***x* = 9.**
- Press [esc] to escape the **Intersection point(s)** tool.



To hide the lines:

- Hover over the line graph of ***f*3(*x*)** and press [ctrl] [menu], then select **Hide** from the pop-up menu.
- Hover over the line graph of ***f*4(*x*)** and press [ctrl] [menu], then select **Hide** from the pop-up menu.

Click the slider and set $a = 1$. Then observe how the points (4, 2) and (9,3) are affected as you change the value of a.



***Note**: Similar files can be constructed to help students compare the graphs of* $y = f\left(n\left(x+b\right)\right)$ *and* $y = f\left(nx+b\right)$. *Students often struggle with the order of transformations in such cases.*

1.3.3 Polynomial functions

Investigating polynomial division

Consider a polynomial $P(x)$ which is divided by a linear polynomial $D(x)$. This will result in a quotient $Q(x)$ and a remainder R. Expressed mathematically:

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R}{D(x)} \quad \Leftrightarrow \quad P(x) = D(x) \cdot Q(x) + R$$

If the divisor can be expressed as $D(x) = x - a$, this leads to the polynomial remainder theorem.

$$\begin{aligned} P(x) &= D(x) \cdot Q(x) + R \\ &= (x-a) \cdot Q(x) + R \\ P(a) &= (a-a) \cdot Q(x) + R \\ P(a) &= R \text{ (Remainder Theorem)} \\ \text{If } P(a) = 0 &\Rightarrow R = 0 \text{ (Factor Theorem)} \end{aligned}$$

Question

For the polynomial division $\dfrac{x^3 - 2x^2 + 11x - 7}{x - 2}$, use the CAS to find $Q(x)$ and R, and verify each line of the above working.

Solution

On a **Calculator** page:

- Enter the polynomial $\boldsymbol{p(x) := x^3 - 2x^2 + 11x - 7}$.
- Enter the divisor $\boldsymbol{d(x) := x - 2}$.
- Express the division as $\boldsymbol{\dfrac{p(x)}{d(x)}}$.
- Press menu > **Algebra >Fraction Tools > Proper Fraction** and then complete the command **PropFrac**$\left(\boldsymbol{\dfrac{p(x)}{d(x)}}\right)$.

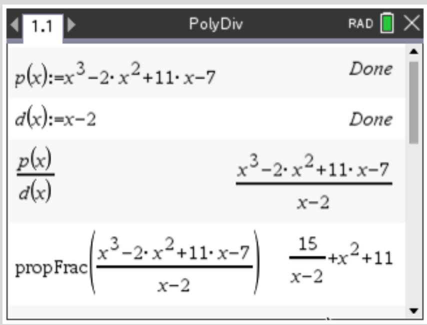
This expresses the division as a quotient (i.e. $x^2 + 11$) and a remainder divided by the divisor (i.e. $\dfrac{15}{x-2}$).

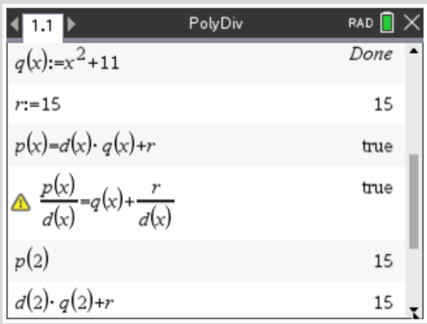
- Enter the quotient as $\boldsymbol{q(x) := x^2 + 11}$.
- Enter the remainder as $\boldsymbol{r := 15}$.

Now we have all the components defined, we can verify the remainder theorem.

- Enter the expression $\boldsymbol{p(x) = d(x) \cdot q(x) + r}$.
- Enter the expression $\boldsymbol{\dfrac{p(x)}{d(x)} = q(x) + \dfrac{r}{d(x)}}$.

These two equations verify the general result.
Note that here both $p(2) = 15$ and $d(2) \cdot q(2) + r = 15$.





***Note**: There are inbuilt commands **polyQuotient** and **polyRemainder** that will return the quotient and remainder for a polynomial division. These commands are shown in section 3.1.1.*

Graphing higher order polynomial functions

Student understanding of the graphs of higher order polynomial functions can be enhanced by considering factored forms, and exploring how the powers of each factor, and the sum of all the powers affect the shape of the graph.

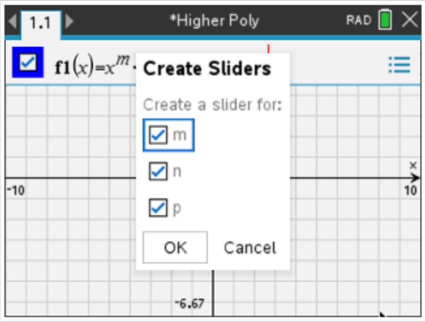
***Notes**: (1) For ease of demonstrating in classrooms, this graphing template is best constructed and viewed using the TI-Nspire CAS Teacher Software, using **Computer Document Preview** mode.*
(2) The axes tick marks and grid settings can be modified using methods shown in Section 1.1.2.

Question

Construct a graphing template for the function $f(x)=x^m(x+1)^n(x-1)^p$, where $m,n,p\in\{1,2,3\}$.
Explore how the values of *m*, *n* and *p* and the sum *m* + *n* + *p* affects the shape of the graph of *f*(*x*).

Solution

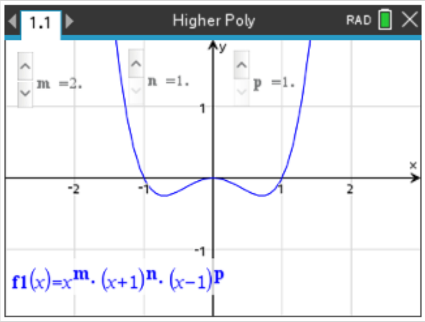
To plot these graphs, on a **Graphs** page:



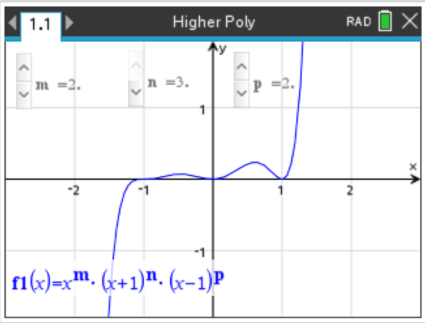
- Enter the rule $\boldsymbol{f1(x)=x^m(x+1)^n(x-1)^p}$.
- Click **OK** to create sliders for the parameter *m, n* and *p*.
- Press enter to locate the sliders on the page.
- Hover the cursor over the slider for *m* and press ctrl menu then select **Settings …**
- In the **Slider Settings** dialog box that follows, enter the following values:
  Value = 1 Minimum = 1 Maximum = 3
  Step Size = 1 Style = Vertical
- Scroll down and check the **Minimised** box.
- Click **OK** to save these slider settings and return to the graph page.
- Repeat the above steps to enter the same slider settings for *n* and *p*.

***Note**: To move the sliders, hover over each slider and press* ctrl menu*, then select **Move**.*

Two example graphs are shown right.



The first example shown right shows the graph of $f(x)=x^2(x+1)(x-1)$. Note the link between the power of each algebraic factor, and the shape of the graph nearby its associated root. The sum of the powers is 4, so this is a polynomial of order 4.



The second example shown right shows the graph of $f(x)=x^2(x+1)^3(x-1)^2$. This is a polynomial of degree 7.

***Note**: As an extension, students could note the number of turning points, stationary points of inflection and x-axis intercepts and their connection with the equation.*

1.3.4 Bisection method for numerical roots of a polynomial function

Estimating the accuracy of the bisection method after n iterations

Question

Let $[a,b]$ be an interval containing a root of a continuous function *f*. The bisection method finds an approximation to the root by repeatedly bisecting the interval, i.e. finding the midpoint $m=(a+b)/2$, calculating $f(m)$, selecting the subinterval in which *f* changes sign, and repeat.

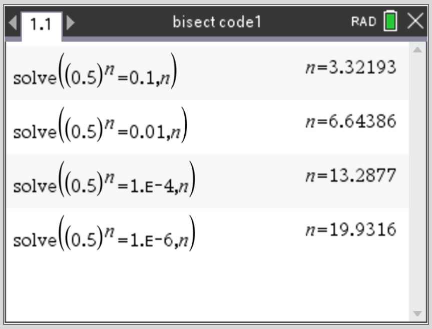
Estimate the approximate number of iterations of this process required to achieve an accuracy to **(i)** 0.1 **(ii)** 0.01, **(iii)** 0.0001 and **(iv)** 0.000001 (i.e. an accuracy of 1, 2, 4 and 6 decimal places).

Solution

Since each iteration halves the interval, if the initial interval width is $w_0=b-a$, then after *k* iterations the interval width is $w_k=(0.5)^k w_0$. Solving $(0.5)^k=0.1$ for *k* gives an estimate of how many iterations are needed to achieve an accuracy of 0.1.

To find *k* for various accuracies, on a **Calculator** page:

- Press menu > **Algebra** > **Solve**. Enter $\mathbf{solve(0.5^k=0.1,k)}$.
- Press ▲▲ enter to copy and paste, then edit and enter $\mathbf{solve(0.5^k=0.01,k)}$. Repeat to solve the equations $\mathbf{0.5^k=0.0001}$ and $\mathbf{0.5^k=0.000001}$.



Answer:

Tolerance	*d.p.*	*Approx. iterations needed*
0.1	1	4
0.01	2	7
0.0001	4	14
0.000001	6	20

***Note**: Depending on the calculator display digits setting, the display of the tolerance value may change to scientific notation (see screen above).*

Implementing the bisection method in the Lists & Spreadsheet application

Question

A student writes the following pseudocode for the bisection method with 20 iterations.

Inputs
define *function* $f(x)$
$\boldsymbol{a}\leftarrow$ lower value
$\boldsymbol{b}\leftarrow$ upper value

(# assumes that [*a*, *b*] is a valid interval containing a root.)

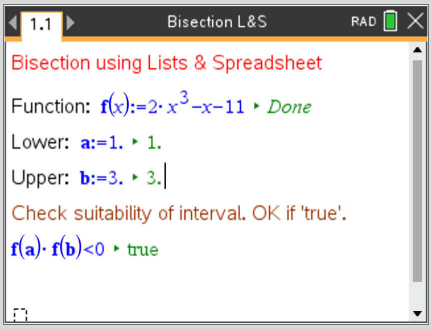
For *k* from 1 to 20
  $m\leftarrow(a+b)/2$
  print *k*, *m*
  if $f(a)\times f(m)<0$ then
    $b\leftarrow m$
  else
    $a\leftarrow m$
  end if
end for
print "approx. root_", *m*

Implement the pseudocode in the Lists & Spreadsheet application using $f(x)=2x^3-x-11$, which has a root in the interval [1, 3]. Determine the approximate root and its accuracy on 14th iteration.

Solution

To enter the user inputs $f(x)$, a and b, on a **Notes** page:

- Enter the labels **Function**, **Lower** and **Upper,** as shown.
- Insert a **Maths Box** next to each label by pressing [ctrl] [M].
- In the first three **Maths Boxes** enter $\boldsymbol{f(x) := 2x^3 - x - 11}$, $\boldsymbol{a := 1.0}$ and $\boldsymbol{b := 3.0}$, as shown.



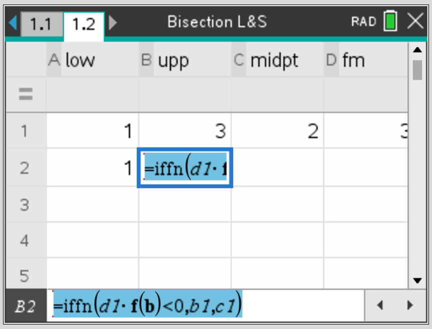
***Note**: Entering the initial values of **a** and **b** as **a = 1.0** and **b = 3.0** forces any calculations using **a** and or **b** to give results as decimal approximations.*

To check interval suitability (opposite signs for $f(a), f(b)$):

- Add a **Maths Box** and enter $\boldsymbol{f(a) \times f(b) < 0}$. If the output returns **true**, then the interval contains a root.

To set up the algorithm, on a **Lists & Spreadsheet** page:

- In the heading row (top row), enter the column names, **low**, **upp**, **midpt**, and **fm**, as shown.
- Enter as follows. Cell A1: **=*a***, cell B1: **=*b***, cell C1: **=(*a*1+*b*1)/2**, and cell D1: **=*f*(*c*1)**.

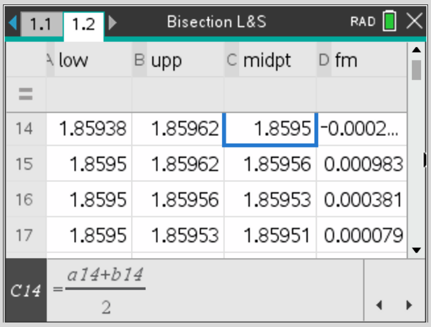


To select the subinterval containing the root:

- In cell A2 enter: **=ifFn(*d*1×*f*(*a*) < 0,*a*1,*c*1)**, pressing [📖] [1] [I] to select **ifFn**(expr, value if true, value if false).
- In cell B2 enter: **=ifFn(*d*1×*f*(*b*) < 0,*b*1,*c*1)**. The **ifFn** command has the same effect as **if** … **then** … **else**.

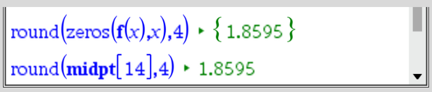
To repeat the halving of the interval for 20 iterations:

- In cell C2 enter: **=(*a*2+*b*2)/2**, and cell D1: **=*f*(*c*2)**.
- To fill these formulas down, navigate to cell A2, hold down the [⇧shift] key and press ▸ across to cell D2.
- Press [ctrl] [menu] > **Fill**. Press ▾ down to row 20, then [enter].



To test the accuracy of the 14th iteration to 4 decimal places:

- On page 1.1, add two **Maths Boxes**. Enter as shown: **round(zeros(*f*(*x*),*x*),4)** and **round(midpt[14],4)**, pressing [📖] [1] [S] to select **round** and [📖] [1] [Z] to select **zeros**.

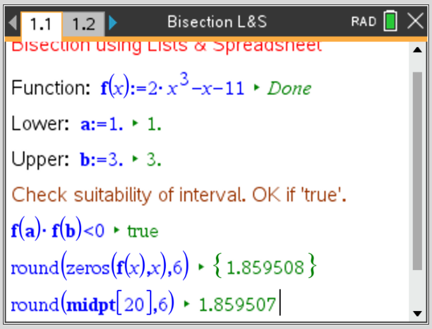


***Note: midpt[14]** selects the 14th element of the **midpt** list.*

Answer: After 14 iterations, $m \approx 1.8594$, to 4 decimal places, the same result as using the '**zeros**' or '**solve**' commands.

Extension. Accuracy of the 20th iteration to 6 decimal places:

- Click the battery icon at the top right of the screen. Select **Document Settings**. Set **Display Digits** to **Float 8**.
- On page 1.1, edit the last two **Maths Boxes**, to: **round(zeros(*f*(*x*),*x*),6)** and **round(midpt[20],6)**.



Answer: Zeros and bisection results: 1.859508 and 1.859507, consistent with the predictions in the previous problem of a tolerance of 0.0001 in the 14th iteration and 0.000001 in the 20th iteration.

Implementing pseudocode for the bisection method in the Python application

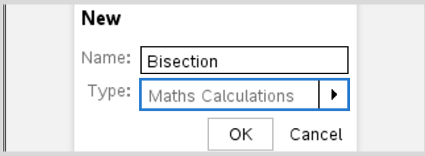
Question

Implement the pseudocode from the previous problem in the Python application. Perform 20 iterations to find the root of $f(x)=x^4-x^3-10x^2-x+1$ contained in the interval $[-1.5,0]$.

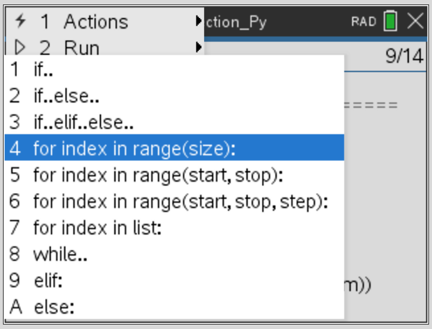
Solution

To start coding, in a new **Document** (or a new **Problem**):

- Select **Add Python > New**.
- In the dialog box that follows, enter as shown.



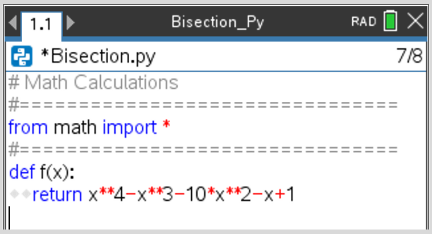
Note: *The* ***Python*** *commands to be used can be accessed by pressing* menu ***> Built-ins*** *then -*
> Function *for: '****def****' (define function) and '****return****'.*
> Control *for: '****if ..else****', '****for*** *index in range(size)', '****while****'*
> Type *for: '****float****', '****int****' and '****round****'*
> I/O *for: '****input****' and '****print****'.*
Text *in quotation marks: press* ?!▸ *to select ".*
Indentation*: ensure correct indentation. Press* tab *to indent.*



To define $f(x)=x^4-x^3-10x^2-x+1$:

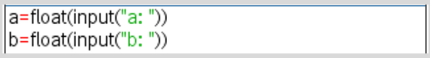
- Enter **def *f*(*x*):** , then **return** $\boldsymbol{x^4-x^3-10\times x^2-x+1}$

Note: *Use the keys* × *for multiplication and* ^ *or* x² *for exponentiation. Output will appear as* ***** *for* ×, ****** *for* ^.



To request user input for interval values, ***a*** and ***b***:

- Enter ***a*** **= float(input("*a*: "))**. For a floating-point value.
- Enter ***b*** **= float(input("*b*: "))**.

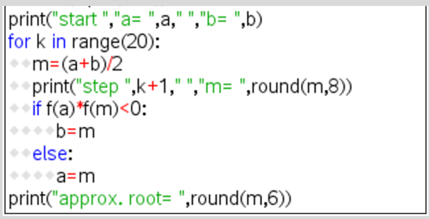


To instruct repetition for **20** iterations using a **for** loop:

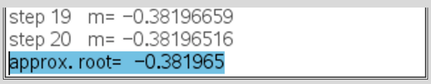
- Enter **print("start= ","a= ",*a*," ","b= ",b)**
- Enter **for *k* in range(20):** , then (with indents as shown)
- Enter $\boldsymbol{m=(a+b)/2}$
- Enter **print("step",*k*+1, " ","*m* =",round(*m*,8))**

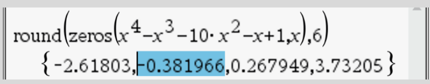
To select the subinterval containing the root:

- Select **if ..else..** and enter, with indentations *as shown*:
- **if** $\boldsymbol{f(a)\times f(m)<0}$**:** followed by $\boldsymbol{b=m}$
- **else:** followed by $\boldsymbol{a=m}$
- Enter **print("approx. root= ",*k*+1, round(*m*,6))**



- Press ctrl R to run the program and check syntax.
- In the **Python Shell** page that follows, follow the prompts to enter ***a*: –1.5**, then ***b*: 0**.





Answer: Root ≈ -0.381965 (using **zeros**, root ≈ -0.381966).

Defining a bisection(a,b,dp) user-defined function in the Python application

Question

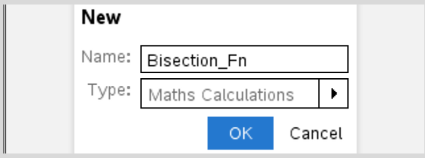
Implement an enhanced version of the pseudocode from the previous problems by utilising a user-defined function in the **Python** application. The code should explicitly include a desired level of accuracy (tolerance), a maximum number of iterations and a check for validity of the interval.

Hence find all roots of $f(x)=x^4-x^3-10x^2-x+1$ using intervals **(a)** $[-3,-1.5]$, **(b)** $[-2,-1]$, **(c)** $[-1.5,0]$, **(d)** $[0,1.5]$ and **(e)** $[2.5,4.5]$. Set tolerances correct to within **(i)** 10^{-4} and **(ii)** 10^{-6}.

Solution

To start coding, in a new **Document** (or a new **Problem**):

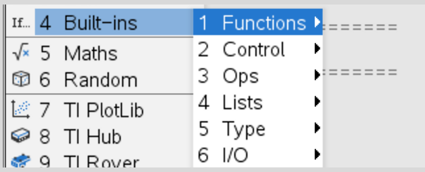
- Select **Add Python** > **New**.
- In the dialog box that follows, enter as shown.



***Note:** Refer to the previous problem for instructions on accessing built-in **Python** commands in* menu *> **Built-ins** > …*

***Text** in quotation marks: press* ?!▸ *to select ".*

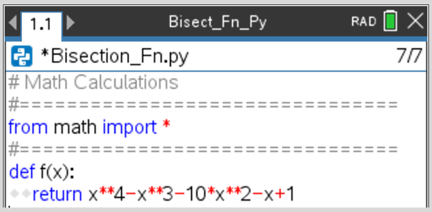
***Indentation**: ensure correct indentation. Press* tab *to indent.*



To define $f(x)=x^4-x^3-10x^2-x+1$:

- Enter **def *f*(*x*):** , then **return** $x^4-x^3-10\times x^2-x+1$

Note:** Use the keys* × *for multiplication and* ^ *or* x² *for exponentiation. Output will appear as* ** *for* ×, ****** *for* ^.



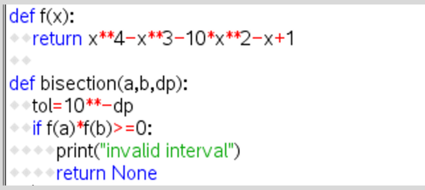
To define the user-defined function **bisection(*a*,*b*,*dp*)**:

- Enter **def bisection(*a*, *b*, *dp*):** then **tol = 10$^{-\text{dp}}$**, as shown.

***Note:** **dp** denotes decimal places and **tol** is the tolerance, so that if **dp = 4** then the root should be correct to within* 10^{-4}.

To check whether the initial interval [***a***, ***b***] captures a root:

- Enter **if $f(a)\times f(b)>=0$:**, pressing ctrl = (≠≥▸) for >=
- Enter **print("invalid interval")**, then **return None**

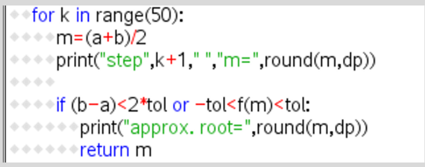


To set up repetition of halving of the interval up to 50 times:

- Enter **for *k* in range(50):** , then (with indents as shown)
- Enter $m=(a+b)/2$
- Enter **print("step",*k*+1, " ","*m* =",round(*m*,*dp*))**

To check if desired accuracy has been achieved (i.e. check if either interval width is twice the tolerance or $f(m)\approx 0\pm tol$):

- Enter **if** $(b-a)<(2\times tol)$ **or** $-tol<f(m)<tol$**:**
- Enter **print("approx. root =",round(*m*,*dp*))**
- Enter **return *m*** , taking note of correct indentations.

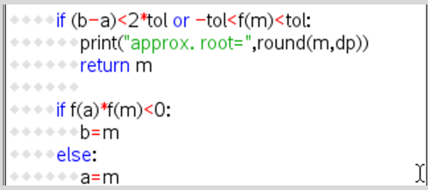


… continued

Solution (continued)

To select the subinterval containing the root:

- Select **if ..else..** and enter, with indentations *as shown*:
- **if** $f(a)\times f(m)<0$**:** followed by $b=m$
- **else:** followed by $a=m$



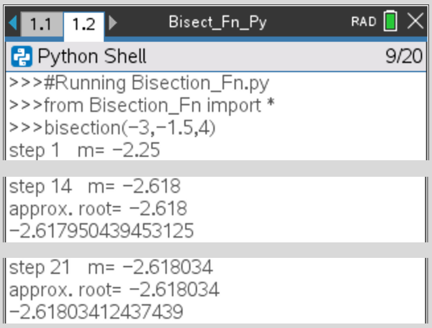
To run the program and check syntax:

- Press ctrl R. A **Python Shell** page should follow.

(a) To search for a root on the interval $[-3,-1.5]$:

- Press var, select **bisection**. Enter **bisection(–3, –1.5,4)**.
- Press var. Enter **bisection(–3, –1.5,6)** (note changed ***dp***).

Answer: (i) $x\approx-2.6180$, 14 iterations.
(ii) $x\approx-2.618034$, 21 iterations



(b) To search for a root on the interval $[-2,-1]$:

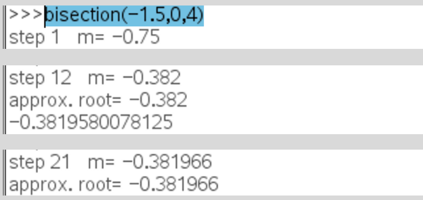
- Press var. Enter **bisection(–2, –1,4)**.

Answer: No root within this interval.

(c) Similarly, to search for a root on the interval $[-1.5,0]$:

- Enter **bisection(–1.5,0,4)**, then **bisection(–1.5,0,6)**

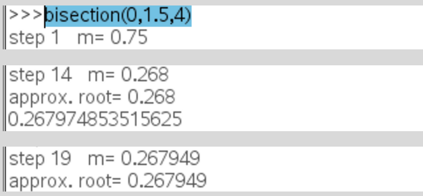
Answer: (i) $x\approx-0.3820$, 12 iterations.
(ii) $x\approx-0.381966$, 21 iterations



(d) Similarly, to search for a root on the interval $[0,1.5]$:

- Enter **bisection(0,1.5,4)**, then **bisection(0,1.5,6)**

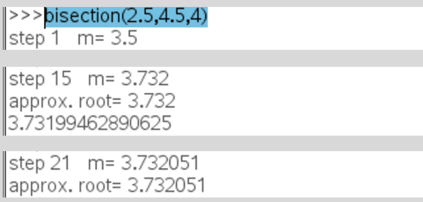
Answer: (i) $x\approx0.2680$, 14 iterations.
(ii) $x\approx0.267949$, 19 iterations



(e) Similarly, to search for a root on the interval $[2.5,4.5]$:

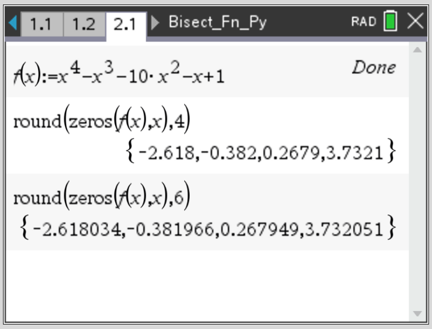
- Enter **bisection(2.5,4.5,4)**, then **bisection(2.5,4.5,6)**

Answer: (i) $x\approx3.7320$, 15 iterations.
(ii) $x\approx3.732051$, 21 iterations



To validate answers, on a **Calculator** page:

- Enter $f(x):=x^4-x^3-10\cdot x^2-x+1$
- Enter **round(zeros(*f*(*x*),*x*),4)**, pressing 📖 1 S to select round and 📖 Z (or **Algebra** menu) to select **zeros**.
- Click the battery icon at the top right of the screen. Select **Document Settings**. Set **Display Digits** to **Float 8**.
- Copy, paste and edit to **round(zeros(*f*(*x*),*x*),6)**.



Answer: The results accord with those obtained using the **bisection(*a*,*b*,*dp*)** function. Tolerances of 10^{-4} and 10^{-6} are consistently achieved in approximately 14 and 20 iterations.

Using the Programme Editor to implement pseudocode for bisection method

Question

(a) Use the **Programme Editor** to implement 20 iterations of the simple pseudocode introduced at the start of this chapter. Test the code using $f(x)=x^3-3x-1$, which has roots on the intervals [–2.5, –1] and [1.5, 2]. Show that the answer is correct to within $0.0001=10^{-4}$.

(b) Modify the code to check the validity of the initial interval and adjust the number of iterations to ensure the accuracy of the solution is correct to the specified number of decimal places.

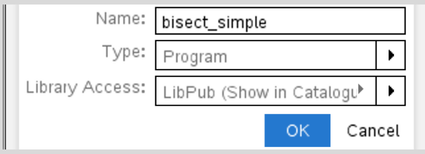
Test the code for an accuracy correct to 2, 4 and 6 decimal places for $f(x)=x^3-3x-1$ with $a=-2.5$, $b=-1$. Compare the results with solutions using the zeros command.

***Note**: A third root for f(x) exists at $x\approx-0.347$, which for brevity will be ignored here.*

Solution

(a) To start coding, in a new **Problem** or a new **Document**:

- Select **Add Programme Editor > New**.
- In the dialog box that follows, enter as shown.



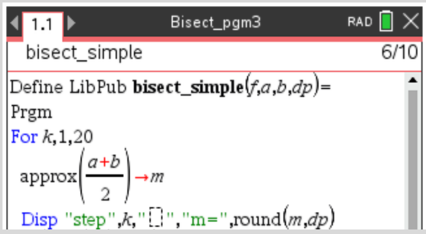
The **Program Editor** will follow, ready to accept the code.

To name the inputs ***f, a, b*** and ***dp*** (decimal places), in line 0:

- Enter **bisect_simple(*f,a,b,dp*)=**

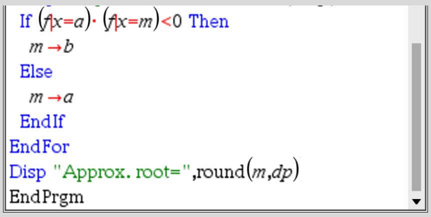
To instruct repetition for 20 iterations using a **For** loop:

- Press [menu] > **Control** > **For … EndFor** and enter **For *k*,1,20** followed by $\mathbf{approx}((a+b)/2)\to m$.
- Press [📖][A] for **approx** and [ctrl][var] ([sto→]) for **store**, →.
- Enter **Disp "step",*k*," ","*m*=",round(*m*,*dp*)** by pressing [menu] > **I/O** > **Disp** and [📖][1][S] to select **round**.



To select the subinterval containing the root:

- Press [menu] > **Control** > **If…Then…Else…EndIf** and enter *as shown*: **If (*f* |*x* = *a*)× (*f* |*x* = *m*) < 0 Then *m* → *b* Else *m* → *a***. Press [ctrl][=] to select **given, |,** and **<**.
- After **EndFor**, enter **Disp "Approx. root=",round(*m*,*dp*)**

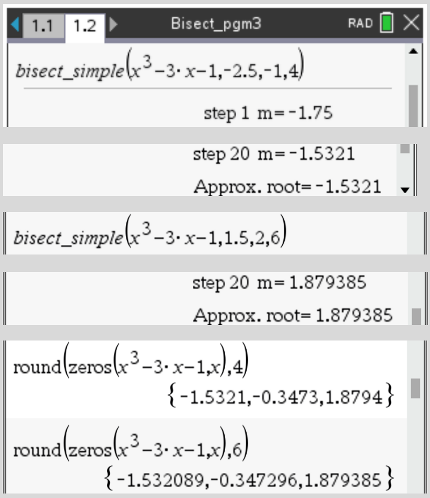


To check, store and run the program for $f(x)=x^3-3x-1$:

- Press [ctrl][B] followed by [ctrl][R].
- **(i)** Enter **bisect_simple(*x*³–3*x*–1,–2.5,–1,4)**.
- **(ii)** Press ▲ to top of page and press [enter] to paste. Edit and enter **bisect_simple(*x*³–3*x*–1,1.5,2,6)**.

To validate the results by comparing with the **zeros** command:

- Enter **round(zeros(*x*³–3*x*–1,*x*),4)**, pressing [📖][1][S] to select round then [📖][Z] (or **Algebra** menu) for **zeros**
- Press ▲▲[enter] to paste and edit to **round(zeros(*x*³–3*x*–1,*x*),6)**.

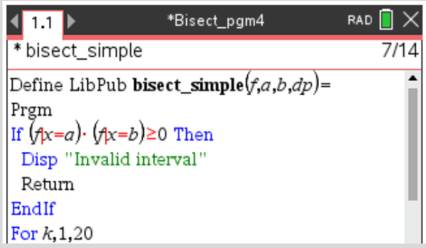


Answer: Roots at $x\approx-1.5321$ and $x\approx1.879385$. A result almost to within 10^{-6} achieved in 20 iterations.

Solution (continued)

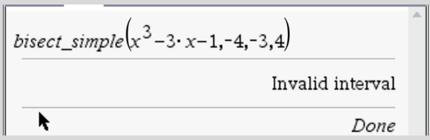
(b) To test the validity of the initial interval [***a***, ***b***]:

- With the cursor at the start of line 1 (i.e. the line after "Prgm"), press menu > **Control** > **If … Then…End If** and enter as follows
  If (*f* |*x* = *a*)× (*f* |*x* = *b*) ≥ 0 Then
  Disp "Invalid interval".
- Enter **Return** by pressing menu > **Transfers** > **Return**.



To test the above modification with an invalid interval:

- Press ctrl B then ctrl R to check, store and run program.
- Enter **bisect_simple(x^3–3x–1,–4, –3,4)**.

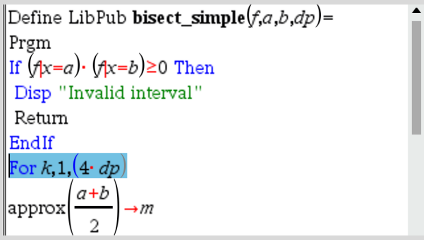


Answer: The program detected the invalid interval and quit.

Note: *As seen earlier, the approximate number of iterations required for a particular accuracy is very predictable. Generally, iterations = (4 × dp) is sufficient.*

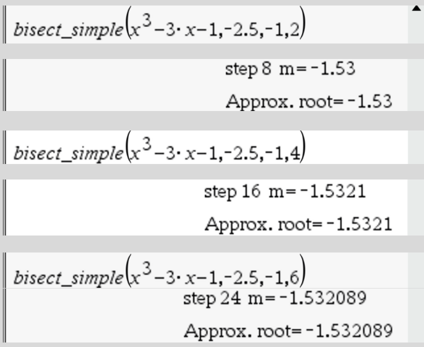
To apply the general '4 × dp' rule for number of iterations:

- Edit line 5 to **For *k*,1,(4×*dp*)**



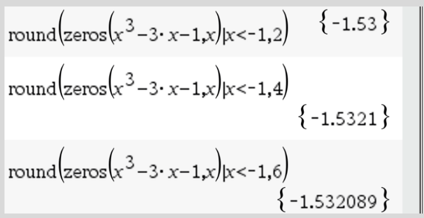
(i) To test the code for $a = -2.5$, $b = -1$ for 2, 4 and 6 dp:

- Press ctrl B followed by ctrl R.
- Enter **bisect_simple(x^3–3x–1,–2.5,–1,2)**.
- Press ▲ to top of page and press enter to paste. Edit ***dp*** value to 4 then to 6: **bisect_simple(x^3–3x–1,–2.5,–1,4)**.



To validate the results by comparing with the **zeros** command:

- Enter **round(zeros(x^3–3x–1,x)|x < –1,2)**.
- Press ▲▲enter to paste. Edit the decimal places to **4**, **6**.



Answer: Root correct to 2, 4, 6 decimal places: identical answers using the program and the zeros command: $x \approx -1.53$, $x \approx -1.5321$ and $x \approx -1.532089$.

1.3.5 Transforming with matrices

***Note**: Matrices are no longer in the Mathematical Methods course, but are included here for interest, and as past VCAA examinations contain questions using matrix methods.*

Matrices can be used to describe and apply transformations to a point, a set of points, and to equations. If a point with coordinates (x, y) is represented as a 2×1 matrix, the coordinates of the image point (x', y') after a series of linear transformations can be represented as a matrix equation as follows.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} \Leftrightarrow \begin{matrix} x' = ax + by + e \\ y' = cx + dy + f \end{matrix}$$

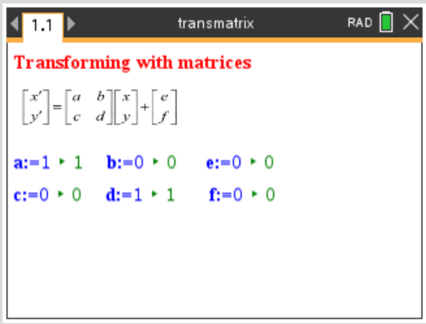
Question

Use the equations above to construct a slider template to demonstrate how each of the transformation parameters $a-f$ transforms a basic shape placed on the Cartesian plane.

Solution

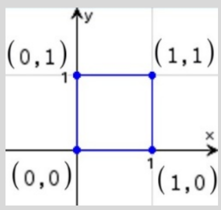
To set up the needed variables, parameters $\boldsymbol{a}$ to $\boldsymbol{f}$ and transformation equations, on a **Notes** page:

- Enter the template title text '**Transforming with matrices**' as shown in the screenshot.
- Press menu > **Insert** > **Maths Box** (or press ctrl M) and enter the command ***a*:=1** then press enter.
- Position the cursor to the right of the Maths Box for ***a***, and press ␣ a few times.
- Press menu > **Insert** > **Maths Box** (or press ctrl M) and enter the command ***b*:=0** then press enter.
- Position the cursor to the right of the Maths Box for ***b***, and press ␣ a few times.
- Press menu > **Insert** > **Maths Box** (or press ctrl M) and enter the command ***e*:=0** then press enter.
- On a new line, repeat the above steps to enter the following (shown right): ***c*:=0** ***d*:=1** ***f*:=0**.



***Note**: A JPEG image of the matrix transformation equation has been inserted onto the Notes page. It is not essential but is included for reference.*

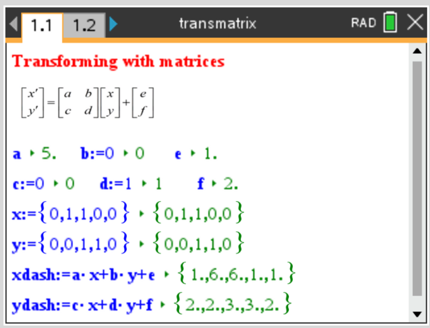
To enter the coordinates of a basic shape, we will use a unit square near the origin (shown right). Starting from the origin, moving anti-clockwise around the vertices of the square, the x and y coordinates of the square can be stored as follows:



- Press menu > **Insert** > **Maths Box** (or press ctrl M) and enter the command ***x*:={0,1,1,0,0}**.
- Press menu > **Insert** > **Maths Box** (or press ctrl M) and enter the command ***y*:={0,0,1,1,0}**.

To enter equations to calculate the coordinates of the transformed points (using the equations above):

- Press menu > **Insert** > **Maths Box** (or press ctrl M) and enter the command ***xdash* := *a* × *x* + *b* × *y* + *e***.
- Press menu > **Insert** > **Maths Box** (or press ctrl M) and enter the command ***ydash* := *c* × *x* + *d* × *y* + *f***.



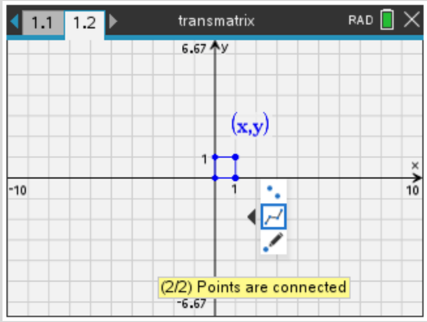
... continued

Solution (continued)

To construct a plot of the square, add a **Graphs** page, then:

- Press [menu] > **Graph Entry/Edit** > **Scatter Plot**
- For *x*, type ***x***, then press ▼ and then for *y*, type ***y***.
- Hover over the plot, press [ctrl] [menu] and select **Attributes**.
- Press ▼ and then ▶ to select **Points are connected**.
- Press [enter] to save this change to the plot attributes.

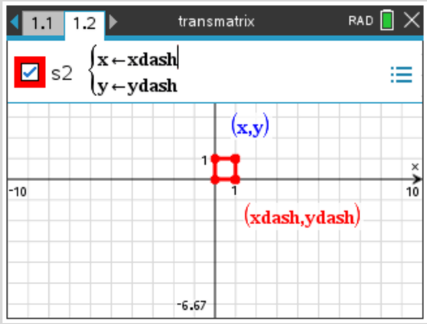
This will display plot of the original square.



To construct a plot of the transformed square, on the same **Graphs** page:

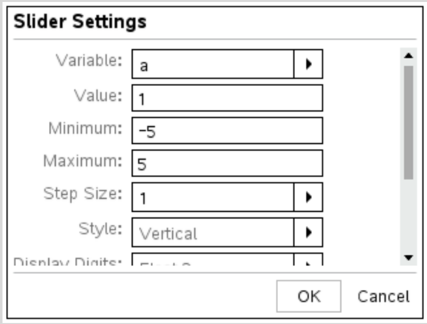
- Press [ctrl] [G] to enter a new scatter plot definition
- For *x*, type ***xdash***, then press ▼ and then for *y*, type ***ydash***.
- Hover over the plot, press [ctrl] [menu] and select **Attributes**.
- Press ▼ and then ▶ to select **Points are connected**.
- Press [enter] to save this change to the plot attributes.

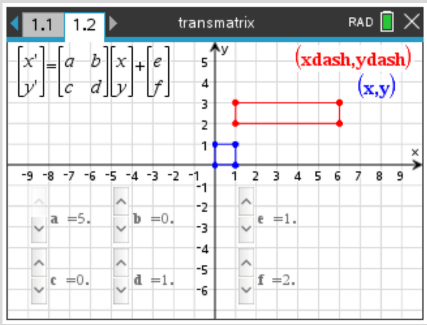
This will display plot of the transformed square. Note that with the current parameter values the original and transformed squares are identical (i.e. no change).



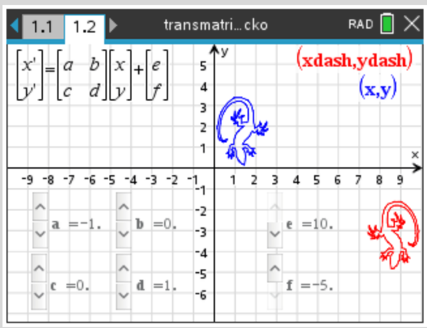
To enter sliders for the transformation parameters ***a* – *f***:

- Press [menu] > **Actions** > **Insert Slider**.
- In the **Slider Settings** dialog box that follows, enter the following values:
  Variable = *a* Value = 1 Minimum = –5
  Maximum = 5 Step Size = 1 Style = Vertical
- Scroll down and check the **Minimised** box.
- Click **OK** to save these slider settings and return to the graph page.
- Use the ◀▶▲▼ keys to move the slider to the desired location.
- Repeat the above steps for the remaining transformation parameters ***b* – *f***. A finished example is shown right.





Notes:
(1) To move the sliders around the viewing window, hover over the slider and press [ctrl] [menu]*, then select* ***Move****.*
(2) The matrix equation is shown here for clarification purposes. It can be added via [menu] *>* ***Actions*** *>* ***Text*** *and entering the required equation.*
(3) In this example, a basic unit square shape has been used. It is possible to use more complex shapes, with more points. For example, in the screen shown right, the x and y coordinates have been taken from an image of a gecko. To change the coordinates of the basic shape, edit the set of x and y coordinates on the ***Notes*** *page.*



1.4 Probability and simulations

1.4.1 Language of events and sets

Using random number generators for simulation of data

Question

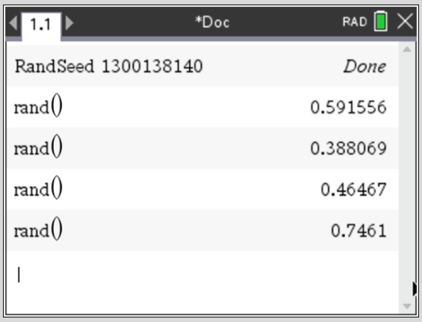
(a) Seed the random number generator and produce some random numbers between 0 and 1.

(b) Produce random integers within a defined range.

(c) Simulate a random sample of 5 counters drawn from a bag of 10 counters, where 3 counters are yellow and 7 counters are black. Consider situations of counters being drawn with and without replacement.

Solution

(a) Before using any of the random number probability tools in the calculator/software, a manually random seed value should be set. Otherwise random generators will all produce the same value across different devices if the devices are all at the same settings (e.g. factory default settings).

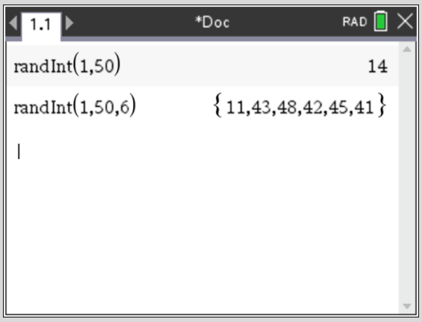


On a **Calculator** page:

- Press [menu] > **Probability** > **Random** > **Seed**.
- Enter a number of your own that will be unknown to the software and different from other users (e.g. your telephone number).
- Press [menu] > **Probability** > **Random** > **Number** then press [enter].

The calculator will display a random number in decimal form that is between 0 and 1. This number will be different from that on a calculator which has been seeded with a different value. Continue to press [enter] to repeat the command and produce more random numbers.

(b) To produce a random integer within the range 1 to 50:



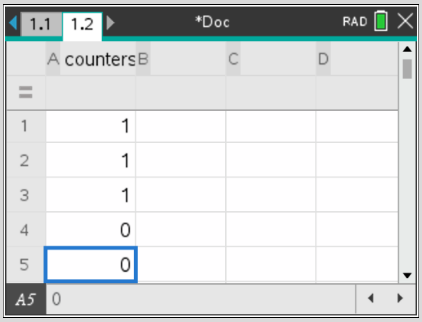
- Ensure that the calculator's random number generator has been seeded. (This only needs to be done once for the lifetime of the calculator unless the calculator has been reset.)
- Press [menu] > **Probability** > **Random** > **Integer**.
- Complete the command **randInt(1,50)**. The calculator will return a randomly selected integer within this range.
- Repeat pressing of the [enter] key will produce more random integers within this range or alternatively **randInt(1,50,6)** will return a set of 6 such values.

... continued

Solution (continued)

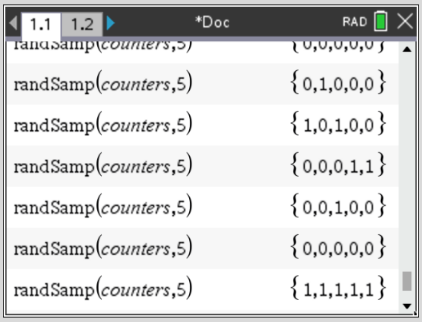
(c) On a **Lists & Spreadsheets** page:

- Label column A **counters** and type **1** into cells A1 to A3 and **0** into cells A4 to A10. This list will represent the 10 counters in the bag with a value of 1 being a yellow counter and value of 0 being black.

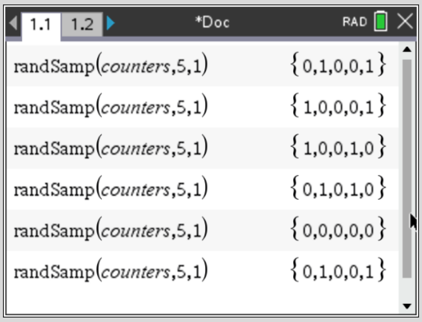


On a **Calculator** page:

- Press menu > **Probability** > **Random** > **Sample**.
- Press var and select **counters**. Complete the command **randSamp(counters,5)**. The set of results from 5 draws will be presented, with 1 representing a yellow counter and 0 representing black. Note that the default setting for this simulation is *with* replacement, so it is possible for a result set to contain more than 3 yellow counters.



In order to modify the simulation to show a set of selections *without replacement*, modify the command to **randSamp(counters,5,1)**. Note that it is now no longer possible for a result set to contain more than 3 yellow counters.



Exploring sample space through random numbers

Question

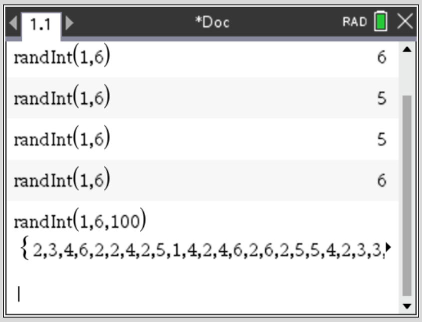
(a) Use a random integer generator to simulate the sample space resulting from the toss of a regular 6-sided die.

(b) If two regular dice are thrown and their numbers added, design a simulation that shows all possible number sums and approximations of their associated probabilities.

Solution

(a) On a **Calculator** page:

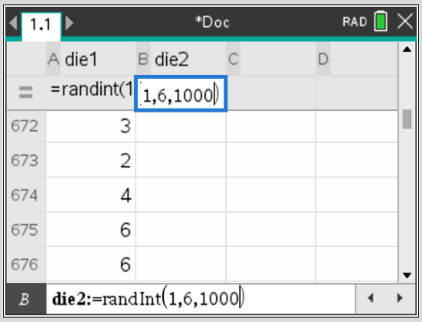
- Press menu > **Probability** > **Random** > **Integer**.
- Complete the command **randInt(1,6)**. The calculator will return a randomly selected integer within this range and can represent the number that is thrown by a regular 6-sided die. Repeated pressing of the enter key can represent repeated dice rolls.
- The command **randInt(1,6,100)** would represent a set of 100 independent dice rolls.



... continued

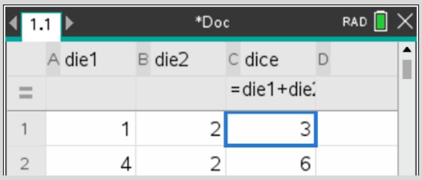
Solution (continued)

(b) On a **Lists & Spreadsheets** page:

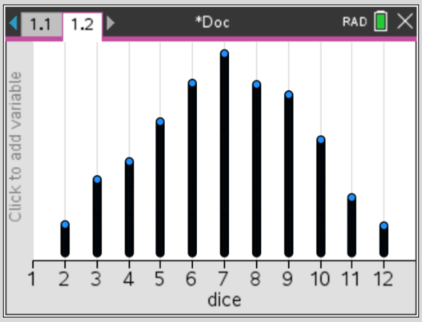


- Label column A as **die1**, column B as **die2**, and column C as **dice**
- Enter a formula for **die1** at the top of column A by entering =**randInt(1,6,1000)**. This will simulate 1000 rolls of a die and paste results into column A.
- Enter a formula for **die2** at the top of column B by entering =**randInt(1,6,1000)**. This will simulate 1000 rolls of a die and paste results into column B.

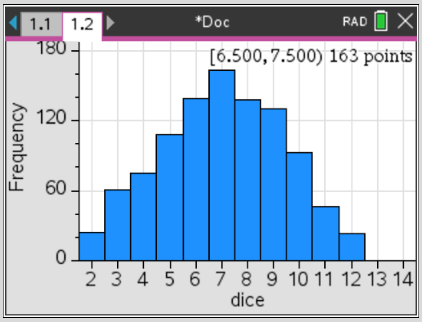
Enter a formula for **dice** at the top of column C by entering [=] **die1+die2**. This will calculate the die sum for each simulated roll of 2 dice.



Add a **Data & Statistics** page and then:



- Select **dice** as the variable on the horizontal axis and do not select a variable for the vertical axis. A dot-plot of the data will appear.
- Press [menu] > **Plot Type** > **Histogram**.
- Press [menu] > **Plot Properties** > **Histogram Properties** > **Bin Settings** > **Equal Bin Width**.
- In the dialog box set **Width=1**, and **Alignment=0.5**.
- The columns of the histogram will now be centred on the numbers 2 to 12 and the relative frequency of each can be read by hovering the cursor over each.



Notes:
(1) These relative frequencies provide approximations of theoretical probability. E.g. in this particular simulation the modal score is 7 with a relative frequency of 163/1000 or 16.3%. Compare this to the theoretical probability 1/6 = 16.7%.
(2) The simulation can be repeated by returning to the ***Lists & Spreadsheets*** *page and then pressing [ctrl] [R]. Doing this will also automatically update the histogram.*

1.4.2 Conditional probability and independence

Exploring conditional probability and independence

Question

By way of simulation of standard playing cards drawn from a 52-card deck, determine relative frequencies of the following events:

(a) Drawing a red card that is less than 10.

(b) Drawing a red card that is less than 10 **or** drawing a black card.

(c) Drawing a card that is less than 10 **given that** a red card is drawn.

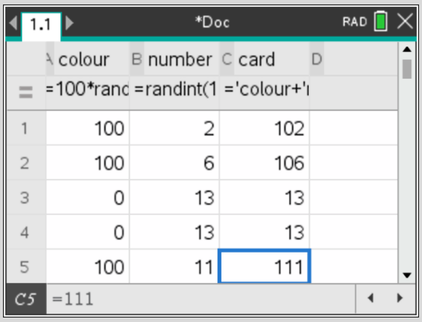
Solution

The card colour (red or black) can be simulated by the command **randInt(0,1)** with 0 representing red and 1 representing black. The card number can be simulated by the command **randInt(1,13)** with 1 representing an Ace, 11 Jack, 12 Queen, 13 King and numbers 2 to 10 simply representing those card numbers.

Note that the full sample space of drawing a card from a deck really contains 52 unique cards. However, for this simulation the sample space contains only 26 different possibilities as we are not differentiating between suits (i.e. diamonds and hearts are red and spades and clubs are black.)

On a **Lists & Spreadsheets** page:

- Label column A **colour**, column B **number** and column C **card**.
- In the formula space for column A type: **=100*randInt(0,1,1000)**.



This will populate the first 1000 cells of column A with either a 0 for red or 100 for black. (Note that the value from **randInt(0,1)** is multiplied by 100. This is to help identify red or black when we examine the final card.)

- In the formula space for column B type: **=randInt(1,13,1000)**.
- In the formula space for column C type: **=colour+number**.

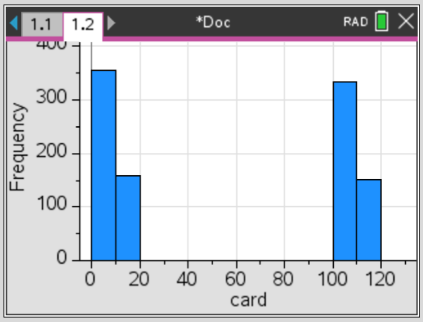
Cell C1 will now show a value that represents which of the 26 possibilities has been drawn. A value within the range (1,13) indicates a red card and a value within the range (101,113) indicates a black card. (e.g. 111 would represent a black Jack)

The 1000 values in column C now provide data from the simulation and may be examined to consider relative frequencies of different outcomes. Note that empirical data from a simulation is an indication of theoretical probabilities, although the relative frequencies may not be the same as theoretical probabilities.

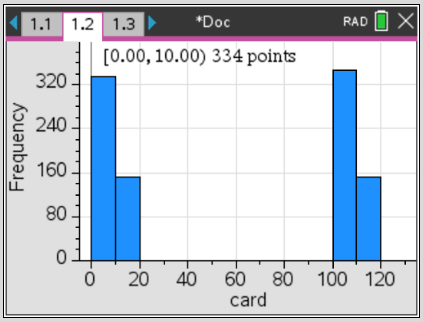
Solution (continued)

Add a **Data & Statistics** page and:

- Select the variable **card** for the horizontal axis and do not select a variable for the vertical axis. A dot-plot of the data will appear.
- Press [menu] > **Plot Type > Histogram**.

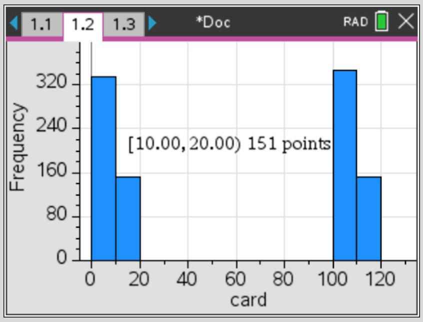


The default bin settings for the histogram will display two columns at the left of the screen and two at the right. The two at the left represent the red cards from the simulation and the two at the right the black cards. The first column in each of these pairs represents cards up to but not including 10 and the other column cards that are 10 or more.



(a) The leftmost column in the histogram represents red cards less than 10.

- Hover the cursor over this column to read its value and state this as a relative frequency of the 1000 trials (e.g. 334/1000 or 33.4%).
- Compare this experimental value to the theoretical probability (18/52 = 34.6%).

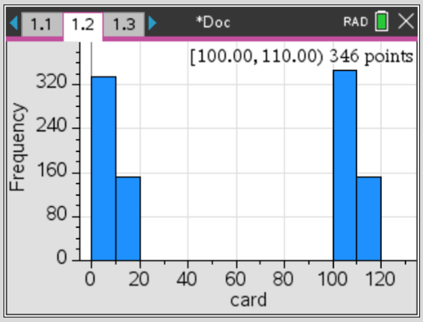


(b) The relative frequency of drawing a red card that is less than 10 **or** drawing a black card is given by adding the value of all columns except the second from the left (red cards $\geq$10). In the example provided this is:

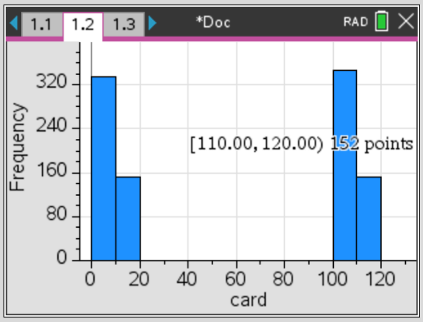
$$334 + 346 + 152 = 832.$$

So the relative frequency is 832/1000 or 83.2%.
Theoretical probability is (18 + 26)/52 = 84.6%.

Note that the formula $\Pr(A) = 1 - \Pr(A')$ could also be used for this calculation.



(c) In the example given the total number of red cards is 334 + 151 = 485. So the relative frequency of a card that is less than 10 **given that** a red card is drawn is 334/485=68.9%. (Theoretical probability is 9/13 = 69.2%)



Note: *The simulation of 1000 trials can be repeated by returning to the* ***Lists & Spreadsheets*** *page and then pressing* [ctrl] [R]*. Doing this will also automatically update the histogram. This is a good way to show variability within samples, even with a sample size of 1000.*

1.5 Combinatorics

1.5.1 Introduction to counting techniques

Using factorial notation

Question

Find the value of each of the following.

(a) $10!$ **(b)** $\dfrac{100!}{98!}$ **(c)** $6\times4!-7\times3!$

Solution

Text in the form of comments can be added to a **Calculator** page.

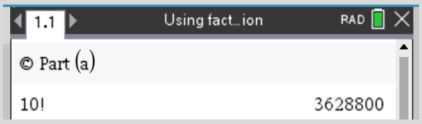
To add a comment such as '© Part (a)':

- Press menu > **Actions** > **Insert Comment**.

To access the factorial symbol:

- Press menu > **Probability** > **Factorial (!)**.

To evaluate parts **(a)**, **(b)** and **(c)**, enter as shown.

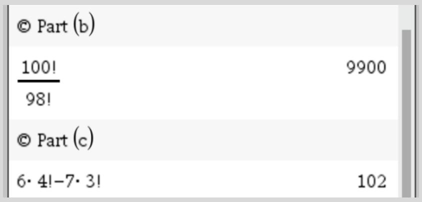


(a) $10!=3628800$ where $10!=10\times9\times8\times...\times3\times2\times1$

- Press ctrl ÷ to access the **Fraction** template.

(b) $\dfrac{100!}{98!}=9900$ where $\dfrac{100!}{98!}=\dfrac{100\times99\times98!}{98!}=100\times99$

(c) $6\times4!-7\times3!=102$



1.5.2 Permutations and combinations

Defining and using permutations

***Note**: Permutations are not formally a topic in the current Mathematical Methods course, but are included here for interest, and to make the link between permuations and combinations.*

Question

Adele has seven different books but there is only room for three of these books on her bookshelf. Find the number of ways Adele can randomly select the books and arrange them on her bookshelf using

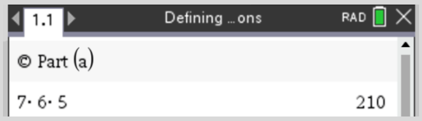
(a) the multiplication principle **(b)** $\dfrac{n!}{(n-r)!}$ **(c)** $^{n}P_{r}$

Solution

To add a comment to a **Calculator** page:

- Press [menu] > **Actions** > **Insert Comment**.

(a) 7 books can be placed in the first position, 6 remain for the second and 5 for the third. So $7\times6\times5=210$.



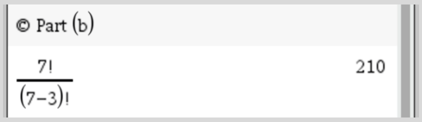
To access the factorial symbol:

- Press [menu] > **Probability** > **Factorial (!)**.

To access the **Fraction** template:

- Press [ctrl] [÷].

(b) $\dfrac{7!}{(7-3)!}=210$ where $\dfrac{7!}{4!}=\dfrac{7\times6\times5\times4!}{4!}=210$



$\dfrac{n!}{(n-r)!}$ can be interpreted as

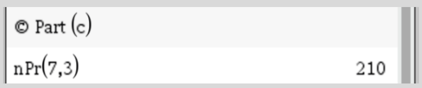
$$\frac{(\text{total number of objects})!}{(\text{total number of objects} - \text{number of objects to be arranged})!}$$

The number of ways to arrange r objects from a total of n objects is $^{n}P_{r}$ where $^{n}P_{r}=\dfrac{n!}{(n-r)!}$.

To access the **Permutations** command:

- Press [menu] > **Probability** > **Permutations**.

(c) $^{7}P_{3}=210$



***Note:** $^{n}P_{r}$ represents the number of ways of selecting r objects from n distinct objects where order is important.*

Solving problems involving permutations

Question

In how many ways can 4 cats and 3 dogs be arranged in a row if
(a) they are placed randomly?
(b) the 4 cats are kept together and the 3 dogs are kept together?
(c) no cat is next to another cat?

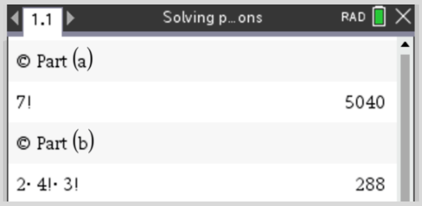
Solution

Note: *To add a comment to a* ***Calculator*** *page, press* menu *>* ***Actions*** *>* ***Insert Comment***.

To access the factorial symbol:

- Press menu > **Probability** > **Factorial (!)**.

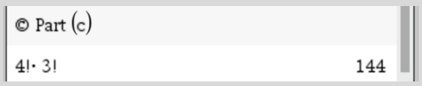
(a) There are $7! = 5040$ arrangements.



(b) There are 4! ways of keeping the 4 cats together and for each of these, 3! ways of keeping the dogs together. Also, there are 2 ways of arranging the group of cats and the group of dogs. The number of ways is $2\times 4!\times 3! = 288$.

(c) If no cat is next to another cat, the arrangement must be CDCDCDC.

There are 4! ways of arranging the 4 cats and for each of these, 3! ways of arranging the 3 dogs.



The number of ways is $4!\times 3! = 144$.

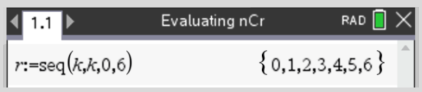
Evaluating ${}^nC_r$

Question

Evaluate ${}^6C_r$ for $r = 0,1,2,3,4,5,6$.

Solution

One way to evaluate ${}^6C_r$ for $r = 0,1,2,3,4,5,6$ is to use the sequence command.



On a **Calculator** page, assign the values of r as a sequence.

To enter $\boldsymbol{r := \text{seq}(k,k,0,6)}$:

- Press ctrl [=] to access the **Assign** [:=] command.
- Press menu > **Statistics** > **List Operations** > **Sequence**.
- Enter as shown.

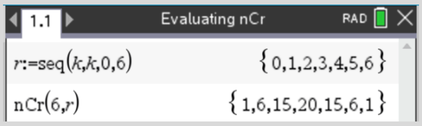
Note: *The syntax for expressing a sequence as a list is* ***seq(Expression, Variable, Low, High[,Step])****.*
The default value for ***Step*** *is 1.*

... continued

Solution (continued)

To access the **Combinations** command:

- Press [menu] > **Probability** > **Combinations**.
- Enter as shown.



$^6C_0 = 1, \; ^6C_1 = 6, \; ^6C_2 = 15, \; ^6C_3 = 20, \; ^6C_4 = 15, \; ^6C_5 = 6, \; ^6C_6 = 1$

Note: *This is the* $n = 6$ *row of Pascal's triangle.*
The $n = 0$ *row of Pascal's triangle is the first row.*

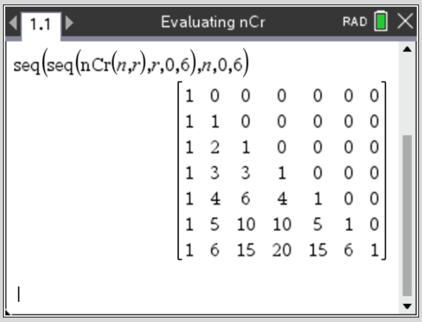
Note: *Alternatively, enter on a* ***Calculator*** *page as shown.*
To access "{}", press [ctrl] [)].

Extension:

A way to generate rows of Pascal's triangle is to write a command for "nested" sequences as shown on the **Calculator** page at right.

The first 7 rows of Pascal's triangle are displayed.

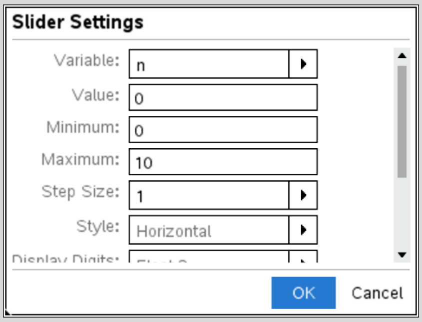
Can you see how it works?



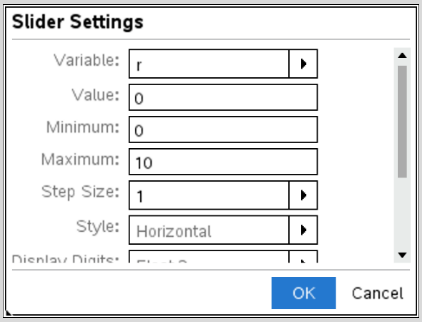
Alternatively, on a **Notes** page:

Insert a **Slider** to control the value of n as follows:

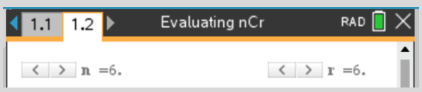
- Press [menu] > **Insert** > **Slider**.
- Set the **Slider Settings** as shown.
- Ensure to check the Minimised box.



Repeat the above instructions to insert a slider for r.



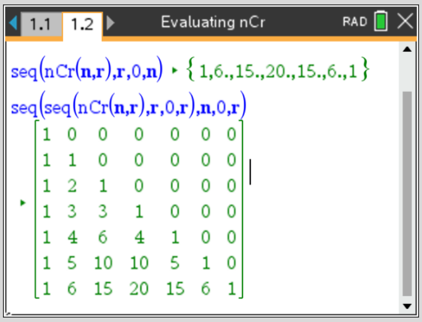
Position the sliders as shown at right.



... continued

Solution (continued)

Insert a **Maths Box** as follows:



- Press menu > **Insert** > **Maths Box**.

***Note:** Alternatively, to insert a **Maths Box**, press* ctrl M.

- Enter **seq(nCr(*n*,*r*),*r*,0,*n*)** into the **Maths Box** as shown.

Insert another **Maths Box** as follows:

- Press menu > **Insert** > **Maths Box**.
- Enter **seq(seq(nCr(*n*,*r*),*r*,0,*r*),*n*,0,*r*)** into this second **Maths Box** as shown.

Click on the sliders to change the value of n and r.

The screenshot at right displays the $n=6$ row of Pascal's triangle and the first 7 rows of Pascal's triangle.

Solving equations involving ^nC_r

Question

Solve $3\times{}^nC_6=11\times{}^nC_4$ for n where n is a positive integer.

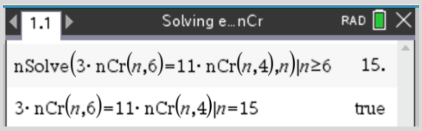
Solution

Note that ${}^nC_6\geq1$ and ${}^nC_4>1$ for $n\in\mathbb{Z}^+, n\geq6$.

On a **Calculator** page:

- Press menu > **Algebra** > **Numerical Solve**.
- Press menu > **Probability** > **Combinations**.

Complete as shown.



To add the constraint $n\geq6$:

- Press ctrl = to access the 'with' or 'given' symbol | and the ≥ symbol.

Solving $3\times{}^nC_6=11\times{}^nC_4$ for n with $n\geq6$ gives $n=15$.

***Note:** Entering the equation with $n=15$ gives the output **'true'**.*

Solving problems involving combinations

Question

A committee of three must be chosen from a cricket team of 11 players.

How many different committees are possible if:

(a) there are no restrictions?

(b) the captain of the team must be on the committee?

Solution

Note: *To add a comment to a* ***Calculator*** *page, press* [menu] *>* ***Actions > Insert Comment****.*

To access the **Combinations** command:

- Press [menu] > **Probability** > **Combinations**.

(a) There are ${}^{11}C_3 = 165$ possible committees.

(b) As the captain of the team must be on the committee, we simply need to select two of the remaining players.

There are ${}^{10}C_2 = 45$ possible committees.

Solving permutations and combinations problems including probability

Question

Twenty balls numbered from 1 to 20 are placed in a barrel.

If two balls are randomly selected, what is the probability that they are both numbered under 10?

Solution

To access the **Fraction** template:

- Press [ctrl] [÷].

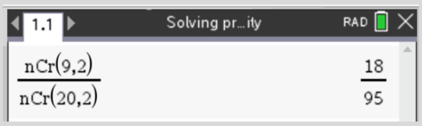
To access the **Combinations** command:

- Press [menu] > **Probability** > **Combinations**.

There are ${}^{9}C_2 = 36$ possibilities for the specified outcome since the two balls must come from those numbered from 1 to 9.

There are ${}^{20}C_2 = 190$ ways of selecting two objects from a set of 20 objects.

$$\Pr(\text{both under } 10) = \frac{{}^{9}C_2}{{}^{20}C_2} = \frac{18}{95}$$



VCE Mathematical Methods Unit 2

2.1 Exponential and logarithmic functions

2.1.1 Indices and index laws

Understanding index laws and auto-simplification

The *TI-Nspire CX II CAS* calculator has several auto-simplification rules that it applies to expressions involving index numbers and radicals.

Question

Use index laws to help interpret the following results obtained from the calculator.

Input → *Output*

(a) $\sqrt[5]{3^6} \rightarrow 3 \cdot 3^{\frac{1}{5}}$

(b) $\dfrac{1}{\sqrt[5]{3^6}} \rightarrow \dfrac{3^{\frac{4}{5}}}{9}$

(c) $\dfrac{1}{\sqrt{2}+\sqrt{5}} \rightarrow \dfrac{\sqrt{5}-\sqrt{2}}{3}$

Solution

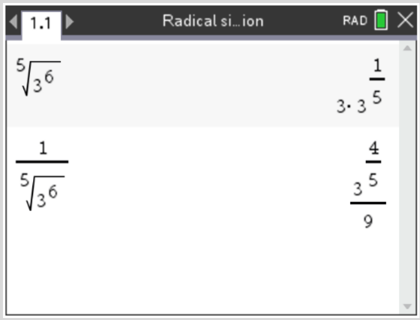
(a) The calculator output can be verified using index laws:

$$\sqrt[5]{3^6} = \left(3^6\right)^{\frac{1}{5}} = 3^{6\times\frac{1}{5}} = 3^{\frac{6}{5}} = 3^{\frac{5}{5}} \times 3^{\frac{1}{5}} = 3 \cdot 3^{\frac{1}{5}}$$

The auto-simplification rules applied consider an index number with an improper fraction index to be less simple than the product of an index number with an integer index and an index number with a proper fraction index.

(b) The calculator output can be verified using index laws:

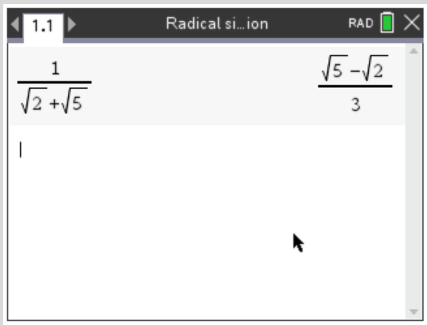
$$\frac{1}{\sqrt[5]{3^6}} = \frac{1}{3^{\frac{6}{5}}} = \frac{1}{3^{\frac{10}{5}-\frac{4}{5}}} = \frac{1}{3^{\frac{10}{5}} \cdot 3^{\frac{-4}{5}}} = \frac{3^{\frac{4}{5}}}{3^{\frac{10}{5}}} = \frac{3^{\frac{4}{5}}}{3^2} = \frac{3^{\frac{4}{5}}}{9}$$



The auto-simplification rules applied consider the simplest form to have a rational denominator, indices expressed in positive form, and any indices which are non-integer fractions to be expressed as proper fractions.

(c) The calculator output can be verified by rationalising the denominator of the input expression:

$$\frac{1}{\sqrt{2}+\sqrt{5}} \times \frac{\sqrt{2}-\sqrt{5}}{\sqrt{2}-\sqrt{5}} = \frac{\sqrt{2}-\sqrt{5}}{-3} = \frac{-\left(\sqrt{2}-\sqrt{5}\right)}{3} = \frac{\sqrt{5}-\sqrt{2}}{3}$$



Using scientific notation

Question

Complete the following calculations and interpret the results obtained.

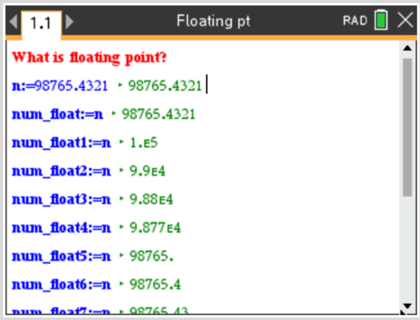
(a) 123456789 × 987654321 **(b)** 1234.5 × 98765 **(c)** $(1.2345 \times 10^{14}) \times (6 \times 10^{-23})$

***Note**: For calculations involving decimals, the screens below show results when the **Display Digits** setting is **Float 6** (that means it will display a maximum of 6 digits).*

Solution

Before completing the calculations, it is worth mentioning that most modern calculators (including the *TI-Nspire CX II CAS*) represent numbers using "Floating point". This calculator uses a maximum of 14 digits to represent a decimal number, or 12 digits for the mantissa and 2 digits for the exponent if the number is expressed using scientific notation.

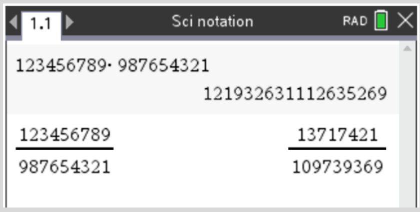
The screen shown right shows the number 98765.4321 displayed at various **Float** settings.



On a **Calculator** page, complete the calculations.

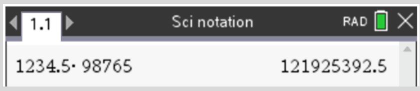
(a) The calculator demonstrates that large integer multiplication can be completed in exact form.

***Note**: For integer division – the calculator will display the result as the simplest equivalent but exact rational form (see example right).*

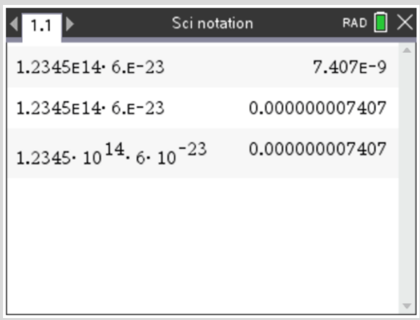


(b) Any calculation where a decimal point is included will provide an answer displayed in decimal form. If the number of digits in the result is **greater** than the display precision (e.g. here the result contains 10 digits, but the display digits setting is Float 6), then the answer will be displayed in scientific notation. Note that the actual result is stored at the full available calculation precision. This can be viewed by selecting the result and pressing [enter] (shown right).

If the display precision is set to a higher setting (e.g. Float 12 as shown right), the calculator may be able to display the answer in the normal decimal form.



(c) Numbers can be added in scientific notation directly by using the [EE] key as shown right.



The first calculation was performed with Display Digits set to Float 6 (and so the 6 display digits permitted was not enough to display the number without using scientific notation).

The second calculation was performed with Display Digits set to Float 12 (and so 12 digits were available to display the result in decimal form).

The third calculation shows the same result with powers of 10.

2.1.2 Introduction to exponential functions

Using recursion to demonstrate exponential growth

It is useful to use a sequence of recursive operations to demonstrate how exponential change works. This process also helps develop an understanding of how to construct a rule for an exponential function based on a constant

Question

Fia has $1000 to invest, and has been offered an interest rate of 7% p.a., compounded annually. If she accepts this offer:

(a) Show how the value of the investment grows in the first 3 years.

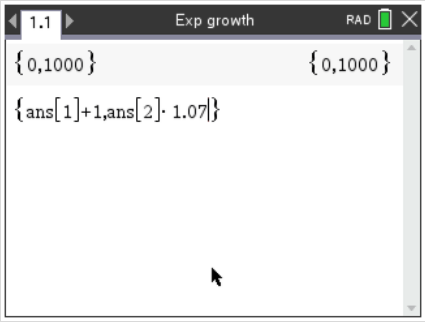
(b) Find the number of years required for the investment to have a value of $2000.

(c) Construct a function V, for the value of the investment after n years, and use this to confirm the answer obtained in part (b).

Solution

(a) To show how the value of the investment grows in the first 3 years, on a **Calculator** page:

- To display the initial investment value, enter **{0,1000}**.
- To display the value for following years using recursion, enter the command **{ans[1]+1,ans[2]×1.07}.**

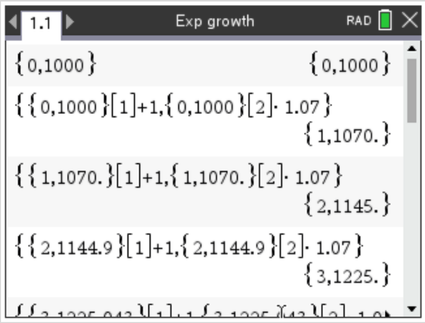


This will calculate and display the 'next' year number based on the previous year (i.e. using ans[1]+1), and the value of the investment at the end of the next year based on a 7% annual increase (i.e. ans[2]×1.07).

To repeat these recursive steps, press enter until the answer {3, 1225.} is displayed

Answer: After 3 years, the investment will be worth $1225.

***Note**: The expressions ans[1] and ans[2] refer to the first and second elements of previous answer. This is useful if the previous answer is expressed as a list of elements.*



(b) To find the number of years required to reach $2000:

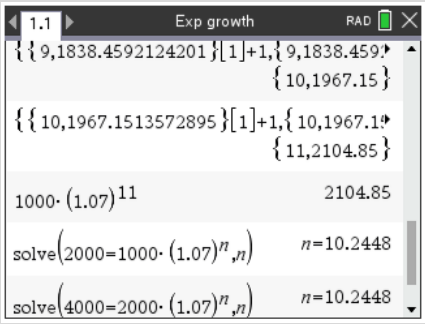
- Press enter until the answer {11, 2104.85} is displayed.

Answer: It takes 11 years for the investment to have a value of over $2000.

(c) The recursive steps show that the value in each year can be worked out by multiplying the previous year's value by 1.07.

So a suitable function is $V(n)=1000(1.07)^n$, where n is the number of completed years. To verify the answer from **(b)**:

- Enter $\mathbf{solve(2000=1000\times(1.07)^{\boldsymbol{n}},\boldsymbol{n})}$.



Answer: The answer $n\approx 10.2448$ means it will take 11 years (annual compounding).

Transforming exponential functions

Question

Let $f(x)=r^{(x-h)}+k,\ r>0.$ What is the effect on the graph of varying the parameter:

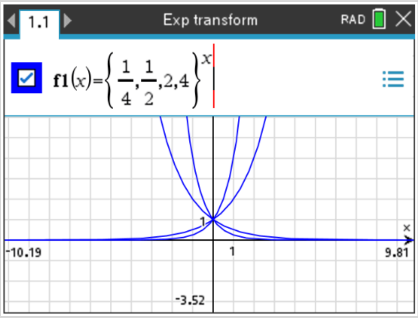
(a) r **(b)** h **(c)** k?

Solution

(a) To explore the effects of varying the value of the parameter r, let $h = 0$ and $k = 0$, on a **Graphs** page

- Enter ***f*1(*x*)={1/4,1/2,2,4}*^x^***

This will generate four different exponential function graphs with the values $r=\frac{1}{4},\frac{1}{2},2$ and 4.



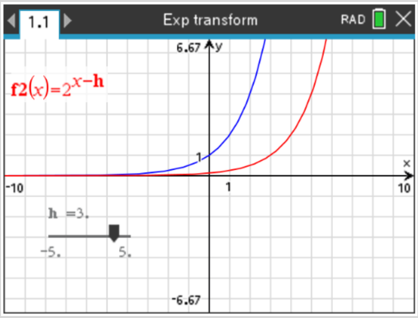
Answer: All graphs have a horizontal asymptote at $y = 0$ and pass through the point (0,1). If $r<1$, the value of y decreases as the value of x increases. If $r>1$, the value of y increases as the value of x increases.

(b) To explore the effects of varying the value of the parameter h, let $r = 2$ and $k = 0$, on a **Graphs** page:

- Enter ***f*1(*x*)=2*^x^*** and ***f*2(*x*)=2*^x − h^***

You will be prompted to create a slider for h, so that you can vary the parameter h as required.

- Move the slider so that $h = 3$.



Notice that for all points on the graph of $f1(x)=2^x$, there is a transformed point on the graph of $f2(x)=2^{x-h}$ which is 3 units to the right.

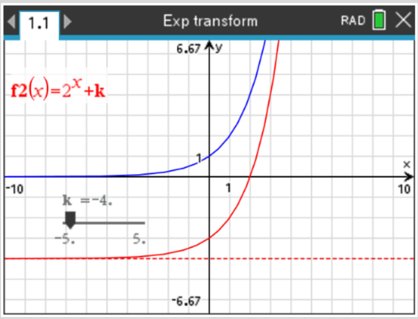
Answer: All graphs have a horizontal asymptote at $y = 0$ and pass through the point $(h,1)$. The general point (x, y) is translated to $(x + h, y)$.

(c) To explore the effects of varying the value of the parameter k, let $r = 2$ and $h = 0$, on a **Graphs** page

- Enter ***f*1(*x*)=2*^x^*** and ***f*2(*x*)=2*^x^*+*k***

You will be prompted to create a slider for k, so that you can vary the parameter k as required.

- Move the slider so that $k = -4$.



*Note: The horizontal asymptote can be defined as **f3(x)=k**, which changes dynamically as the slider is moved. This is shown on the screen above.*

Notice that for all points on the graph of $f1(x)=2^x$, there is a transformed point on the graph of $f2(x)=2^x-4$ which is 4 units down.

Answer: All graphs have a horizontal asymptote at $y = k$, and have a y-intercept at $y = 1 + k$. The general point (x, y) is translated to $(x, y + k)$.

Modelling with exponential functions

Question

A student deposits \$1500 into an account offering 6% per annum interest, compounded monthly.

(a) Construct an exponential relationship between the account balance B and t, the number of years over which the money is invested.

(b) Use your answer to part **(a)** to find the account balance after each of the first 5 years (to the nearest dollar).

(c) Find to the nearest month, the time that it would take for the account balance to double.

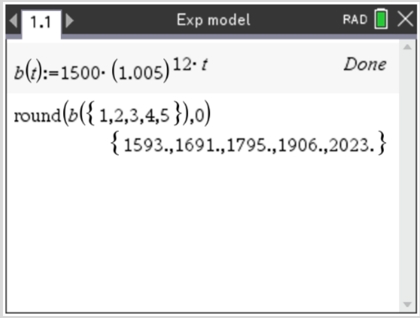
(d) Display your answer to part **(c)** graphically.

Solution

(a) Since compounding occurs monthly, a suitable function model can be defined as:

$$B(t) = 1500 \times \left(1 + \frac{6/12}{100}\right)^{12t} = 1500(1.005)^{12t}$$

On a **Calculator** page, enter ***b*(*t*):=1500*(1.005)**^{12t}

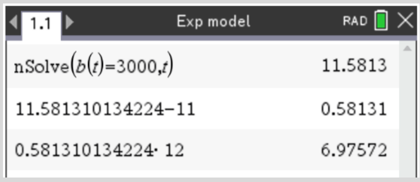


(b) To calculate the balance of the investment each year for the first 5 years, enter the command **round(*b*({1,2,3,4,5}),0).**

Answer: The balances after each of the first five years are \$1593, \$1691, \$1795, \$1906 and \$2023.

(c) The time required to double the account balance to \$3000 can be found using the numerical solve command as follows:

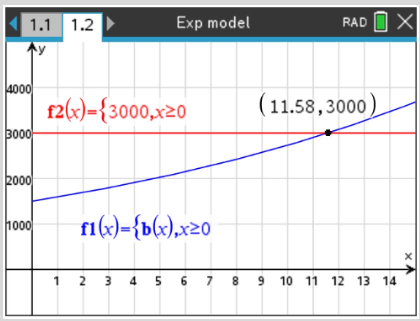
- Press menu > **Algebra > Numerical Solve.**
- Enter the command **nsolve(*b*(*t*)=3000,*t*)**
- Convert the decimal part to months as shown right.



Answer: It will take 11 years and 7 months for the loan balance to double in value.

(d) On a **Graphs** page, enter ***f*1(*x*)=*b*(*x*)|*x*≥0** and then enter the constant function ***f*2(*x*)=3000**.

- Press menu > **Window/Zoom > Window Settings**
- In the dialog box that follows, enter the following values:
  XMin= –1 Xmax = 15 XScale = 1
  YMin = –1000 YMax = 5000 YScale = 1000
- To show the point of intersection, press menu, select **Analyse Graph** and **Intersection** and follow the prompts.



__Notes__: (1) The "|" and "≤" symbols are found by pressing ctrl = *. To display the grid, press* menu *and select **Settings**. In the **Grid** options, select **Lined Grid**.*

(2) The multiple scale labels can be toggled on/off by hovering over one of the axes, and then pressing ctrl menu*. Then select **Attributes** and modify the attribute shown right.*

2.1.3 Logarithms and logarithmic laws

Understanding simplest form via logarithmic laws

Question

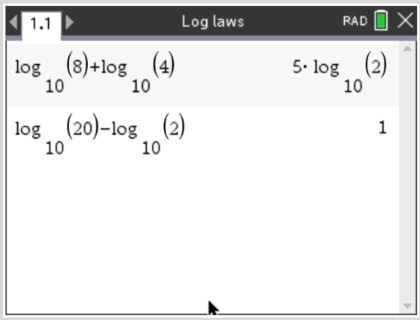
The calculator uses auto-simplification rules including logarithmic laws to write answers involving logarithms in their simplest form. Use these laws to explain why the calculator produces the following results.

	Input	*Output*		*Input*	*Output*
(a)	$\log_{10} 8 + \log_{10} 4$	$5 \cdot \log_{10} 2$	**(b)**	$\log_{10} 20 - \log_{10} 2$	1
(c)	$\dfrac{\log_{10} 8}{\log_{10} 4}$	$\dfrac{3}{2}$	**(d)**	$\dfrac{1}{\log_{10} 7} = \log_7 10$	true

Solution

(a) The output can be explained by applying the log law for adding logs and the law applying to logs of index numbers.

$$\begin{aligned}\log_{10} 8 + \log_{10} 4 &= \log_{10}(8 \times 4)\\ &= \log_{10}(2^5)\\ &= 5\log_{10} 2\end{aligned}$$

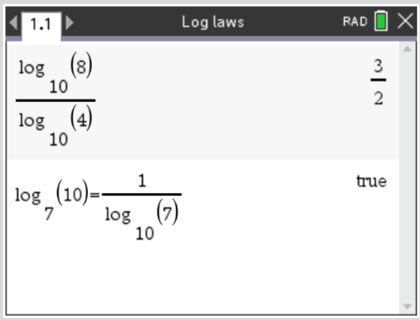


(b) The output can be explained by applying the log law for subtracting logs and the law applying to logs of index numbers.

$$\begin{aligned}\log_{10} 20 - \log_{10} 2 &= \log_{10}\left(\frac{20}{2}\right)\\ &= \log_{10}(10)\\ &= 1\end{aligned}$$

(c) The output can be explained by the law applying to logs of index numbers and cancelling the common factor of the numerator and denominator.

$$\begin{aligned}\frac{\log_{10} 8}{\log_{10} 4} &= \frac{\log_{10}(2^3)}{\log_{10}(2^2)}\\ &= \frac{3\log_{10} 2}{2\log_{10} 2}\end{aligned}$$



(d) In this case the output indicates that the equation is true, and this can be shown with the *change of base* rule for logarithms.

$$\log_a x = \frac{\log_b x}{\log_b a} \Leftrightarrow \frac{1}{\log_a x} = \frac{\log_b a}{\log_b x} = \log_x a$$

So $\log_x a = \dfrac{1}{\log_a x}$

2.1.4 Logarithmic functions

Transforming logarithmic functions

Question

Let $f(x)=\log_a(x-h)+k,\ a>1$. What is the effect on the function graph of varying the parameter:

(a) a **(b)** h **(c)** k?

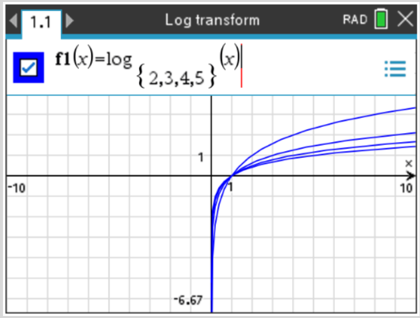
Solution

(a) To explore the effects of varying the value of the parameter a, let $h = 0$ and $k = 0$, on a **Graphs** page:

- Enter $\boldsymbol{f1(x)=\log_{\{2,3,4,5\}}(x)}$

This will generate 4 different logarithmic function graphs with base values $a = 2, 3, 4$, and 5. Note that:

- all of the graphs have a vertical asymptote at $x = 0$, and pass through the point (1,0).
- For higher a values, the gradient of the graph decreases more rapidly for $x > 0$.



(b) To explore the effects of varying the value of the parameter h, let $a = 2$ and $k = 0$, on a **Graphs** page

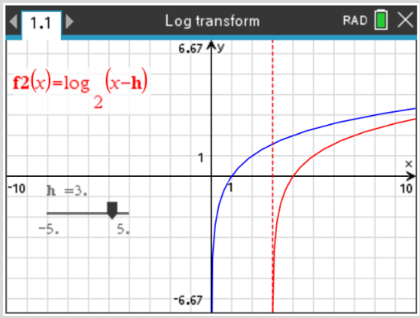
- Enter $\boldsymbol{f1(x)=\log_2(x)}$ and $\boldsymbol{f2(x)=\log_2(x-h)}$

You will be prompted to create a slider for h, so that you can vary the parameter h as required.

- Move the slider so that $h = 3$.

Notice that for all points on the graph of $\boldsymbol{f1(x)=\log_2(x)}$, there is a transformed point on the graph of $\boldsymbol{f2(x)=\log_2(x-h)}$ which is 3 units to the right. By varying the value of h :

- all the graphs have a vertical asymptote at $x = h$ and have an x-intercept at $y = 1 + h$
- the general point (x, y) is translated to $(x + h, y)$.



Note: you can also define the vertical asymptote as $x = h$, which changes as the slider is moved. To do so, press menu *>* ***Graph Entry/Edit*** *>* ***Relation***, *and enter the relation* $\boldsymbol{x = h}$.

(c) To explore the effects of varying the value of the parameter k, let $a = 2$ and $h = 0$, on a **Graphs** page:

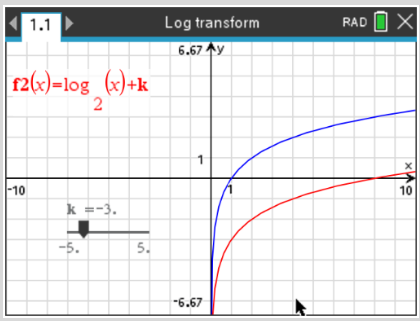
- Enter $\boldsymbol{f1(x)=\log_2(x)}$ and $\boldsymbol{f2(x)=\log_2(x)+k}$

You will be prompted to create a slider for k, so that you can vary the parameter k as required.

- Move the slider so that $k = -3$.

Notice that for all points on the graph of $\boldsymbol{f1(x)=\log_2(x)}$, there is a transformed point on the graph of $\boldsymbol{f2(x)=\log_2(x)-3}$ which is 3 units down. By varying the value of k, it can be seen that:

- all the graphs have a vertical asymptote at $x = 0$
- As k increases, the x-intercept approaches 0.
- the general point (x, y) is translated to $(x, y + k)$.
- all graphs pass through the point $(1,k)$.



Modelling with logarithmic functions

Question

The intensity of a sound depends on the energy of the sound wave. It is measured as power per unit area, usually in watts/m^2 . An alternative measure is the loudness level, measured in decibels (dB).

The loudness L dB is related to the intensity I watts/m^2 by the formula $L = 10\log_{10}\left(\dfrac{I}{10^{-12}}\right)$.

(a) The threshold for human hearing (i.e. the intensity of a barely audible sound) serves as a base measure for decibels. The intensity is 10^{-12} watts/m^2. What is the corresponding loudness in decibels?

(b) The human eardrum is in danger of rupturing at a loudness level of 160 dB. What is the intensity in watts/m^2 at which the human eardrum is in danger of rupturing?

(c) Normal conversation takes place at an intensity of about 10^{-6} watts/m^2. Traffic on a busy street might be 10 times the sound intensity of a normal conversation. Show that this does not mean that the loudness level is 10 times greater, and hence explain how the loudness level changes when the sound intensity doubles.

(d) Explore ways to visually represent the relationship between loudness level and sound intensity.

Solution

On a **Calculator** page, define the function by entering

***louddb*(*i*):=10*log((*i*/10^{-12}),10)**

(a) To find the loudness level for an intensity of 10^{-12}:

- Enter ***louddb*(10^{-12})**

(b) To find the intensity for a loudness level of 160 dB:

- Enter **nsolve(*louddb*(i)=160,i)**

Note: *Either the **solve** or **nsolve** is suitable here.*

(c) To find the loudness level for intensities of 10^{-6} and 10^{-5}:

- Enter ***louddb*({10^{-6},10^{-5}})**

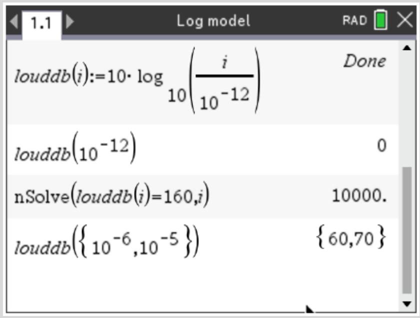
If the sound intensity is multiplied by a factor of 10, the loudness level is increased by 70 – 60 = 10 dB.

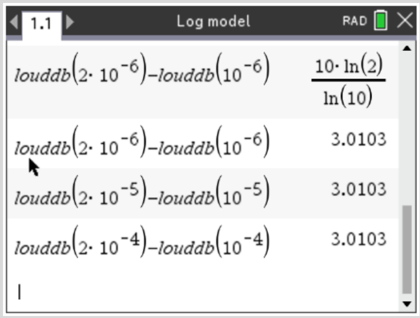
To find how much the loudness level would increase if the sound intensity is doubled:

- Enter ***louddb*(2*10^{-6})–*louddb*(10^{-6})**

Note: *Press* ctrl enter *to find the answers as decimal approximations.*

If the sound intensity is doubled, the loudness level would increase by about 3 dB. This is true for doubling all sound intensities, which is verified in the screen right, and can be proven algebraically.





... continued

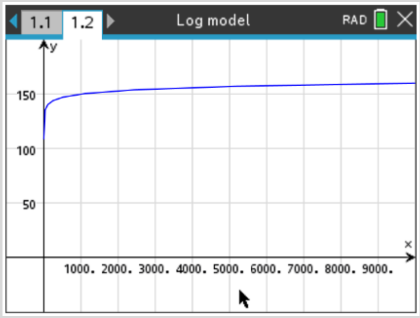
Solution (continued)

(d) Add a **Graphs** page and enter the commands as follows:

- Enter ***f*1(*x*) = *louddb*(*x*)**
- Press [menu], select **Window/Zoom** and **Window Settings**

In the dialog box that follows, enter the following values:
XMin = –1000 Xmax = 10000 XScale = 1000
YMin = –50 YMax = 200 YScale = 50

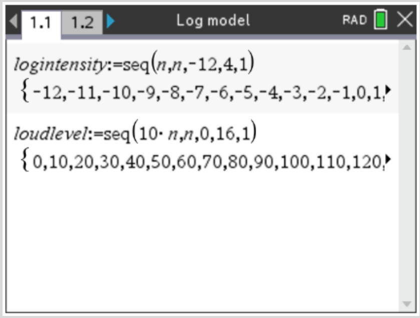
Note that this view obscures the graph for lower values of I.



As it is known the values of I change more rapidly than those of L, try plotting the $\log_{10}(I)$ values against the loudness level. From above, a reasonable range for human hearing is from $I = 10^{-12}$ (threshold of hearing) to $I = 10^4$ (ear bleeds). The associated values of L are from $L = 0$ dB to $L = 160$ dB.

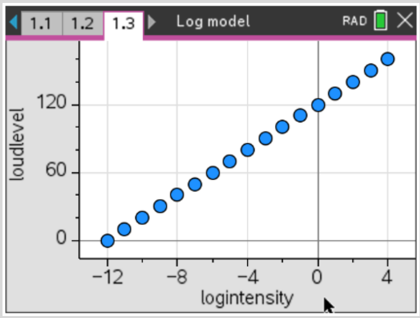
Generate these sequences of values on a **Calculator** page as follows:

- Enter ***logintensity*:=seq(*n*,*n*,–12,4,1)**
- Enter ***loudlevel*:=seq(10*n*,*n*,0,16,1)**



To create a scatter plot of ***loudlevel*** vs ***logintensity***, add a **Data & Statistics** page and then:

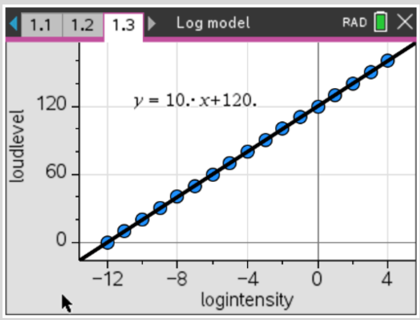
- Select ***logintensity*** for the horizontal axis.
- Select ***loudlevel*** for the vertical axis.



The plot shows a perfect linear relationship between the loudness level and the logarithm (base 10) of sound intensity.

To find the equation of the line relating loudness level and the logarithm (base 10) of sound intensity:

- Press [menu] > **Analyse** > **Regression** > **Show Linear (mx+b)**



So according to the regression analysis,

$$loudlevel = 10 \times \log_{10}(intensity) + 120$$

***Note**: This linear relationship between the loudness level and the logarithm of the sound intensity can also be illustrated via the **Lists and Spreadsheet** application, as the screen below highlights.*

	A intensity	B log_intensity	C loudness_db
=	=seq(10^n,n,-12,4,1)	=log(intensity,10)	=louddb(intensity)
1	1/1000000000000	-12	0
2	1/100000000000	-11	10
3	1/10000000000	-10	20
4	1/1000000000	-9	30
5	1/100000000	-8	40
6	1/10000000	-7	50
7	1/1000000	-6	60
8	1/100000	-5	70

This result can be shown via the log laws to be an alternative form of the function L, as follows:

$$L = 10\log_{10}\left(\frac{I}{10^{-12}}\right)$$
$$= 10\left(\log_{10} I - \log_{10}\left(10^{-12}\right)\right)$$
$$= 10\left(\log_{10} I + 12\log_{10}(10)\right)$$
$$= 10\left(\log_{10} I + 12\right)$$
$$= 10\log_{10} I + 120$$

2.2 Circular functions

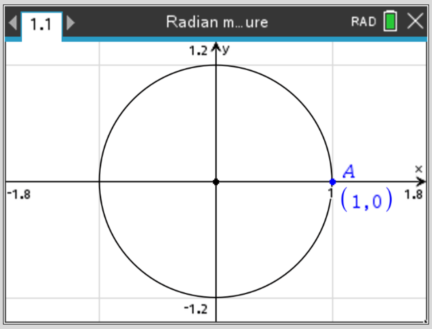
2.2.1 The unit circle, arc length and radian measure

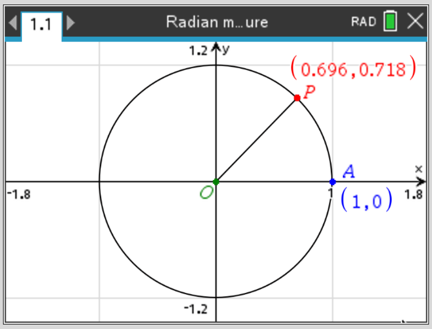
Understanding radian measure and its relationship to the unit circle

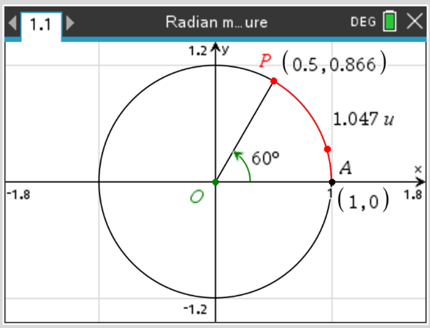
Question

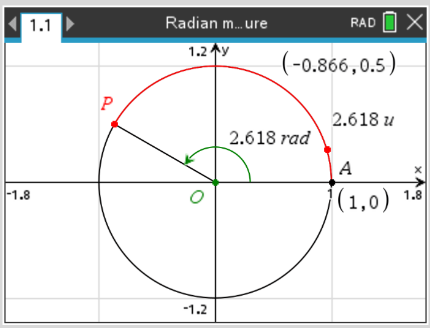
Construct an interactive model to display the arc length on a unit circle for angles subtended at the centre. Comment on observed arc length and angle measurements in degrees and radians.

Solution









To construct a model for measuring the angle subtended at the centre by an arc in a unit circle, on a **Graphs** page:

- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows enter the values:

  XMin = – 1.8   XMax = 1.8   XScale = 1
  YMin = – 1.2   YMax = 1.2   YScale = 1

- Press [P] > **Point by Coordinates** and enter (**1**, **0**). Hover over this point, press [ctrl] [menu] > **Label** and enter label *A*.
- Similarly, place a point at the origin, and enter label *O*.
- Hover over coordinates (0,0) and press [ctrl] [menu] > **Hide.**
- Press [menu] > **Geometry** > **Shapes** > **Circle**.
- Click (i.e. press [click]) the origin, *B*, then point *A*.
- Press [menu] > **Geometry** > **Points & Lines** > **Segment**.
- Click the centre of the circle, then a point on the circumference (in the first quadrant) and press [esc]. Hover over this point, press [ctrl] [menu] > **Label** and enter label *P*.
- Hover over point *P* and press [ctrl] [menu] > **Coordinates …**

To measure the arc length *AP* and the angle *AOP*:

- Press [menu] > **Geometry** > **Points & Lines** > **Circle arc**. Click point *A*, then a point on the circumference between *A* and *P*, then click point *P*. Press [esc] to exit the tool.
- To measure arc length *AP*, hover over the arc and press [ctrl] [menu] > **Measurement** > **Length**.
- To toggle angle settings, click on the **DEG** or **RAD** setting at top right of the screen. Select **DEG**.
- To measure angle *AOP*, press [menu] **Geometry** > **Measurement** > **Directed Angle** then click points *A*, *O* and *P* in that order. Press [esc] to exit the tool.
- Move point *P* around the circle by hovering over the point and pressing [ctrl] [grab] to grab and [esc] release the point.
- After moving *P* to a new position, toggle the **DEG** or **RAD** setting and observe the arc length.

Answer: The magnitude of angle *AOB* in radians is numerically equal to the arc length *AB*. If arc length $AB = \theta$ units, then by definition $\measuredangle AOB = \theta^c$ (θ radians).

Determining exact values of common angles in radians

Question

The previous problem established that if the arc length on the unit circle is θ, the angle is θ^c.

The circumference of the unit circle is 2π, therefore $360^\circ = 2\pi^c$ and $\theta^\circ = \left(\frac{2\pi}{360}\times\theta\right)^c = \left(\frac{\pi}{180}\times\theta\right)^c$.

Create an interactive Notes page to convert angle measurements between degrees and radians.

(a) Determine the exact values in radians for the sequence of angles $30^\circ, 60^\circ, 90^\circ, ..., 360^\circ$.

(b) Find the equivalent angle measurements in degrees for the sequence $\frac{\pi}{4}^c, \frac{\pi}{2}^c, \frac{3\pi}{4}^c, ..., 2\pi^c$.

Solution

To set up an interactive conversion page, on a **Notes** page:

- Enter the headings and labels, as shown.
- Press [ctrl] [M] to insert a **Maths Box** next to each label.

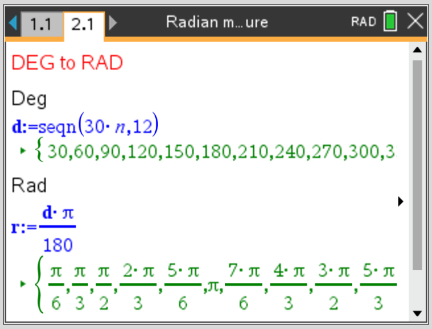
(**a**) To generate the angles $30^\circ, 60^\circ, ...,$ in the top **Maths Box**:

- Enter ***d***:**= seqn(30*n*,12)** by pressing [book] [1] [S] and selecting **seqn**. The syntax is **seqn(**Expr(*n*),*n*Max**)**.

*Note: An alternative sequence command is **seq(30n,n,1,12)**.*

To convert to radians, in the second **Maths Box**:

- Enter $r := \frac{d\times\pi}{180}$.



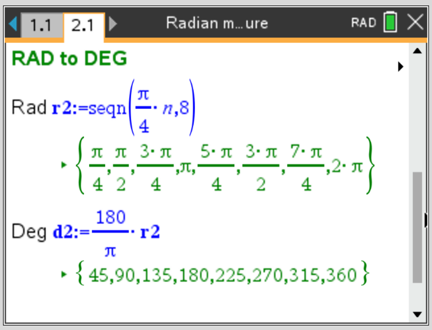
Answer: $\{30^\circ, 60^\circ, 90^\circ ...\} = \left\{\frac{\pi}{6}^c, \frac{\pi}{3}^c, \frac{\pi}{2}^c, ...\right\}$

(**b**) To generate angles $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, ...,$ in the third **Maths Box**:

- Enter ***r2***:**= seqn(*π**n*/4,8)**.

To convert the angles to degrees, in the fourth **Maths Box**:

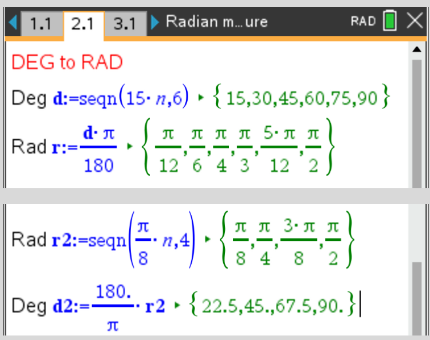
- Enter $d2 := \frac{r2\times 180}{\pi}$



Answer: $\left\{\frac{\pi}{4}^c, \frac{\pi}{2}^c, \frac{3\pi}{4}^c, ...\right\} = \{45°, 90°, 135°...\}$

*Note: (1) Edit the first or third **Maths Boxes** to convert other angle measurements from degrees to radians, or vice versa.*

*(2) To obtain a decimal answer, edit the conversion formula to include a decimal point by changing the '**180**' to '**180.0**'.*



Converting between degrees and radians, including cases involving DMS values

Question

(a) Convert $60°45'15''$ to radians, correct to four decimal places.

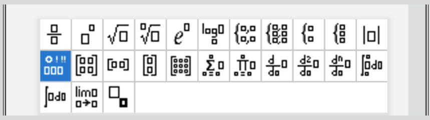
(b) Convert 1.5 radians to degrees, minutes and seconds, correct to the nearest second.

(c) Convert $3\pi / 7^c$ to decimal degrees, correct to three decimal places.

Solution

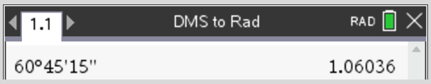
To change the **Calculation Mode**, on a **Calculator** page:

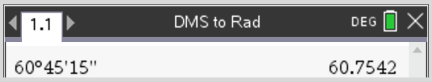
- Click the **battery** icon at the top right corner of the screen.
- Select **Document Settings > Calculation Mode > Approximate**, then click **OK**.



(a) To convert $60°45'15''$ to radians:

- Press [templates key], select the □°□′□″ template and enter **60°45′15″**.
- Toggle between Degree and Radian mode by clicking **RAD** or **DEG** at the top right of the screen.

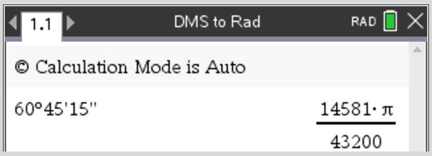




Answer: $60°45'15'' = 1.0604^c$, obtained in **RAD** mode.

In **DEG** mode, $60°45'15'' = 60.7542°$ (to four decimal places).

***Note:** If the **Calculation Mode** is set to **Auto** and an exact value is returned, press ctrl enter for a decimal Answer:*



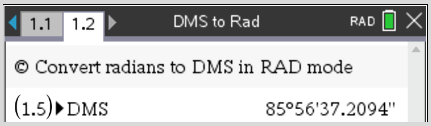
For conversion of $60°45'15''$ to radians in **DEG** mode:

- Enter **60°45′15″** then press [catalog] 1 R and select ▸Rad.

Answer: $60°45'15'' = 1.06036...^c$, displayed as $(1.06036...)^r$.

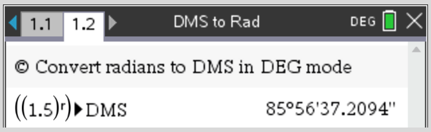
(b) To convert 1.5^c to degrees in **RAD** mode:

- Enter **1.5** then press [catalog] 1 D and select **▸DMS**.



To convert 1.5 radians to degrees in **DEG** mode:

- Enter **1.5** then press [π▸] and select the ^{r} symbol.
- Press [catalog] 1 D and select **▸DMS**.



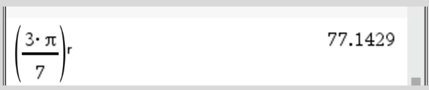
Answer: $1.5^c = 85°56'37''$.

(c) To convert $3\pi / 7^c$ to decimal degrees in **RAD** mode:

- Enter **3π / 7** then press [catalog] 1 D and select **▸DD**.

To convert $3\pi / 7^c$ to decimal degrees in **DEG** mode:

- Enter **3π / 7** then press [π▸] and select the ^{r} symbol.
- Optional in **DEG** mode - press [catalog] 1 D and select **▸DD**.



Answer: $3\pi / 7^c = 77.143°$ (three decimal places).

2.2.2 Definition and properties of circular functions and their graphs

Defining sine and cosine functions from the unit circle

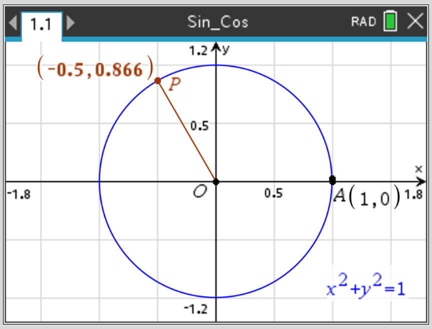
Question

(a) Construct an interactive unit circle model displaying values of θ, $\sin(\theta)$ and $\cos(\theta)$. Use this model to find the values of θ, where $\theta \in [0, 2\pi]$, such that **(i)** $\sin(\theta) = \frac{1}{2}$ **(ii)** $\cos(\theta) = -\frac{\sqrt{2}}{2}$.

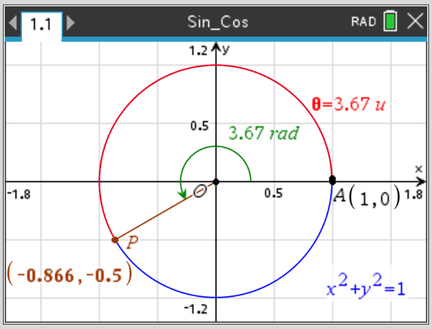
(b) Use the model to show that **(i)** $\cos^2(\theta) + \sin^2(\theta) = 1$ **(ii)** $\sin(\theta) \approx \theta$ for small values of θ.

Solution

(a) To construct an interactive unit circle model to show the values of θ, $\sin(\theta)$ and $\cos(\theta)$, on a **Graphs** page:

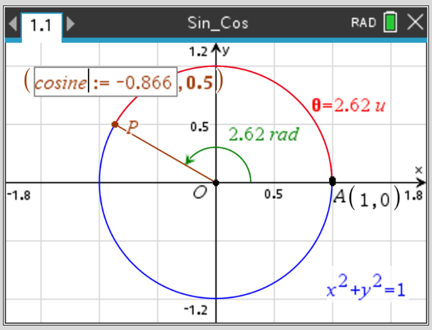


- Press [menu] > **Graph Entry/Edit > Relation**.
- Enter $\boldsymbol{x^2 + y^2 = 1}$, the equation of a circle $C(0, 0)$, $r = 1$.
- Press [menu] > **Window/Zoom > Window Settings**.
  In the dialog box that follows enter the following values:
  XMin = – 1.8 XMax = 1.8 XScale = 0.5
  YMin = – 1.2 YMax = 1.2 YScale = 0.5
- Press [P] > **Point by Coordinates** and enter (**1**, **0**). Hover over this point, press [ctrl] [menu] > **Label** and enter label ***A***.
- Similarly, place a point at the origin, and enter label ***O***.
- Hover over coordinates (0,0) and press [ctrl] [menu] > **Hide.**
- Press [menu] > **Geometry > Points & Lines > Segment**.
- Click the centre of the circle, then a point on the circumference (in the first quadrant) and press [esc]. Hover over this point, press [ctrl] [menu] > **Label** and enter label ***P***.
- Hover over point P and press [ctrl] [menu] > **Coordinates & Equations**. The coordinates of P will be displayed.



To measure the arc length AP and the angle AOP:

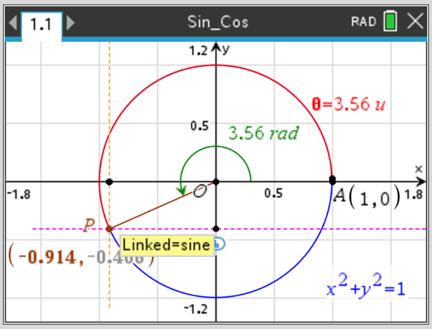
- Press [menu] > **Geometry > Points & Lines > Circle arc**. Click point A, then a point on the circumference just beyond A, then click point P. Press [esc] to exit the tool.
- To measure arc length AP, hover over the arc and press [ctrl] [menu] > **Measurement > Length**.
- To toggle angle settings, click on the **DEG** or **RAD** setting at top right of the screen. Select **RAD**.
- To measure angle AOP, press [menu] > **Geometry > Measurement** > **Directed Angle** then click points A, O and P in that order. Press [esc] to exit the tool.
- Hover over arc length measurement, press [ctrl] [menu] > **Store** and enter the variable name θ by pressing [ctrl] [📖] to select θ.
- Store value of $\boldsymbol{\cos(\theta)}$. Hover over the x-coordinate of P, press [ctrl] [menu] > **Store** and enter variable name **cosine**.



... continued

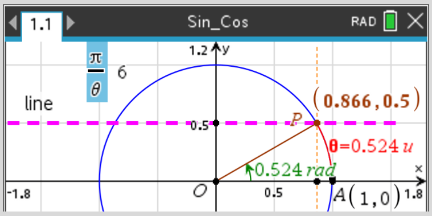
Solution (continued)

To store the values of $\sin(\theta)$ and illustrate it on the y-axis:



- Hover over the y-coordinate of P, press ctrl menu > **Store** and enter the variable name **sine**.
- Press menu > **Geometry** > **Construction** > **Perpendicular**. Click the y-axis then point P. Click the x-axis then point P. Press esc to exit the tool.

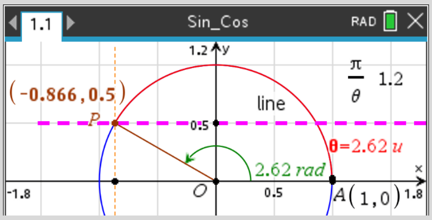
(i) To find values of θ such that $\sin(\theta)=0.5$:



- With point P in the first quadrant, click the y-coordinate of P so that it is editable. Enter the value **1/2**. Move P to the second quadrant and similarly edit the y-coordinate to **1/2**.

To find $\boldsymbol{n}$ such that the values of $\theta=\pi/n$:

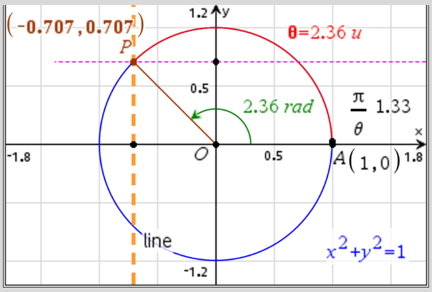
- Press ctrl menu > **Text**. In the textbox, enter $\boldsymbol{\pi/\theta}$.
- Press menu > **Actions** > **Calculate**. Click the text, for the prompt '**Select** $\boldsymbol{\theta}$ **?**'. Click the arc length measurement θ, move the answer next to the text, then press esc.



Answer: If $\sin(\theta)=\dfrac{1}{2}$, $\theta\in[0,2\pi]$ then $\theta=0.524...=\dfrac{\pi}{6}$ or

$\theta=2.62...=\dfrac{\pi}{1.2}=\dfrac{5\pi}{6}$.

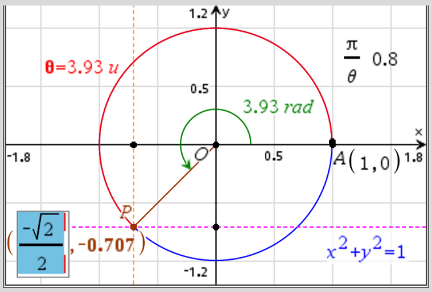
(ii) To find values of θ such that $\cos(\theta)=-\dfrac{\sqrt{2}}{2}$:



- With point P in the second quadrant, click the x-coordinate of P so that it is editable.
- Enter the value $-\boldsymbol{\sqrt{2}/2}$. Move P to the third quadrant and similarly edit the x-coordinate to $-\boldsymbol{\sqrt{2}/2}$.

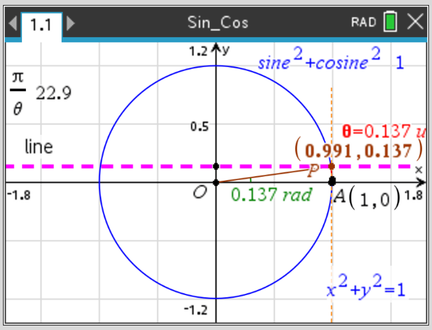
Answer: If $\cos(\theta)=-\dfrac{\sqrt{2}}{2}$, $\theta\in[0,2\pi]$ then $\theta=2.36...=\dfrac{3\pi}{4}$

or $\theta=3.93...=\dfrac{\pi}{0.8}=\dfrac{5\pi}{4}$.



(b) To show **(i)** $\cos^2(\theta)+\sin^2(\theta)=1$:

- Press ctrl menu > **Text**. Enter $\boldsymbol{cosine^2+sine^2}$.
- Press menu > **Actions** > **Calculate**. Click the text. When prompted: '**Select** ***cosine*?**', click the x-coordinate of P.
- When prompted: '**Select** ***sine*?**', click the y-coordinate of P, then press esc to exit the tool.
- Observe the calculated value for different positions of P.



(ii) To show that $\sin(\theta)\approx\theta$, compare the values of arc length, θ, and y-coordinate of P for $0\le\theta<0.2$.

Answer: (i) The sum of the squares of the coordinates of P always equals 1, confirming $\cos^2(\theta)+\sin^2(\theta)=1$

(ii) The arc length, θ, and y-coordinate of P for $0\le\theta<0.2$ are approximately equal to at least two decimal places.

Plotting sine and cosine graphs by capturing values from the unit circle

Question

Use the interactive unit circle model from the previous problem to capture the values θ, $\sin(\theta)$ and $\cos(\theta)$ in a spreadsheet and plot the ordered pairs $(\theta, \sin(\theta))$ and $(\theta, \cos(\theta))$.

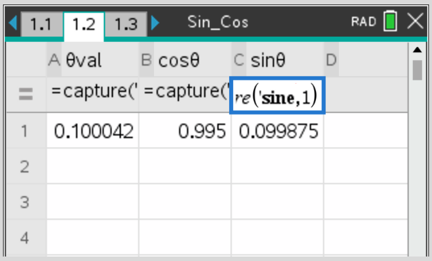
Comment on key features of the graphs, including:

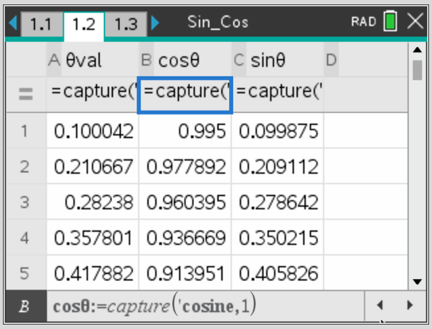
(a) Periodicity **(b)** Complementary relations.

Solution

To capture the values of θ, $\sin(\theta)$ and $\cos(\theta)$:

- Open the document from the previous problem.
- Edit the x-coordinate of P to **0.995**.
- Press [ctrl] [doc▾] [+page]. Select **Add Lists & Spreadsheet**.
- In the heading row (top row), enter the column names, ***θval***, ***cosθ*** and ***sinθ***, as shown.
- Navigate to the column A formula cell, press [menu] > **Data** > **Data Capture** > **Automatic**.
- Press [var] and select θ for the variable name.
- Similarly, in the columns B and C formula cells, capture the variables **cosine** and **sine**, as shown.





To populate the spreadsheet, navigate to **page 1.1**, then:

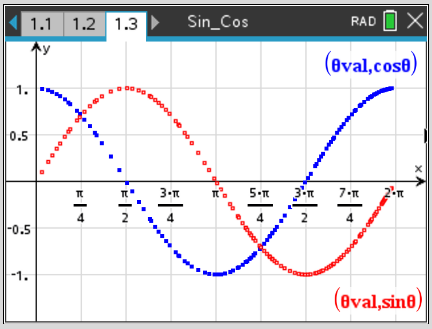
- Grab point P (long press of [click] key) and move point P anticlockwise a full revolution around the circle.

To create detailed scatter plots, add a **Graphs** page, then:

- Press [menu] > **Window/Zoom** > **Window Settings**. In the dialog box that follows enter the following values:

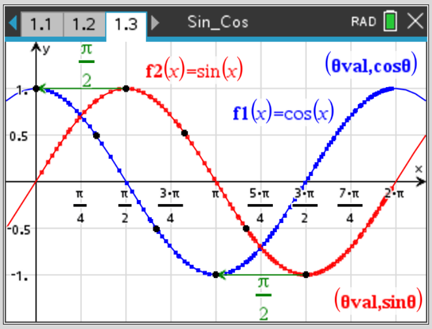
  XMin = – $\pi/6$ XMax = $13\pi/6$ XScale = $\pi/4$
  YMin = – 1.5 YMax = 1.5 YScale = 0.5

- Press [menu] > **Graph Entry/Edit** > **Scatter Plot**, then [var].
- Enter **s1**: $x \leftarrow \boldsymbol{\theta val}$, $y \leftarrow \boldsymbol{\cos\theta}$; **s2**, $x \leftarrow \boldsymbol{\theta val}$, $y \leftarrow \boldsymbol{\sin\theta}$



To graph continuous functions containing all plotted points:

- Press [menu] > **Graph Entry/Edit** > **Function**.
- Enter $\boldsymbol{f1(x) = \cos(x)}$, $\boldsymbol{f2(x) = \sin(x)}$.



Answer: (a) Periodicity. Both sine and cosine plots and graphs are periodic with a period of 2π, corresponding to a revolution of point P around the unit circle.
(b) Complementary relations. Derived from the unit circle:

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right) \text{ and } \cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$$

Understanding the tangent function and plotting $y = \tan(\theta)$ from the unit circle

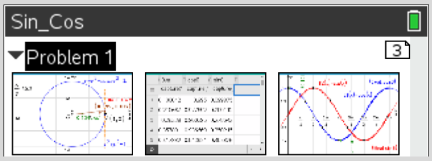
Question

Use the interactive unit circle model from the previous problem to capture the values θ, $\tan(\theta)$ in a spreadsheet and plot the ordered pairs $(\theta, \tan(\theta))$. Comment on key features of the function, including: **(a)** Asymptotes, **(b)** Periodicity, **(c)** Domain and range, **(d)** The ratio $\sin(\theta)/\cos(\theta)$.

Solution

To make an editable copy of the previous problem:

- Press [ctrl] ▲ and navigate to heading **Problem 1**.
- To copy and paste **Problem 1**, press [ctrl] [C] then [ctrl] [V].

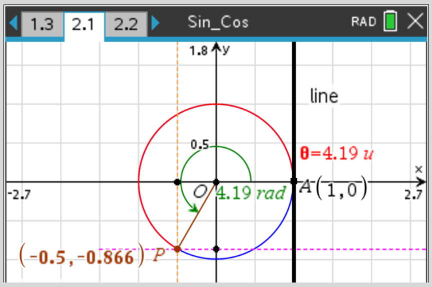


To modify **Problem 2** to include tangent, on page **2.1**:

- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows enter the following values:

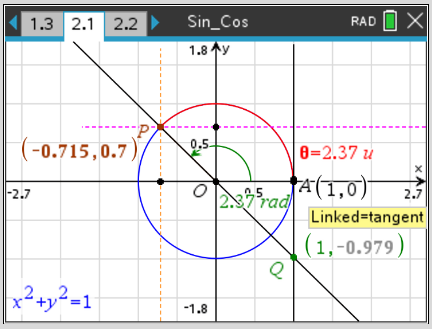
  XMin = – 2.7 XMax = 2.7 XScale = 0.5
  YMin = – 1.8 YMax = 1.8 YScale = 0.5

- Hover over any objects that are not necessary to display, and press [ctrl] [menu] > **Hide**.
- Add a tangent by pressing [menu] > **Geometry** > **Construction** > **Parallel**. Click the *y*-axis then point *A*.



To find the point of intersection of line *OP* and the tangent:

- Press [menu] > **Geometry** > **Points & Lines** > **Line**.
  Click point *O* then point *P*. Click on the ends of the line, and drag the ends to extend the line.
- Press [P] > **Point**. Click **intersection point** of $\overline{OP}$ and tangent. Hover over the point, press [ctrl] [menu] > **Coordinates & Equations**. Continue hovering over point.
- Press [ctrl] [menu] > **Label** and enter the label *Q*.
- Hover over the *y*-coordinate of *Q*, press [ctrl] [menu] > **Store** and enter variable name ***tangent***.
- Edit the *x*-coordinate of point *P* to **0.95**.

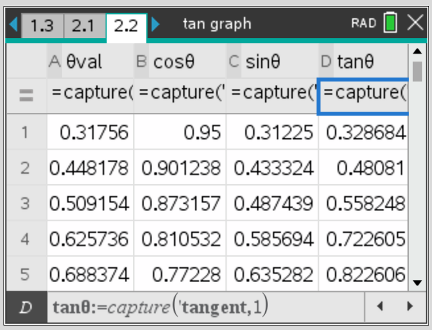


To capture the tangent value, $\tan(\theta)$, and arc length θ:

- On page **2.2**, clear the lists by navigating to a formula cell and pressing [ctrl] [menu] > **Clear Data**.
- Enter the column name ***tanθ*** for column D.
- Navigate to the column D formula cell, press [menu] > **Data** > **Data Capture** > **Automatic**.
- Press [var] and select ***tangent*** for the variable name.

To populate the spreadsheet, navigate to page **2.1**, then:

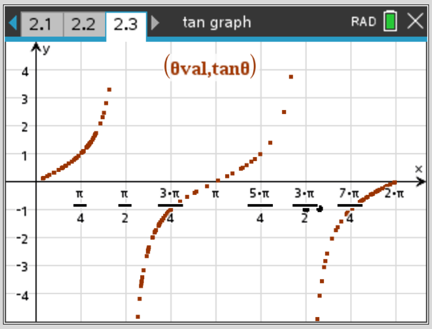
- Grab point *P* (long press of [click] key) and move point *P* anticlockwise a full revolution around the circle.
- To rename the document, press [doc▾] > **File** > **Save As …**


1.3 2.1 2.2 tan graph RAD

	A θval	B cosθ	C sinθ	D tanθ
=	=capture(	=capture('	=capture(	=capture(
1	0.31756	0.95	0.31225	0.328684
2	0.448178	0.901238	0.433324	0.48081
3	0.509154	0.873157	0.487439	0.558248
4	0.625736	0.810532	0.585694	0.722605
5	0.688374	0.77228	0.635282	0.822606

D tanθ:=capture('tangent,1)

Solution (continued)



To create a detailed scatter plot, add a **Graphs** page, then:

- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows enter the following values:

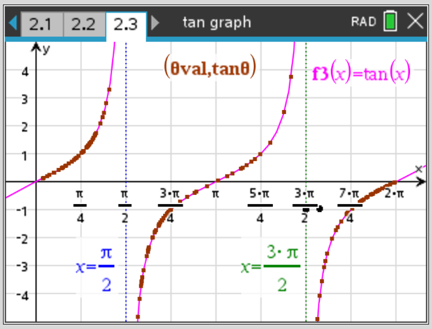
  XMin = $-\pi/6$ XMax = $13\pi/6$ XScale = $\pi/4$
  YMin = – 5 YMax = 5 YScale = 1

- Press [menu] > **Graph Entry/Edit** > **Scatter Plot**, then [var].

  Enter **s3**: $x \leftarrow \boldsymbol{\theta val}$, $y \leftarrow \boldsymbol{tan\theta}$

To graph a function containing all plotted points:

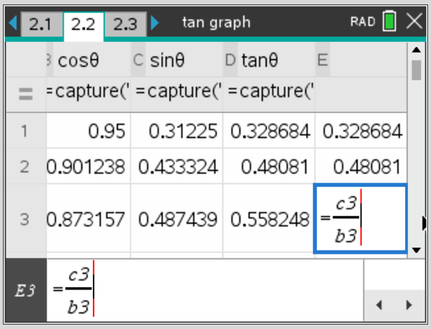
- Press [menu] > **Graph Entry/Edit** > **Function**.
- Enter $f3(x) = \tan(x)$
- Similarly, enter the **Relation** $x = \pi/2$ and $x = 3\pi/2$



Answer: (a) Asymptotes. These occur at $\theta = \pi/2 + n\pi, n \in Z$, corresponding to where the line *OP* on the unit circle is parallel to the tangent at *A*(1, 0).
(b) Periodicity. The graph repeats with a period of π, corresponding to the interval between asymptotes.
(c) Domain is *R* except for odd multiples of $\pi/2$. Range is *R*.

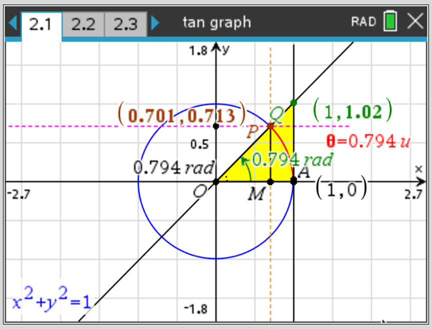
(d) To calculate the ratio $\sin(\theta)/\cos(\theta)$, on **page 2.2**:

- In cell **E1**, enter the formula, ***=c1/b1***.
- Navigate to cell **E1**, press [ctrl] [menu] > Fill.
- Press ▼ key to fill down, then [enter] to lock-in the formulas.
- To clear or reset captured data, navigate to the formula cell and press [ctrl] [menu] > **Clear Data**.



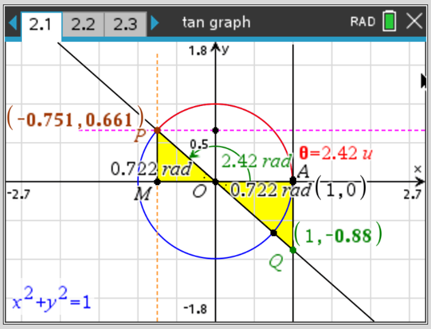
*Note: The cell references **c1** and **b1** are relative to the cell location. When filled down, it renews to: **=c2/b2**, **=c3/b3** etc.*

The results in column E are identical to those in column D, confirming that $\sin(\theta)/\cos(\theta) = \tan(\theta)$.



From the unit circle diagram, the similarity of triangles *POM* and *QOA* is apparent, meeting the AAA condition. That is, three corresponding angles of the two triangles are equal. It follows that:

$$\frac{d(PM)}{d(OM)} = \frac{d(QA)}{d(OA)} \Rightarrow \frac{\sin(\theta)}{\cos(\theta)} = \frac{\tan(\theta)}{1} \text{ or } \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$



These problems can be used to illustrate that although $\sin(\theta)$ and $\sin(\theta^c)$ are numerically equal, $\sin(\theta)$ is the sine of a real number, not the sine of an angle. The number θ can be represented on a number line by the length of an arc on the unit circle. In modelling situations involving circular functions, the variable may be a quantity such as time, rather than an angle.

2.2.3 Graphical and analytical solution of trigonometric equations

Graphing y = Af(ϴ) and solving Af(ϴ)=b, where f is sine or cosine

Question

(a) Graph the functions **(i)** $f(\theta)=2\sqrt{2}\cos(\theta), \theta\in[-\pi,2\pi]$, **(ii)** $g(x)=-3\sin(x), x\in\left[-\pi,\frac{\pi}{2}\right]$.

(b) Use a graphical method to solve the following equation:

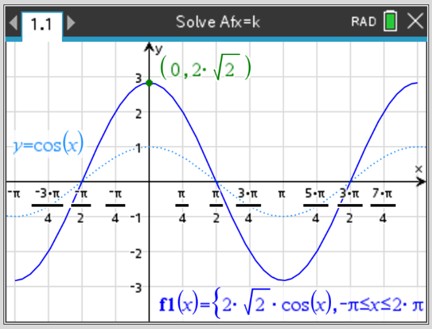
(i) $2\sqrt{2}\cos(\theta)=2,\ \theta\in[-\pi,2\pi]$ **(ii)** $-3\sin(x)=3/2, x\in\left[-\pi,\frac{\pi}{2}\right]$

Solution

(a) (i) To graph $g(x)=2\sqrt{2}\cos(\theta)$, on a **Graphs** page:

- Enter $\boldsymbol{f1(x)=2\sqrt{2}\cos(x)\,|-\pi\le x\le 2\pi}$, pressing ctrl = to select the inequality, ≤, and **given**, |, symbols.
- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows enter the following values:

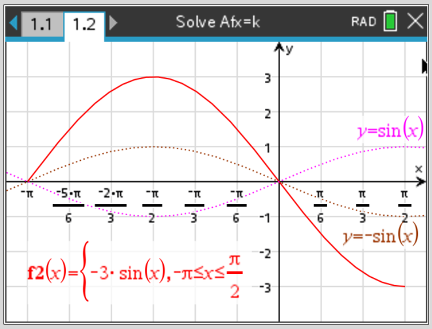
XMin = – 17π/16 XMax = 33π/16 XScale = π/4
YMin = – 4 YMax = 4 YScale = 1



(ii) To graph $g(x)=-3\sin(x)$, add a **Graphs** page, then:

- Enter $\boldsymbol{f2(x)=-3\sin(x)\,|-\pi\le x\le \pi/2}$.
- Press menu > **Window/Zoom** > **Window Setting**.
  In the dialog box that follows enter the following values:

XMin = – 13π/12 XMax = 7π/12 XScale = π/6
YMin = – 4 YMax = 4 YScale = 1

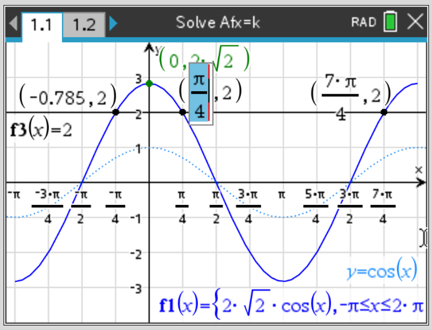


(b) (i) To solve $2\sqrt{2}\cos(\theta)=2,\ \theta\in[-\pi,2\pi]$, on page **1.1**:

- Enter $\boldsymbol{f3(x)=2}$, then press menu > **Geometry** > **Points & Lines** > **Intersection Points**. Click graphs ***f*1** and ***f*3**.

To test the exact x-coordinates of the intersection points:

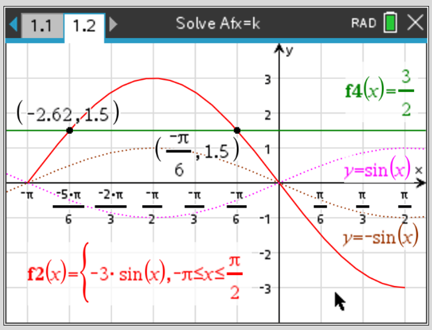
- Edit the x-coordinate of middle point to ***π*/4**, then press enter enter. The change is accepted and confirmed.



(b) (ii) To solve $-3\sin(x)=3/2$, on page **1.2**:

- Enter $\boldsymbol{f4(x)=3/2}$. Find intersection points, as above.
- Test the exact x-coordinates of the intersection points, as above. The grid points suggest **–5*π*/6** and **–*π*/6**.

Answer: (b) (i) $\theta=-\frac{\pi}{4},\frac{\pi}{4},\frac{7\pi}{4}$ **(ii)** $x=-\frac{5\pi}{6},-\frac{\pi}{6}$



Graphing y =A f(nθ)+k and solving Af(nθ)+k = b, f is sine, cosine or tangent

Question

(a) Graph the functions $f(t)=8-8\cos\left(\frac{\pi t}{6}\right),\ t\in[0,18]$ and $g(x)=\frac{1}{2}\tan(2x)+1,\ x\in\left[-\frac{\pi}{2},\ \pi\right]$.

Hence use a graphical method to solve: **(i)** $f(t)=4,\ t\in[0,18]$ **(ii)** $g(x)=\frac{3}{2},\ x\in\left[-\frac{\pi}{2},\ \pi\right]$.

(b) Confirm the results by solving the equations using the 'solve' command.

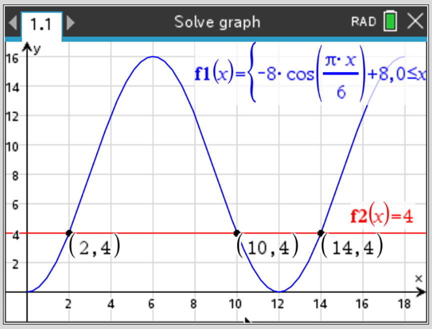
Solution

(a)(i) To graph $y=f(t)$, on a **Graphs** page:

- Enter $\boldsymbol{f1(x)=8-8\cos(\pi x/6)\,|\,0\le x\le 18}$.
- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows enter the following values:

  XMin = – 1 XMax = 19 XScale = 2
  YMin = – 2 YMax = 17 YScale = 2

- Enter $\boldsymbol{f2(x)=4}$, then press menu > **Geometry** > **Points & Lines** > **Intersection Points**. Click graphs ***f*1** and ***f*2**.



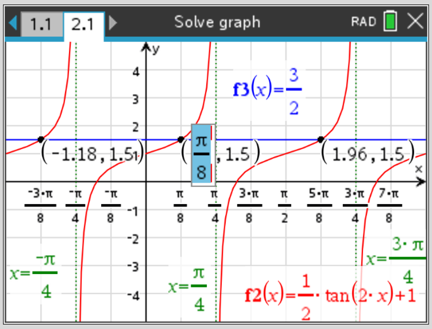
Answer: $f(t)=4,\ t\in[0,18]$ at $t=2,10,14$.

(ii) To graph $y=g(x)$, on a **Graphs** page:

- Enter $\boldsymbol{f3(x)=1/2\tan(2x)+1\,|\,-\pi/2\le x\le\pi}$.
- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows enter the following values:

  XMin = – π/2 XMax = π XScale = π/8
  YMin = – 5 YMax 5 YScale = 1

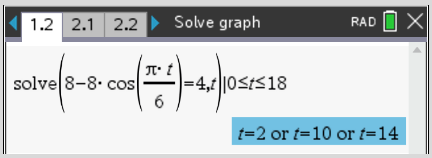
- Enter $\boldsymbol{f4(x)=3/2}$. Press menu > **Geometry** > **Points & Lines** > **Intersection Points**. Click graphs ***f*3** and ***f*4**.



Answer: $g(x)=3/2$ at $x=-3\pi/8,\ \pi/8,\ 5\pi/8$

(b) (i) To solve $f(t)=4,\ t\in[0,18]$, on a **Calculator** page:

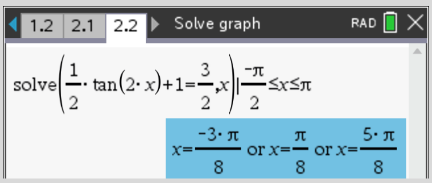
- Press menu > **Algebra** > **Solve**.
- Enter $\textbf{solve}\boldsymbol{(8-8\cos(\pi t/6)=4,t)\,|\,0\le t\le 18}$.



Answer: $f(t)=4,\ t\in[0,18]$ at $t=2,10,14$.

(ii) To solve $g(x)=3/2,\ x\in[-\pi/2,\ \pi]$:

- Enter $\textbf{solve}\boldsymbol{(1/2\tan(2x)+1=3/2,x)\,|\,-\pi/2\le x\le\pi}$.



Answer: $g(x)=3/2$ at $x=-3\pi/8,\ \pi/8,\ 5\pi/8$

2.3 Differentiation

2.3.1 Average and instantaneous rates of change

Comparing average rates of change with instantaneous rate of change

***Note**: Section 2.3.1 on average and instantaneous rates is part of Unit 1 Area of Study 3, but is included here as it links closely with other differential calculus sub-topics.*

Question

The volume of water V remaining in an 8000 L tank t minutes after a tap is opened can be modelled by the function $V(t) = (20-t)^3$, $0 \le t \le 20$.

(a) Find the average rate of change of volume in the **(i)** first 10 minutes; **(ii)** second 10 minutes.

(b) Estimate the rate of change of volume at t = 10 minutes.

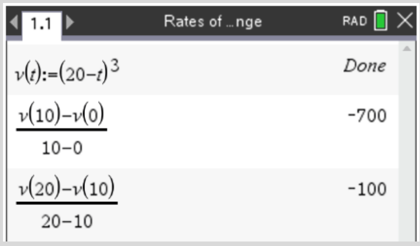
(c) Construct a graph of the above function and visualise the rates of change over the 20 minutes.

Solution

(a) On a **Calculator** page, enter the commands as follows:

- Enter the function $\boldsymbol{v(t):=(20-t)^3}$
- Enter the expression $\dfrac{\boldsymbol{v(10)-v(0)}}{\boldsymbol{10-0}}$.
- Enter the expression $\dfrac{\boldsymbol{v(20)-v(10)}}{\boldsymbol{20-10}}$.

Answer: (i) –700 L/min. **(ii)** –100 L/min.

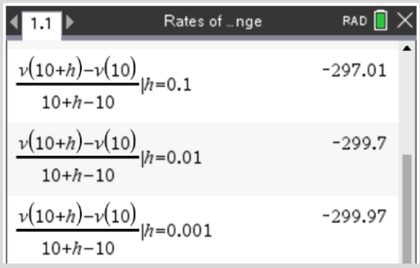


(b) To estimate the rate of change at t = 10 minutes:

- Enter the expression $\dfrac{\boldsymbol{v(10+h)-v(10)}}{\boldsymbol{(10+h)-10}}\,\boldsymbol{|\,h=0.1}$.

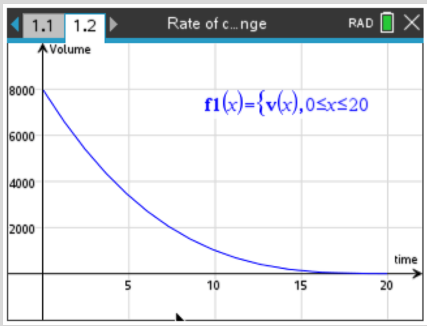
By experimenting with very small values of h, it can be estimated that at t = 10 minutes, the water in the tank is emptying at a rate of 300 L/min.

Answer: Estimated rate at t = 10 minutes is –300 L/min.



(c) To plot a graph of $V(t)$ and a line segment to represent the rates calculated, drawn on the graph of $V(t)$ over the relevant domain, add a **Graphs** page and then:

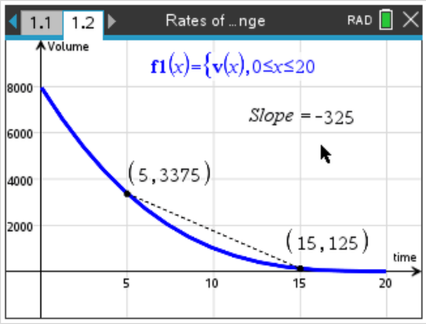
- Enter the rule $\boldsymbol{f1(x)=v(x)|0\le x\le 20}$
- Press menu > **Window/Zoom** > **Window Settings**. Adjust the window settings as shown.
  XMin = −2 Xmax = 22 XScale = 5
  YMin = −2000 YMax = 10000 YScale = 2000



***Note**: Modifying graph attributes such as line thickness and dashes can be done by hovering over a graph, pressing* ctrl menu*, selecting **Attributes** and then modifying the desired attribute. Axes titles can be modified by clicking on them and editing their properties. See section 1.1.2 for more about this.*

... continued

Solution (continued)



To add a line segment on the curve:

- Press [menu] > **Geometry** > **Points & Lines** > **Point on**.
- Click twice on the curve to add a point and its coordinates.
- Repeat last step to add a second point and its coordinates, then press [esc] to exit the **Point on** tool.
- Press [menu] > **Geometry** > **Points & Lines** > **Segment**.
- Click on the two points added to construct a line segment between them, then press [esc] to exit the **Segment** tool.
- Press [menu] > **Geometry** > **Measurement** > **Slope** and then click on the line segment to display the value of the slope. Then press [esc] exit the **Slope** tool.
- Add Label Text via [menu] > **Actions** > **Text** and enter the text "**Slope =**' (position the text near the value of the slope)

Now drag either point an observe how the value of the slope changes.

Note: *There is also a calculator command* ***avgRC*** *which will return the average rate of change. As an example, to find the average rate of change of $y = x^2$ from $x = 5$ to $x = 8$ (i.e. $h = 3$), enter the command* ***avgRC(x²,x=5,3)***.
To find the average rate of change for a set of different h values, enter a command such as ***avgRC(x²,x=5,{3,2,1,0.1,0.01})***. *If no h value is entered, the default value of $h = 0.001$ is used.*

2.3.2 Finding and graphing derivatives

Visualising first principles differentiation

Using the rate of change notation, the derivative of a graph of $y = f(x)$ at a given x value can be approximated by the formula:

$$f'(x) \approx \frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$$

This is sometimes called the *forward difference approximation* to the derivative, since it refers to a point $(x+h, f(x+h))$, which is just 'forward' of the point $(x, f(x))$.

There is also a similar approximation, where:

$$f'(x) \approx \frac{f(x)-f(x-h)}{x-(x-h)} = \frac{f(x)-f(x-h)}{h}$$

This is sometimes called the *backward difference approximation* to the derivative, since it refers to a point $(x-h, f(x-h))$ which is just 'backward' of the point $(x, f(x))$.

Question

Construct a visualisation of the *forward* and *backward difference approximations* to the derivative of $f(x) = x^2$ at $x = 5$ as $h \to 0$.

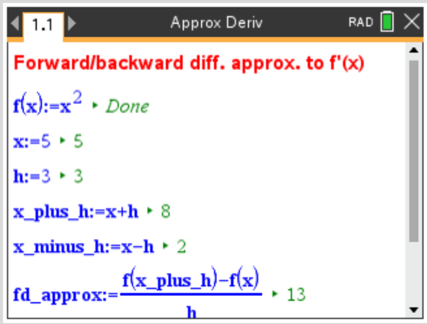
Note: *This construction is best attempted using the TI-Nspire CAS Teacher Software rather than on the handheld device, and used as a visual demonstration to students.*

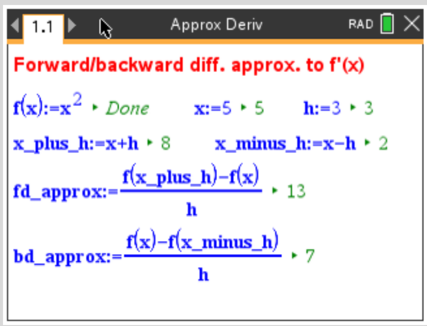
Solution

Before constructing the visualisation in the **Graphs** application, first set up the needed variables on a **Notes** page:

- Enter the template title text '**Forward/backward diff. approx. to f ′(x)**' as shown in the screenshot.
- Press ctrl M to insert a **Maths Box** and enter the $\boldsymbol{f(x):=x^2}$.
- For each of the following commands, in a **Maths Box**:
  - Enter ***x*:=5**.
  - Enter ***h*:=3**.
  - Enter ***x_plus_h*:=*x*+*h***.
  - Enter ***x_minus_h*:=*x*–*h***.
  - Enter $\boldsymbol{fd\_approx:=\dfrac{f(x\_plus\_h)-f(x)}{h}}$
  - Enter $\boldsymbol{bd\_approx:=\dfrac{f(x)-f(x\_minus\_h)}{h}}$

Notes: (1) The underscore character '_' can be entered by pressing ctrl ␣.
*(2) To rearrange any **Maths Boxes** so that they can be viewed on the same screen, position the cursor to the left of each relevant **Maths Box** and press* ␣ *or* del *keys as needed. See screenshot right.*



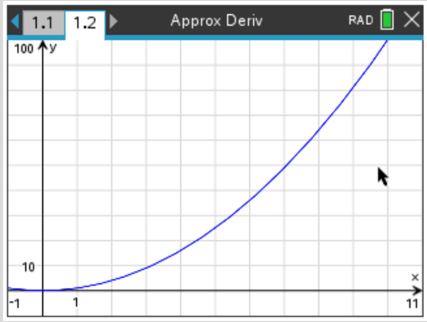


... continued

Solution (continued)

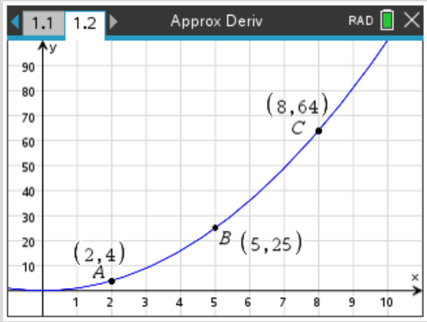
To plot a graph of $y = f(x)$ and line segment to represent the approximate values calculated, add a **Graphs** page and then:

- Enter the rule ***f1*(*x*):=*f*(*x*)**
- Press [menu] > **Window/Zoom** > **Window Settings**.
  Adjust the window settings as shown.
  XMin = −1 Xmax = 11 XScale = 1
  YMin = −10 YMax = 100 YScale = 100



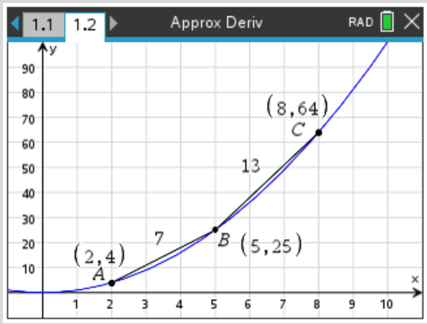
To place and label the coordinates of $A(2,4)$, $B(5,25)$ and $C(8,64)$ on the graph:

- Press [menu] > **Trace** > **Graph Trace**
- Press [2], then press [enter] twice to place a point and coordinates at $x = 2$.
- Press [5], then press [enter] twice to place a point and coordinates at $x = 5$.
- Press [8], then press [enter] twice to place a point and coordinates at $x = 8$.
- Press [esc] to exit **Trace** mode.
- Hover over the point (2,4), press [ctrl][menu] then select **Label** and enter the label ***A***.
- Hover over the point (5,25), press [ctrl][menu] then select **Label** and enter the label ***B***.
- Hover over the point (8,64), press [ctrl][menu] then select **Label** and enter the label ***C***.
- Click and rearrange the coordinates labels as shown right.



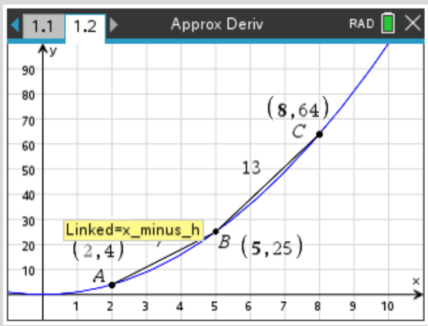
To construct line segments *AB* and *BC*, and find their slopes:

- Press [menu] > **Geometry** > **Points & Lines** > **Segment**.
- Click on point *A* and then on point *B*.
- Click on point *B* and then on point *C*.
- Press [esc] to exit the **Segment** tool.
- Press [menu] > **Geometry** > **Measurement** > **Slope,** then click twice on the line segment *AB* to display its slope.
- Click twice on the line segment *BC* to display its slope.
- Press [esc] to exit the **Slope** tool.



To link the x–coordinates of A, B and C to the values of $x - h$, x, and $x + h$ (as defined on the Notes page):

- Hover over the x–coordinate of A, press [ctrl][menu] then select **Variables** > **Link to:** > ***x_minus_h***.
- Hover over the x–coordinate of B, press [ctrl][menu] then select **Variables** > **Link to:** > ***x***.
- Hover over the x–coordinate of C, press [ctrl][menu] then select **Variables** > **Link to:** > ***x_plus_h***.



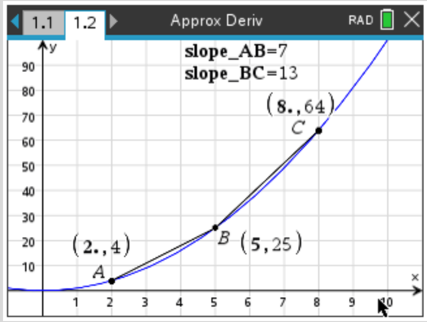
Each of the x–coordinates will now be linked to the variables on the **Notes** page.

... continued

Solution (continued)

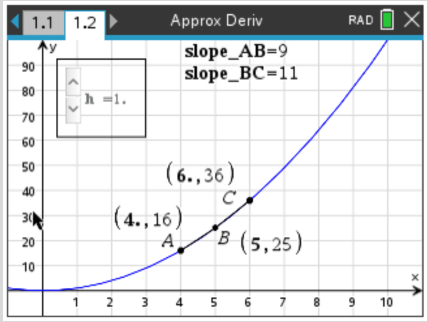
To label the slopes of *AB* and *BC*:

- Hover over the value of the slope of *AB*, press ctrl menu, then select **Store** and type ***slope_AB***.
- Hover over the value of the slope of *BC*, press ctrl menu, then select **Store** and type ***slope_BC***.
- Drag the labels to the top of the page (so they don't obscure the graphs and their labels).

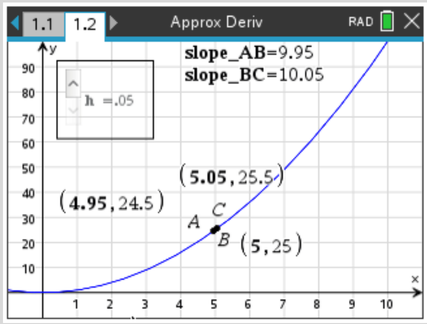


To insert a slider for *h* to vary its value and see how it affects the values of the *forward difference approximation* and *backwards difference approximation* to the derivative at $x = 5$:

- Press menu > **Actions** > **Insert Slider**.
- In the **Slider Settings** dialog box that follows, enter the following values:
  Variable = *h* Value = 3 Minimum = 0.05
  Maximum = 3 Step Size = 0.05 Style = Vertical
- Scroll down and check the **Minimised** box.
- Click **OK** to save these slider settings and return to the graph page.
- Position the slider on the top left of the page.

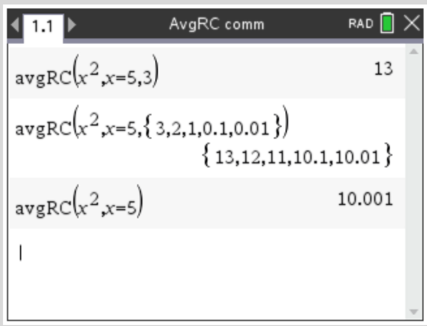


Click on the slider to view how the values of the *forward difference approximation* (slope of *BC*) and *backwards difference approximation* (slope of *AB*) to the derivative at $x = 5$ approach the same value of 10 as *h* approaches 0.



Save the file as "Approx Deriv.tns". The file will be added to in the next example.

***Note**: There is also a calculator command **avgRC** which will return the average rate of change using the forward difference approximation. As an example, to find the average rate of change of $y = x^2$ from $x = 5$ to $x = 8$ (i.e. $h = 3$), enter the command **avgRC(x²,x=5,3)**. To find the average rate of change for a set of different h values, enter a command such as **avgRC(x²,x=5,{3,2,1,0.1,0.01})**. If no h value is entered, the default value of $h = 0.001$ is used. See screenshot at right for these examples.*



Visualising the central difference approximation

The *central difference approximation* is a preferred method for approximating the gradient of a curve at a point, as it can be shown to reduce error more so than the *backward* or *forward difference approximation* methods. Rather than using the point x directly, it calculates the gradient of the line segment between the two points either side of x, that is $\left(x-h, f(x-h)\right)$ and $\left(x+h, f(x+h)\right)$. The *central difference approximation* is calculated as

$$f'(x) \approx \frac{f(x+h)-f(x-h)}{(x+h)-(x-h)} = \frac{f(x+h)-f(x-h)}{2h}$$

Question

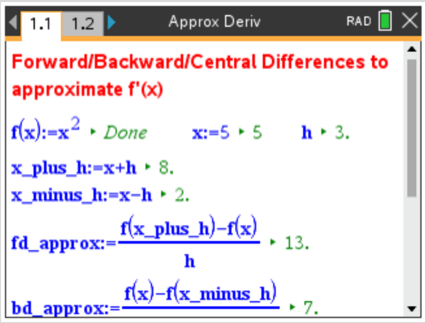
Open the "**Approx Deriv.tns**" file from the previous example for the *backward* or *forward difference approximation* methods, and add a visualisation of the *central difference approximation* to the derivative of $f(x) = x^2$ at $x = 5$ as $h \to 0$. Show that the *central difference method gives a* more accurate approximation, and then show algebraically that the *central difference approximation* is the mean of the *backward* and *forward difference approximation* methods.

***Note:** This construction is best attempted using the TI-Nspire CAS Teacher Software rather than on the handheld device and used as a visual demonstration to students.*

Solution

Locate and open the file named "**Approx Deriv.tns**", and then:

- Change the template title text to '**Forward/Backward/Central Differences to approximate f ′(x)**' as shown in the screenshot.
- Set the values for ***f*(*x*)**, ***x*** and ***h*** as used previously and shown on the screen right.



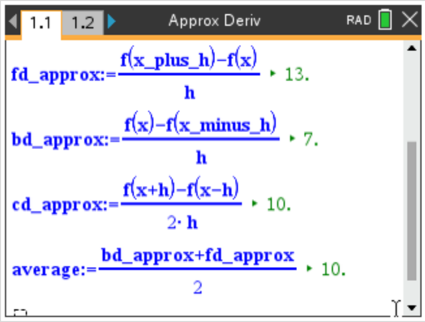
In the line below the bd_approx definition, add the following in Maths boxes.

- For the *central difference approximation*, enter

$$\boldsymbol{cd\_approx := \frac{f(x\_plus\_h) - f(x\_minus\_h)}{2h}}$$

- For the average of the *backwards* and *forwards difference approximations*, enter

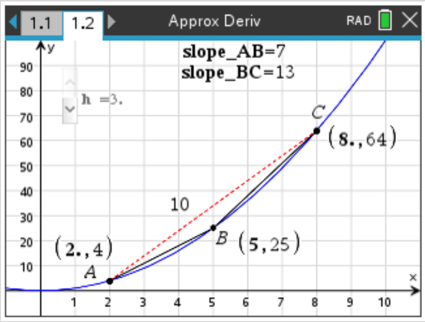
$$\boldsymbol{average := \frac{bd\_approx + fd\_approx}{2}}$$



To construct the line segment associated with the central difference approximation, move to the **Graphs** page and then:

To construct line segment *AC,* and find its slope:

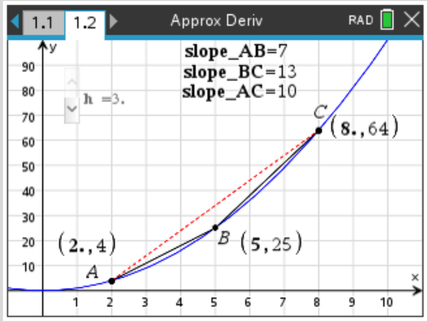
- Press menu > **Geometry** > **Points & Lines** > **Segment**.
- Click on point *A* and then on point *C*.
- Press esc to exit the **Segment** tool.
- Press menu > **Geometry** > **Measurement** > **Slope,** then click twice on the line segment *AC* to display its slope.
- Press esc to exit the **Slope** tool.



Solution (continued)

To label the slope of AC:

- Hover over the value of the slope of AC, press ctrl menu, then select **Store** and type ***slope_AC***.
- Drag the label to the top of the page (so they don't obscure the graph or line segments). See example screen right.
- To change the line segment AC to a dashed line, hover over it, then press ctrl menu and select Attributes. Press ▼ to select the second row of attributes, which is Line Style.
- Select **Line Style is Dashed** and press enter to save this style.

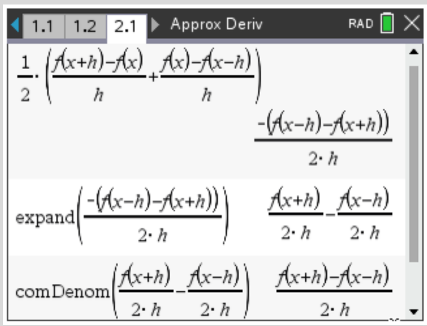


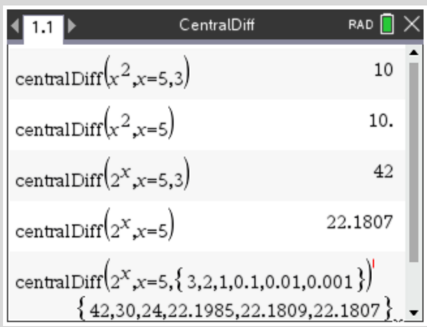
Click on the slider to change the value of h, and view how the *central difference approximation* (slope of AC) is the mean of the *forward* and *back differences approximations*.

This can be shown algebraically, starting with the mean of the *forward* and *back differences approximations.*

$$
\begin{aligned}
f'(x) &\approx \frac{1}{2}\left(\frac{f(x+h)-f(x)}{h}+\frac{f(x)-f(x-h)}{h}\right)\\
&=\frac{1}{2}\left(\frac{f(x+h)-f(x)+f(x)-f(x-h)}{h}\right)\\
&=\frac{1}{2}\left(\frac{f(x+h)-f(x-h)}{h}\right)\\
&=\frac{f(x+h)-f(x-h)}{2h}
\end{aligned}
$$

The CAS can be used to show this result on the **Calculator** application as shown right, using the **Expand** and **Common Denominator** commands to change the form of the result. These commands can be found via menu





*Note: There is also a calculator command **centralDiff()** which will return the numerical derivative using the central difference approximation. As an example, to find the approximate value of the derivative of $y = x^2$ at $x = 5$ using $h = 3$, enter the command **centralDiff(x²,x=5,3)**. If no h value is entered, the default value of $h = 0.001$ is used. See the screenshot for some examples for $y = x^2$ and for $y = 2^x$.*

Calculating the derivative using first principles

Although not required in the current course, calculating the derivative using limits provides students with a glimpse under the hood of the process of differentiation.

Question

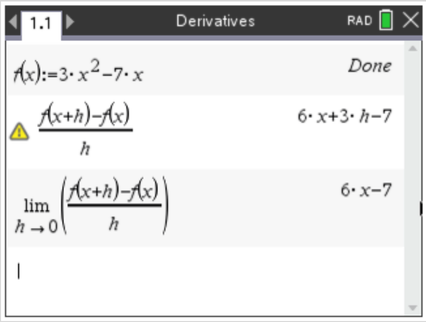
Use the first principles approach to find the derivative of the following functions.

(a) $f(x) = 3x^2 - 7x$ **(b)** $f(x) = \frac{1}{x}$

Solution

(a) To find the derivative of $f(x) = 3x^2 - 7x$ by first principles, make a **New Document**, then on a **Calculator** page:

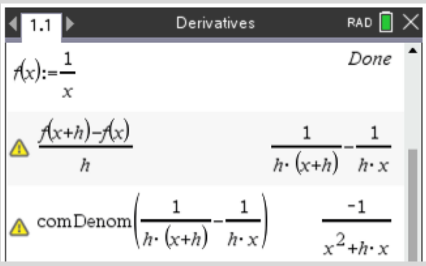
- Enter the function rule $\boldsymbol{f(x) := 3x^2 - 7x}$.
- Enter the command $\boldsymbol{\frac{f(x+h) - f(x)}{h}}$.
- Press [menu] > **Calculus** > **Limit** and complete the command $\boldsymbol{\lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)}$.

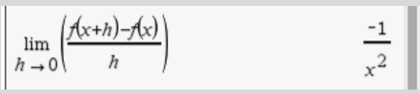


Answer: If $f(x) = 3x^2 - 7x$, $f'(x) = 6x - 7$.

(b) To find the derivative of $f(x) = \frac{1}{x}$ by first principles, on a **Calculator** page:

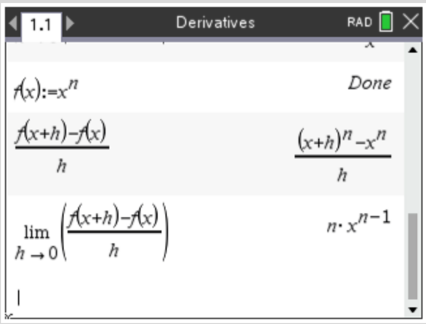
- Enter the function rule $\boldsymbol{f(x) := \frac{1}{x}}$.
- Enter the command $\boldsymbol{\frac{f(x+h) - f(x)}{h}}$.
- Press [menu] > **Algebra** > **Fraction Tools** > **Common Denominator** and enter the previous answer as shown.
- Press [menu] > **Calculus** > **Limit** and complete the command $\boldsymbol{\lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)}$.





Answer: If $f(x) = \frac{1}{x}$, $f'(x) = \frac{-1}{x^2}$.

Note: A similar method can be used to verify the power rule for differentiation (see screen right). The algebra required to show the calculation of the limit by hand is beyond the scope of the Mathematical Methods course.



Graphing the derivative function

The derivative function can be graphed alongside the graph of the original function to illustrate the link between two graphs.

Question

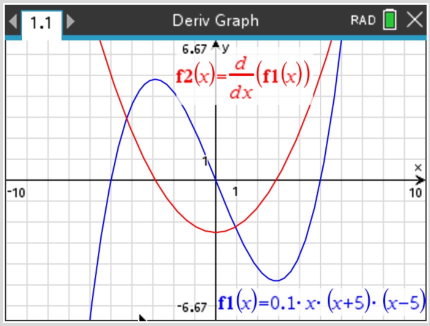
Graph the function and its derivative function with the following rules:

(a) $f(x) = 0.1x(x+5)(x-5)$ **(b)** $f(x) = \begin{cases} -x(x+4) & x \le -2 \\ 2-x & -2 < x \le 4 \\ 3 & x > 4 \end{cases}$

Solution

(a) To graph $f(x) = 0.1x(x+5)(x-5)$ and its derivative graph, on a **Graphs** page:

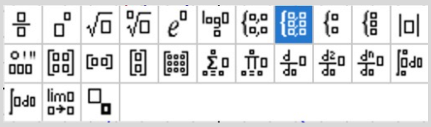
- Enter the function rule $\boldsymbol{f1(x) = 0.1x \cdot (x+5) \cdot (x-5)}$.
- Press ctrl G to enter the derivative rule in ***f*2(*x*)**, then press ⇧shift − to paste in the derivative template, then complete the rule as $\boldsymbol{f2(x) = \frac{d}{dx}(f1(x))}$.

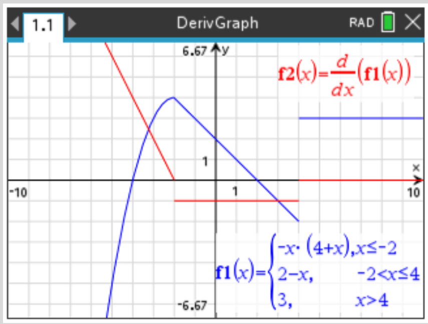


(b) To graph the piecewise function and its derivative graph, on the same **Graphs** page:

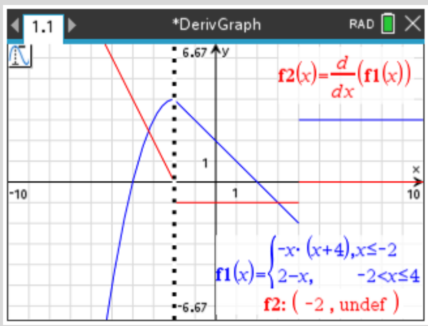
- Press ctrl G and press ▲ until the rule for ***f*1(*x*)** is displayed.
- Delete the existing rule and enter the hybrid rule as follows:
  - Press the template key and select the **Piecewise function** template as shown right.
  - In the dialog box, select **Number of pieces = 3**
  - Enter the piecewise rule

$$\boldsymbol{f1(x) = \begin{cases} -x \cdot (x+4) & x \le -2 \\ 2-x & -2 < x \le 4 \\ 3 & x > 4 \end{cases}}$$





Assuming that ***f*2(*x*)** is still defined as the derivative of ***f*1(*x*)**, both graphs will now be displayed.



***Note**: The **Trace** function can be used to highlight key points on the derivative graph where the value of the derivative is undefined, for example at x = –2 (see screen right).*

2.4 Applications of differentiation

2.4.1 Tangent lines and instantaneous rates of change

Exploring where a function is increasing/decreasing using a moveable tangent

Question

Consider the function $f(x)=\frac{1}{10}(x-4)^2(x+2), x \in R$.

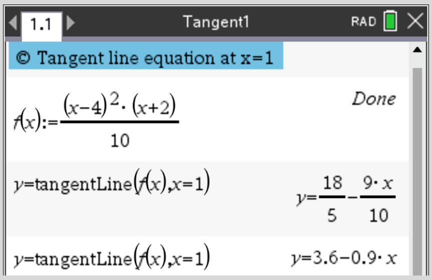
(a) Determine the equation of the tangent line to the graph of *f* at $x=1$.

(b) Use the gradient of a moveable tangent on the graph of *f* to explain where the function is:
(i) increasing, **(ii)** decreasing or **(iii)** stationary.

Solution

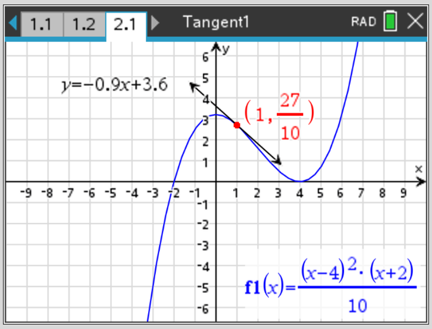
(a) To determine the equation of the tangent line using a non-graphical approach, on a **Calculator** page:

- Enter $\boldsymbol{f(x):=(x-4)^2\cdot(x+2)/10}$.
- Key in $\boldsymbol{y=}$, then press menu > **Calculus** > **Tangent Line**. Enter ***y* = tangentLine(*f* (*x*), *x*=1)**.
- Press ctrl enter to obtain an answer with decimal values.



To draw a tangent on the graph of *f*, on a **Graphs** page:

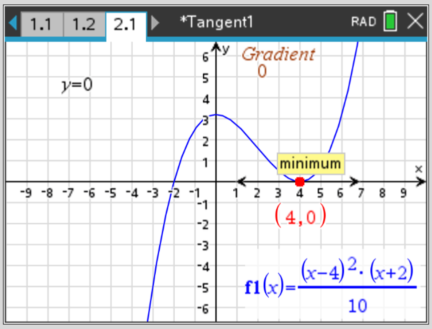
- Enter $\boldsymbol{f1(x)=(x-4)^2\cdot(x+2)/10}$.
- Press menu > **Geometry** > **Points & Lines** > **Tangent**.
- Hover the cursor over the curve, press enter, then press enter again to locate the tangent and its equation.
- Press esc to exit the tangent tool.
- Hover over the contact point, press ctrl menu > **Coordinates & Equations** to display its coordinates.
- Edit the *x*-coordinate to **1** and press enter.



Answer: The equation of the tangent line is $y=-0.9x+3.6$.

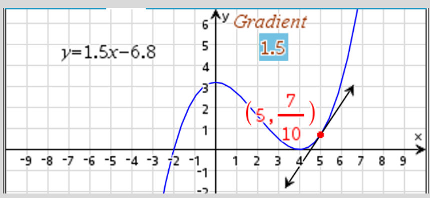
(b) To measure the gradient of the tangent:

- Press menu > **Geometry** > **Measurement** > **Slope**.
- Click the tangent line. Press enter then esc.



To observe variations in the value of the gradient:

- Drag the point along the curve, taking notice when points of interest (i.e. **minimum** or ***y*-intercept**) are displayed.
- Edit the *x*-coordinate of the point to 'jump' to a particular location on the graph. The *y*-coordinate may be displayed as an exact fraction.



Answer: *f* is: **(i)** increasing where the gradient, $m > 0$. That is, $x \in (-\infty, 0) \cup (4, \infty)$; **(ii)** decreasing where $m < 0$, i.e. $x \in (0, 4)$; **(iii)** stationary where $m = 0$, i.e. $x = 0$ or $x = 4$.

Making connections between gradient of the tangent and the graph of y = f '(x)

Question

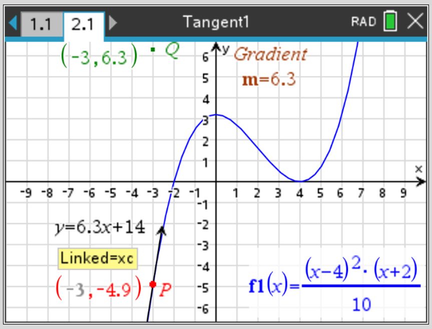
(a) Using the moveable tangent on the graph of $f(x)=\frac{1}{10}(x-4)^2(x+2), x\in R$ from the previous problem, plot the gradient of the tangent against the *x*-coordinate of the contact point.

(b) Show that the graph of $y=f'(x)$ contains all the plot points from part **(a)** above. Hence state the connection between $y=f'(x)$ and where *f* is increasing, decreasing or stationary.

Solution

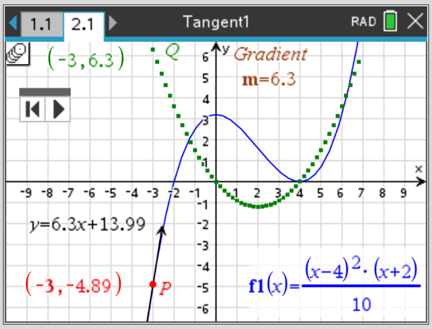
(a) To set up an interactive plot of the gradient and *x*-intercept, on the **Graphs** page from the previous problem:

- Edit the *x*-coordinate of the contact point to **–3**.
- Hover over this point, press ctrl menu > **Label**, then enter the label ***P***.
- Hover over the *x*-coordinate of *P*, press ctrl menu > **Store** and enter the variable name ***xc***. Hover over the value of the gradient, press ctrl menu > **Store** and enter variable ***m***.



To add a point with coordinates (***xc***, ***m***) and observe its locus:

- Press P > **Point by Coordinates**, then enter (***xc***, ***m***).
- Hover over this point, press ctrl menu > **Label**, then enter the label ***Q***.
- Hover over point *P*, press ctrl menu > **Attributes**. Press ▼ then enter animation speed of **1** by pressing 1 enter enter.
- Use the control buttons to start, pause or reset animation.

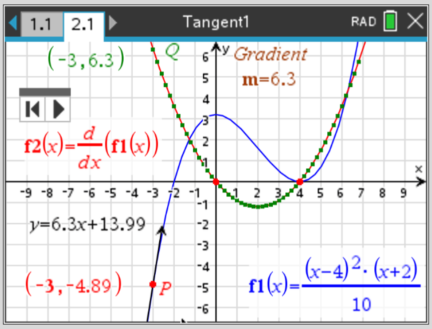


To obtain a trace of point *Q* as *P* moves along $y=f1(x)$:

- Hover over point *Q*, press ctrl menu > **Geometry Trace**.
- Start the animation and observe the plotted path of *Q*.
- Press esc to exit **Geometry Trace** tool.

(b) To show that graph of $y=f'(x)$ contains the plot points:

- Enter $\boldsymbol{f2(x)=\frac{d}{dx}(f1(x))}$, pressing ⇧shift − to insert the derivative template.



Observe the relationship between the stationary points $y=f(x)$, the plot points and the *x*-intercepts of $y=f'(x)$.

Answer: At $x=0$ and $x=4$: *f* is stationary. The gradient of the tangent line is $m=0$ and $f'(x)=0$.

For $x\in(-\infty,0)\cup(4,\infty)$: *f* is increasing. The gradient of the tangent is $m>0$ and $f'(x)>0$ ($y=f'(x)$ above the *x*-axis).

For $x\in(0,4)$, *f* is decreasing, the gradient of the tangent line, $m<0$, and $f'(x)<0$ (graph of *f* ′ is below the *x*-axis).

Note**: To erase the trace, press* ctrl menu > ***Erase Geometry Trace.

Calculating and interpreting rate of change at a point of inflection

Question

In a manufacturing process, the temperature inside a chamber is regulated such that t minutes into the process the temperature T°C is modelled by $T(t)=80-4t+\frac{t^2}{2}-\frac{t^3}{32}, t\in[0,16]$.

(a) Calculate the average rate of change of temperature for **(i)** $t\in\left[0,\frac{16}{3}\right]$, **(ii)** $t\in\left[\frac{16}{3},16\right]$.

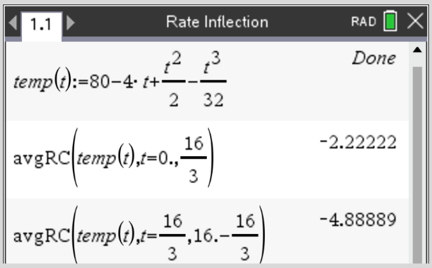
(b) Calculate the instantaneous rate of change of temperature at $t=4$, $t=\frac{16}{3}$ and $t=8$.

(c) Graph the functions T and T' on the same set of axes. Explore the gradient of a moveable tangent to the graph of T in the proximity of $t=\frac{16}{3}$, and interpret the significance of this point.

Solution

(a) To calculate average rate of change, on a **Calculator** page:

- Enter $\boldsymbol{temp(t):=80-4t+t^2/2-t^3/32}$.
- Press [book] [1] [B]. Select **avgRC**(Expr, Var[=Value] [,Step]).
- Enter the inputs as shown for **(i)** $t\in\left[0,\frac{16}{3}\right]$, **(ii)** $t\in\left[\frac{16}{3},16\right]$.



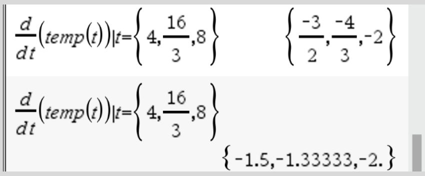
Answer: **(i)** Av rate $=-\frac{20}{9}\approx-2.22$°C / min, $t\in\left[0,\frac{16}{3}\right]$

(ii) Average rate $=-\frac{44}{9}\approx-4.89$°C / min, $t\in\left[\frac{16}{3},16\right]$

*Note: for **(a)(ii)**, the **Step** from t = 16/3 to t = 16 is Step = 16 – 16/3.*

(b) To calculate instantaneous rates of change at $t=4,\frac{16}{3},8$:

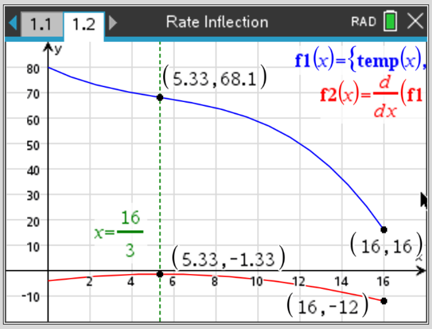
- Press [shift] [−] for the derivative template.
- Enter $\frac{d}{dt}(temp(t))\,|\,t=\{4,16/3,8\}$.



Answer: Rates at $t=4,\frac{16}{3},8$ are $-1.5,-\frac{4}{3}\approx-1.33,-2$°C / min

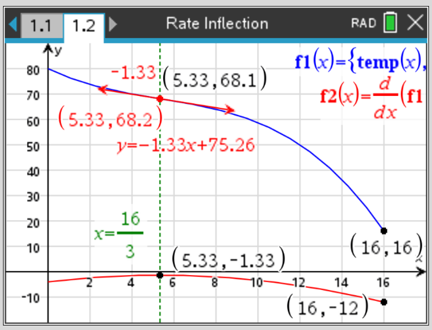
(c) To graph functions T and T', add a **Graphs** page, then:

- Enter $\boldsymbol{f1(x)=temp(x)\,|\,0\le x\le 16}$.
- Enter $\boldsymbol{f2(x)=\frac{d}{dx}(f1(x))}$
- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values.
  XMin = –2 XMax = 18 XScale = 2
  YMin = –20 YMax = 90 YScale = 10



To find the point of inflection and add a moveable tangent:

- Press [menu] > **Analyse Graph** > **Inflection**.
- Click graph $\boldsymbol{f1}$ then click left and right bounds.
- Press [menu] > **Analyse Graph** > **Maximum**.
- Click graph $\boldsymbol{f2}$ then click left and right bounds.
- Press [menu] > **Geometry** > **Points & Lines** > **Tangent**.
- Click graph $\boldsymbol{f1}$, press [enter] then [esc].



Answer: At $t=\frac{16}{3}$ the cooling rate 'flips', that is, at that time, the rate at which the temperature is dropping changes from slowing down, to speeding up.

2.4.2 Stationary values of functions

Locating stationary points of a polynomial function using various approaches

Question

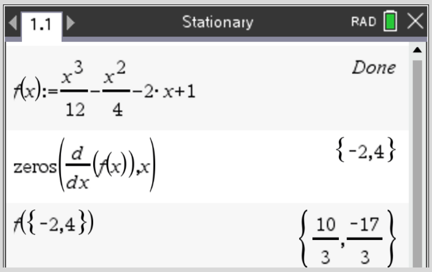
(a) Determine the coordinates of the stationary points of $f(x)=\frac{x^3}{12}-\frac{x^2}{4}-2x+1, x\in R$ using a variety of approaches from the Algebra menu: **(i)** Zeros, **(ii)** Solve, **(iii)** Polynomial Tools.

(b) Confirm the stationary values and their nature from the graph of $y=f(x)$.

Solution

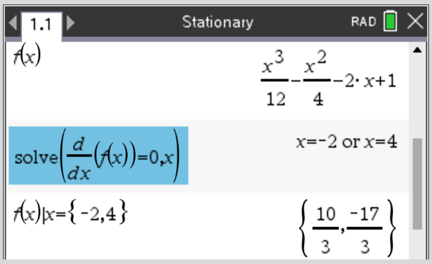
(a) (i) To find the points using **Zeros**, on a **Calculator** page:

- Enter $\boldsymbol{f(x):=x^3/12-x^2/4-2x+1}$.
- Press [menu] > **Algebra** > **Zeros**, then press [⇧shift] [−].
- Enter $\mathbf{zeros}\left(\frac{d}{dx}(f(x)),x\right)$ for the *x*-coordinates.
- Enter $\boldsymbol{f1(\mathbf{ans})}$ (use the [ans] key) for the *y*-coordinates.



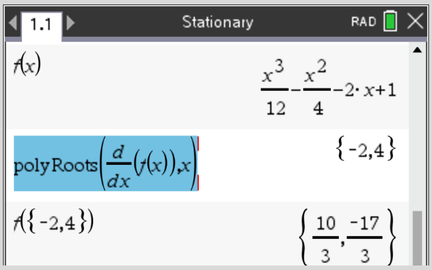
(ii) To use **Solve** to find stationary points:

- Press [menu] > **Algebra** > **Solve**, then press [⇧shift] [−].
- Enter $\mathbf{solve}\left(\frac{d}{dx}(f(x))=0,x\right)$ for the *x*-coordinates.
- Enter $f(x)\,|\,x=\{-2,4\}$ or $f(\{-2,4\})$ for *y*-coordinates.



(iii) To use **Polynomial Tools** to find stationary points:

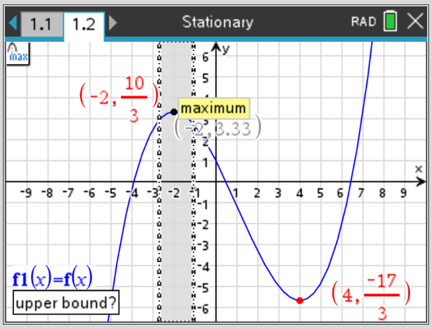
- Press [menu] > **Algebra** > **Polynomial Tools** > **Real Roots**...
- Press [⇧shift] [−] then enter $\mathbf{polyRoots}\left(\frac{d}{dx}(f(x)),x\right)$.
- Enter $\boldsymbol{f1(\mathbf{ans})}$ (use the [ans] keys) for the *y*-coordinates.



Answer: Each of the above methods confirm the coordinates of the stationary points at $\left(-2,\frac{10}{3}\right)$ or $\left(4,-\frac{17}{3}\right)$.

(b) To graph *f* and find stationary points, on a **Graphs** page:

- Enter $\boldsymbol{f1(x)=f(x)}$.
- Press [menu] > **Analyse Graph** > **Minimum**. Click at the left of the local minimum then click at right of the point.
- Press [menu] > **Analyse Graph** > **Maximum**. Click at the left of the local maximum then click at right of the point.



Answer: Local maximum: $\left(-2,\frac{10}{3}\right)$, local minimum $\left(4,-\frac{17}{3}\right)$.

***Note**: To test an exact value for a y-coordinate (for example $y=-10/3$ at the local minimum), edit the y-value to $-\frac{17}{3}$ and press* [enter] *to make the change, and then press* [enter] *again to confirm the change.*

Analysing turning and inflection points of a quartic polynomial function

Question

(a) Graph the function $f(x) = -\frac{x^4}{4} - x^3 + 4x + 2, x \in R$ and determine the coordinates of all turning points and points of inflection. Interpret the nature of any inflection points.

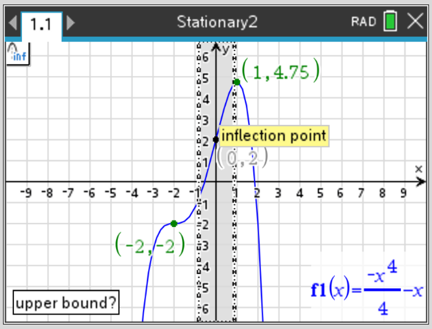
(b) Confirm the coordinates of the stationary points using a non-graphical approach.

(c) Graph $y = f'(x)$. Interpret the relationship between key features of the graphs of *f* and *f*′.

Solution

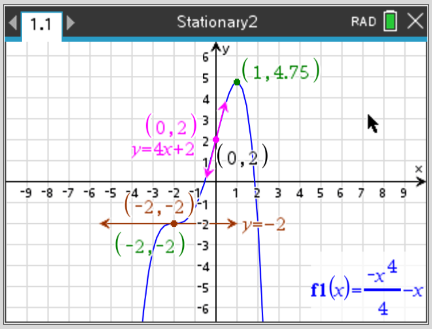
(a) To graph *f* and locate stationary points, on a **Graphs** page:

- Enter $\boldsymbol{f1(x) = -x^4/4 - x^3 + 4x + 2}$.
- Press [menu] > **Analyse Graph** > **Maximum**. Click left and right of the local maximum.
- Press [menu] > **Analyse Graph** > **Inflection**. Click near $x = -3$ then near $x = -1$. Repeat with bounds at $x = -1$ and $x = 1$.



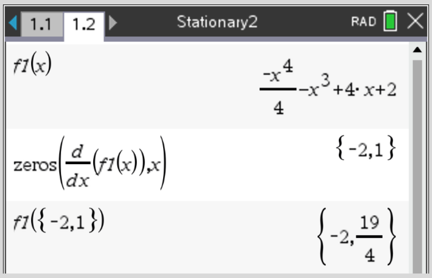
To add a tangent line and test whether a point is stationary:

- Press [menu] > **Geometry** > **Points & Lines** > **Tangent**.
- Click the graph, press [enter] then [esc]. Hover over contact point. Press [ctrl] [menu] > **Coordinates & Equations**.
- Drag this point and note where the tangent is horizontal. Jump to points of interest by editing the *x*-coordinate.



(b) To confirm the coordinates of the stationary points, on a **Calculator** page:

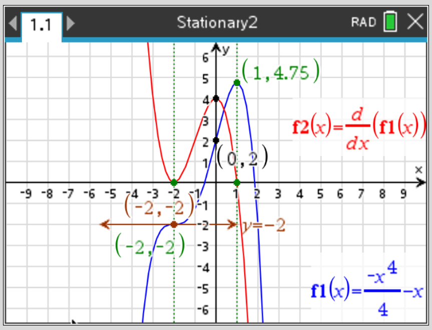
- Press [menu] > **Algebra** > **Zeros**, then press [shift] [–].
- Enter $\mathbf{zeros}\left(\frac{d}{dx}(f1(x)), x\right)$ for the *x*-coordinates.
- Enter $\boldsymbol{f1(\mathbf{ans})}$ (use the [ans] key) for the *y*-coordinates.



Answer: Stationary points: maximum at $\left(1, \frac{19}{4}\right) = (1, 4.75)$; stationary point of inflection at $(-2, -2)$. Non-stationary point of inflection at $(0, 2)$.

(c) To graph $y = f'(x)$ on the previous **Graphs** page:

- Enter $\boldsymbol{f2(x) = \frac{d}{dx}(f1(x))}$.



Answer: Relationship between key features of the graphs of *f* and *f*′. For turning point at $(1, 4.75)$ and inflection point at $(-2, -2)$ on graph of *f*, $f'(x) = 0$. The stationary values with *x*-intercepts on graph of *f*′. The non-stationary inflection point of *f* at $(0, 2)$ is associated with a turning point on $y = f'(x)$.

2.4.3 Maximum and minimum optimisation problems

Demonstrating an optimisation problem through a graphical approach

Question

A garden design requires the inclusion of a vegetable patch with an area of $16\ \text{m}^2$.

(a) Use Geometry tools to draw a dynamic rectangle representing the possible dimensions.

(b) For rectangles of varying widths, capture the perimeter and width of the rectangles. Hence obtain a plot of perimeter against width. Assume that the width is at least 1 metre.

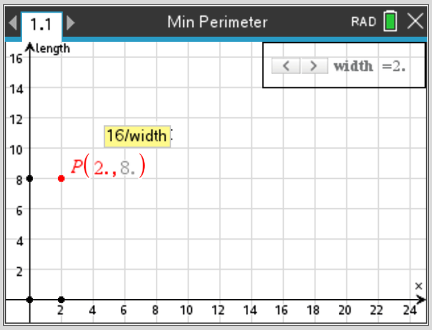
(c) Graph a continuous function containing all plot values and use a graphical method to find the width of the rectangle with minimum perimeter.

Solution

Area = 16. If width = x, then length = $\frac{16}{x}$, perimeter = $2x+\frac{32}{x}$

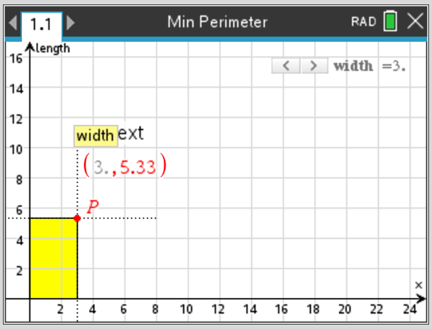
(a) To set up for a dynamic rectangle, on a **Graphs** page:

- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values.
  XMin = –1.5 XMax = 25 XScale = 2
  YMin = –1.5 YMax = 17 YScale = 2
- Press menu > **Actions** > **Insert Slider**.
  Enter slider settings as follows:
  Variable = *width* Value = 1 Minimum = 1,
  Maximum = 16 Step Size = 1 Click Minimised
- Press P > **Point by Coordinates**. Enter x, y coordinates $(\boldsymbol{width}, \mathbf{16}/\boldsymbol{width})$, pressing var to select ***width***.
- Hover over the point, press ctrl menu > **Label**. Enter ***P***.



To draw the dynamic rectangle:

- Press menu > **Geometry** > **Construction** > **Perpendicular**. Click x-axis then point ***P***, then click y-axis followed by P. Press esc to exit the tool.
- Press menu > **Geometry** > **Shapes** > **Polygon**. Click point P, then, in turn, the intersection point of the perpendicular line and the x-axis, the origin, the intersection point of the perpendicular line and the y-axis, then P. Press esc.
- Hover over the rectangle, press ctrl menu > **Colour** and choose a **Fill** colour. Press esc.
- Hover over each perpendicular line, press ctrl menu > **Hide**.

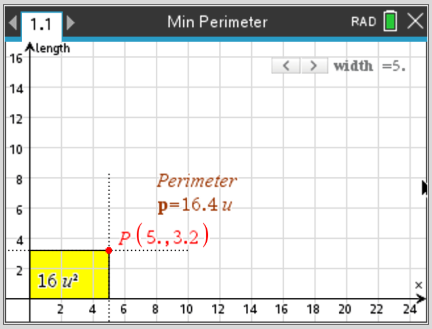


To measure the area and perimeter as the width varies:

- Hover over the rectangle and press ctrl menu > **Measurement**. Select **Area** then repeat to select **Length**.

To store the value of the perimeter as a variable, ***p***:

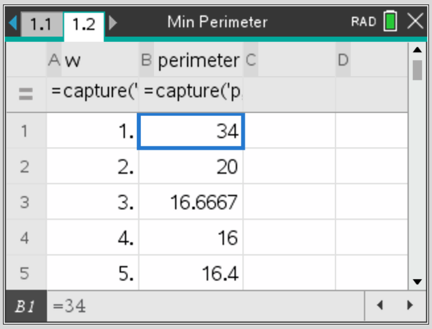
- Hover over the perimeter value and press ctrl menu > **Store**. Enter the variable name, ***p***.



Solution (continued)

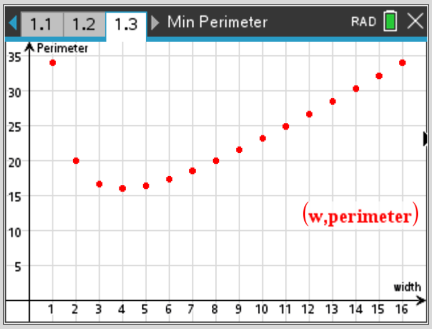
(b) To capture the measurements, set width to **1**, add a **Lists & Spreadsheet** page, then:

- Enter the headings ***w*** and ***perimeter***, as shown.
- Navigate to the column A formula cell, press [ctrl] [menu] > **Data Capture > Automatic**. Press [var] and select ***width*** and then press [enter].
- Navigate to the column B formula cell, press [ctrl] [menu] > **Data Capture > Automatic**. Press [var] and select ***p***.
- Adjust the rectangle's width by changing the slider value.



To plot width against perimeter, on the **Graphs** page:

- Press [menu] > **Graph Entry/Edit > Scatter Plot**.
- For **s1**, press [var] to enter $x \leftarrow \boldsymbol{w}$ and $y \leftarrow \boldsymbol{perimeter}$.
- Press [menu] > **Window/Zoom > Window Settings**. In the dialog box that follows, enter the following values.
  XMin = –1 XMax = 17 XScale = 1
  YMin = –3 YMax = 37 YScale = 5

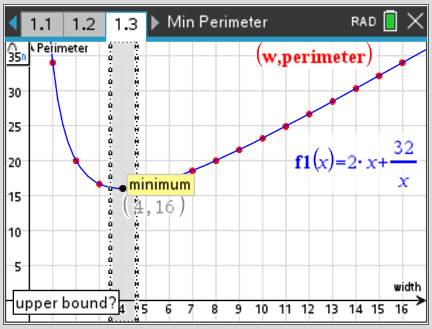


(c) To graph a continuous function for perimeter $= 2x + \dfrac{32}{x}$:

- Press [menu] > **Graph Entry/Edit > Function**.
- Enter $\boldsymbol{f1(x) = 2x + 32/x}$.

To graphically find the minimum value of the perimeter:

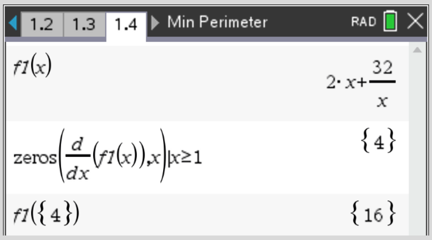
- Press [menu] > **Analyse Graph > Minimum**. Click to the left then to the right for the lower and upper bounds.



Answer: Minimum perimeter: $16\,\text{m}^2$ at width = length = 4 m.

To confirm the result using calculus, on a **Calculator** page:

- Press [menu] > **Algebra > Zeros** then press [⇧shift] [–].
- Enter $\textbf{zeros}\left(\dfrac{d}{dx}\big(f1(x)\big), x\right) \mid x \geq 1$ for the width.
- Enter $\boldsymbol{f1(\text{ans})}$ (use the [ans] key) for the perimeter.



Answer: Confirmed that minimum perimeter is $16\,\text{m}^2$.

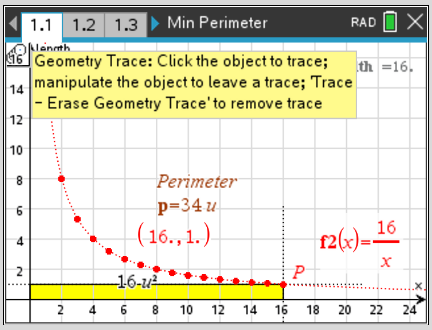
***Note:** A possible extension is to trace the locus of point P.*

To trace the locus of point *P*, on page 1.1:

- Hover over point *P*, press [ctrl] [menu] > ***Geometry Trace***.
- Use the slider to vary the width. Press [esc] to exit the tool.

To graph the continuous curve traced out by point *P*:

- Enter $\boldsymbol{f2(x) = 16/x}$.



Answer: The locus of point *P* is the curve with rule $y = \dfrac{16}{x}$.

Maximising the area of a triangle and minimising the length of its hypotenuse

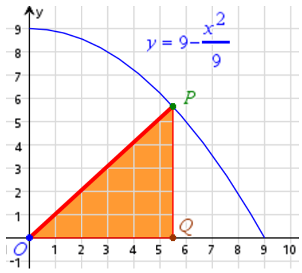
Question

Let OP be the hypotenuse of right-angled triangle OPQ, where O is the origin, P is a point on the graph of $f(x) = 9 - \frac{x^2}{9}, 0 < x < 9$, and Q is a point on the x-axis.

Use a **(i)** graphical and **(ii)** calculus method to determine the following, correct to two decimal places.

(a) The maximum area of OPQ and the value of x where this occurs.

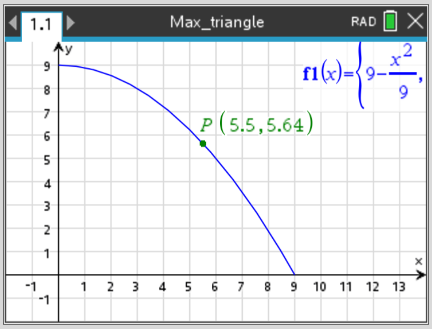
(b) The minimum length of OP and the value of x where this occurs.



Solution

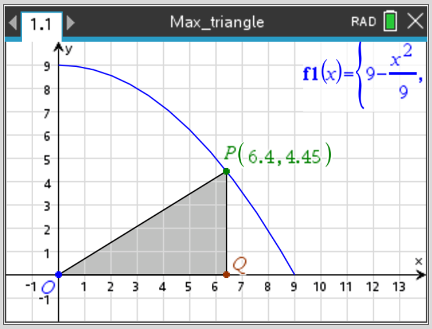
To set up a model of this context, on a **Graphs** page:

- Enter $\boldsymbol{f1(x) = 9 - x^2 / 9 \mid 0 < x < 9}$
- Press menu > **Window/Zoom** > **Window Settings**. In the dialog box that follows, enter the following values.
  XMin = –1 XMax = 14 XScale = 1
  YMin = –1 YMax = 10 YScale = 1
- Press P > **Point**. Click graph ***f*1**, then press esc to exit.
- Hover over the point, press ctrl menu > **Label**. Enter ***P***.



To draw triangle OPQ:

- Press menu > **Geometry** > **Construction** > **Perpendicular**. Click the x-axis then point P. Press esc.
- Press menu > **Geometry** > **Shapes** > **Triangle**. Click point P, then the intersection point of the perpendicular line and the x-axis, then the origin. Press esc to exit.
- Hover over the triangle, press ctrl menu > **Colour** and choose a **Fill** colour. Label points O and Q as shown. Press esc.
- Hover over the perpendicular line, press ctrl menu > **Hide**.

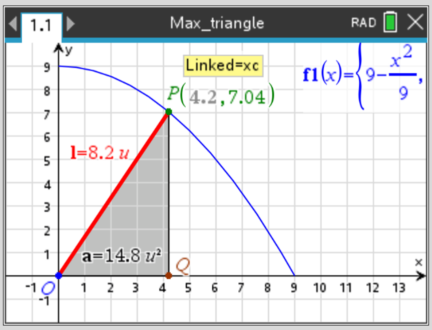


(a)(i) and **b(i)** To measure the area of OPQ and length OP:

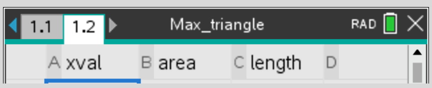
- Press menu > **Geometry** > **Points & Lines** > **Segment**.
- Click the point O then point P. Press esc.
- Hover over the triangle. Press ctrl menu > **Measurement**. Select **Area**. Repeat for segment OP but select **Length**.

To store the values of the area, length and x-coordinate:

- Hover over the area value and press ctrl menu > **Store**. Enter the variable name, ***a***. Similarly, store the length OP as variable ***l*** and the x-coordinate of P as variable ***xc***.



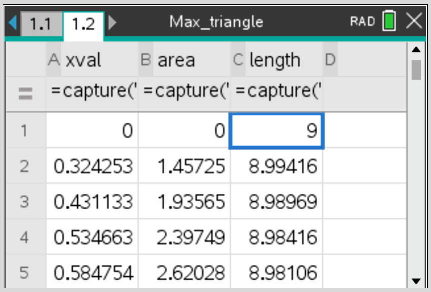
To set up lists of the variables xc, a and l, add a **Lists & Spreadsheet** page with the column headings as shown.



Solution (continued)

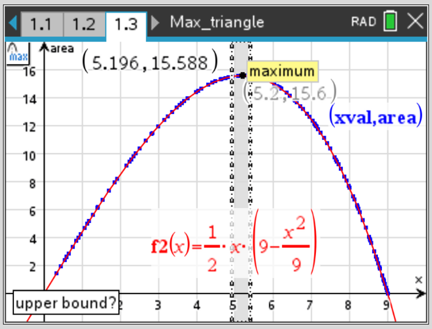
To capture the values of variables *xc*, *a* and *l*:

- Navigate to the column A formula cell, press [ctrl] [menu] > **Data Capture > Automatic**. Press [var] and select *xc*.
- Similarly, in the column B formula cell, capture variable *a*. In the column C formula cell, capture variable *l*.
- On page 1.1, drag point *P* along the curve. The lists on page 1.2 will be populated by the captured values.



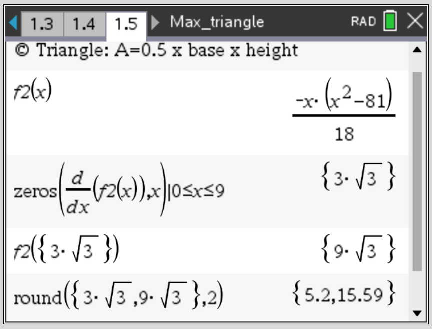
(a) To plot and find maximum area, add a **Graphs** page, then:

- Press [menu] > **Graph Entry/Edit > Scatter Plot**.
- For **s1**, press [var] to enter $x \leftarrow \boldsymbol{xval}$ and $y \leftarrow \boldsymbol{area}$.
- Adjust the window settings as shown.
- Press [menu] > **Graph Entry/Edit > Function**.
- Enter $f2(x)=\frac{1}{2}x\cdot(9-x^2/9)$ $\left(\text{by } Area(\triangle)=\frac{1}{2}\times b\times h\right)$
- Press [menu] > **Analyse Graph > Maximum**. Click to the left then to the right for the lower and upper bounds.



(a)(ii) To find the exact minimum area, on a **Calculator** page:

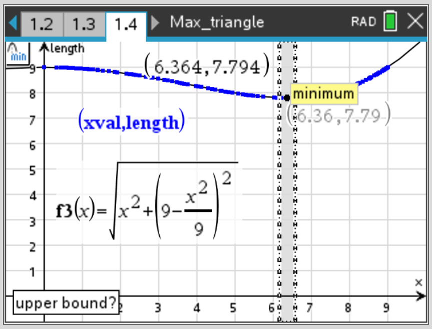
- Press [menu] > **Algebra > Zeros** then press [⇧shift] [−].
- Enter $\textbf{zeros}\left(\frac{d}{dx}(f2(x)),x\right)|0\le x\le 9$ for the *x* value.
- Enter $f2(\textbf{ans})$ (use the [ans] key) for the area.



Answer: (a) Maximum area: $9\sqrt{3}\approx 15.59$ at $x=3\sqrt{3}\approx 5.20$.

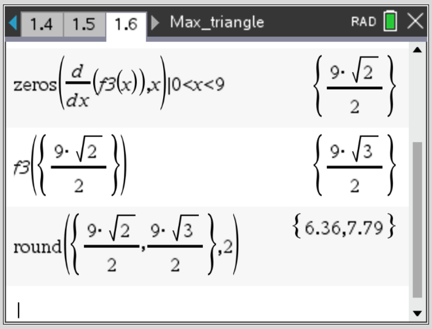
(b) To plot and find minimum length *OP*, on a **Graphs page**:

- Press [menu] > **Graph Entry/Edit > Scatter Plot**.
- For **s2**, press [var] to enter $x \leftarrow \boldsymbol{xval}$ and $y \leftarrow \boldsymbol{length}$.
- Adjust the window settings as shown.
- Press [menu] > **Graph Entry/Edit > Function**.
- Enter $f3(x)=\sqrt{x^2+(9-x^2/9)^2}$ (by Pythag. Theorem).
- Press [menu] > **Analyse Graph > Minimum**. Click to the left then to the right for the lower and upper bounds.



(b)(ii) To find exact minimum length, on a **Calculator** page:

- Press [menu] > **Algebra > Zeros** then press [⇧shift] [−].
- Enter $\textbf{zeros}\left(\frac{d}{dx}(f3(x)),x\right)|0< x< 9$ for the *x* value.
- Enter $f3(\textbf{ans})$ (use the [ans] key) for the length.



Answer: (b) Min. length: $\frac{9\sqrt{3}}{2}\approx 7.79$ at $x=\frac{9\sqrt{2}}{2}\approx 6.36$.

Graphing the position and velocity of a particle at time t

Question

Consider a particle moving in a straight line. Its position s metres from a fixed point O at time t seconds is modelled by the function $s(t)=3+5t-t^2$, $0\le t\le 6$.

(a) Find **(i)** the position at $t = 4$, **(ii)** the time, correct to two decimal places, when $s = 0$.

(b) Draw the graph of s and find the maximum and minimum values of s.

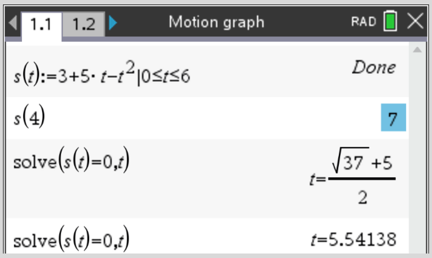
(c) Determine the velocity, v, at time t and draw the graph of the velocity of the particle.

Solution

(a) To perform the calculations, on a **Calculator** page:

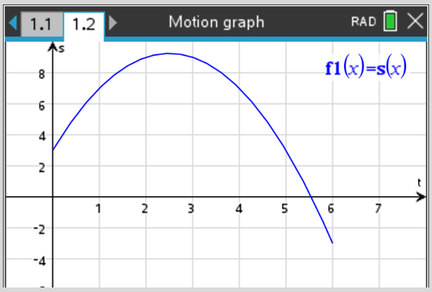
- Enter $\boldsymbol{s(t):=3+5t-t^2\,|\,0\le t\le 6}$, then enter $\boldsymbol{s(4)}$.
- Press menu > **Algebra** > **Solve**.
- Enter $\textbf{Solve}(\boldsymbol{s(t)=0,t})$, then press ctrl enter.

Answer: (i) At $t = 4$, $s = 7$, **(ii)** $s = 0$ when $t = 5.54$ (2 d.p.).



(b) To draw the graph of s, add a **Graphs** page, then:

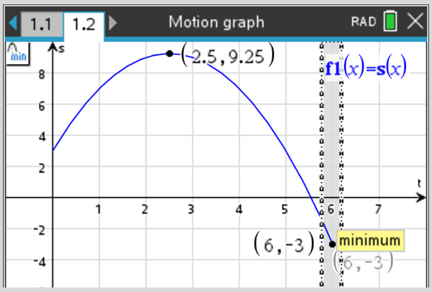
- Enter $\boldsymbol{f1(x)=s(x)}$.
- Press menu > **Window/Zoom** > **Window Settings**. In the dialog box that follows, enter the following values.
  XMin: –1 XMax: 8 XScale: 1
  YMin: –8 YMax: 10 YScale: 2



To find the minimum and maximum values of s:

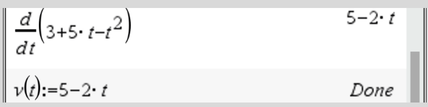
- Press menu > **Analyse Graph** > **Maximum**.
  Click to the left then to the right of the turning point.
- Press menu > **Analyse Graph** > **Minimum**.
  Click to the left then to the right of the endpoint at $t = 6$.

Answer: Maximum value: $s = 9.25$ (at $t = 2.5$).
Minimum value: $s = -3$ (at endpoint, $t = 6$).



(c) To determine the velocity at time t, on page **1.1**:

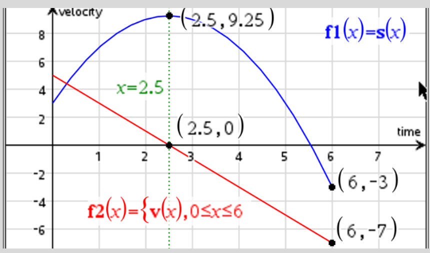
- Press shift −, enter $\boldsymbol{\frac{d}{dt}\left(3+5t-t^2\right)}$ then $\boldsymbol{v(t):=}\textbf{ans}$.



To graph the velocity at time t, on page **1.2**:

- Enter $\boldsymbol{f2(x)=v(x)\,|\,0\le x\le 6}$.

Answer: $v(t)=5t-2t$, $0\le t\le 6$. For $0\le t<2.5$, $v>0$.
At $t=2.5$, $v=0$ and s is maximum. For $2.5<t<6$, $v<0$.



2.4.4 Newton's method for finding numerical roots

Note: Refer to ***Section 3.1.4*** *for Newton's method, with pseudocode and working code examples in the Calculator, Program Editor, and Python applications.*

2.5 Antiderivatives

2.5.1 Antidifferentiation

Finding an antiderivative

Antidifferentiation is the inverse process of differentiation and can be described using the following conventional notation variants:

Leibnitz' notation

If $y = x^n$, then $\dfrac{dy}{dx} = nx^{n-1} \Leftrightarrow$ an antiderivative of $\dfrac{dy}{dx}$ is $y = \int \dfrac{dy}{dx}\,dx = \int nx^{n-1}dx = x^n$ $(n \neq 1)$.

Newton's notation

If $f(x) = x^n$, then $f'(x) = nx^{n-1} \Leftrightarrow$ an antiderivative of $f'(x)$ is $f(x) = \int f'(x)\,dx = \int nx^{n-1}dx = x^n$.

Also, an antiderivative of $f(x) = x^n$ can be found as follows:

If $f(x) = x^n$, then an antiderative is $\int f(x)\,dx = \dfrac{1}{n+1}x^{n+1} = \dfrac{x^{n+1}}{n+1}$.

Question

Find an antiderivative of the following functions:

(a) $y = x^3 - x^2 - x + 7$ **(b)** $y = x^n$ **(c)** $y = x^n$, for $n \in \{1, 2, 3, 4\}$

Solution

(a) To find $\int \left(x^3 - x^2 - x + 7\right) dx$, on a **Calculator** page:

- Press and select the **antiderivative** template as shown.
- Enter the command $\int\left(x^3 - x^2 - x + 7\right) dx$ as shown.

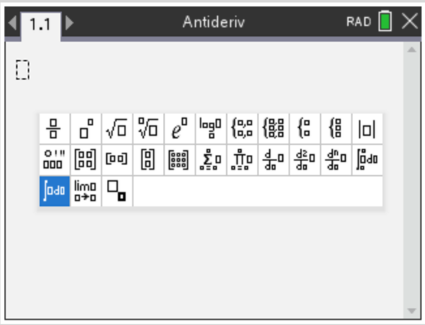
(b) To find $\int x^n\ dx$, on a **Calculator** page:

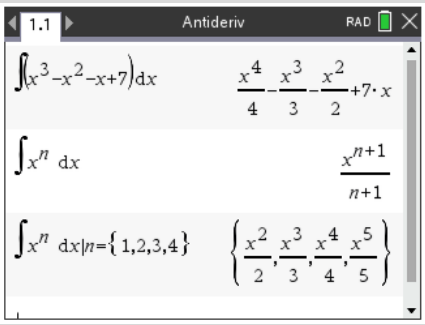
- Press and select the **antiderivative** template.
- Enter the command $\int\left(x^n\right) dx$ as shown.

(c) To find $\int x^n\ dx$ for these n values, on a **Calculator** page:

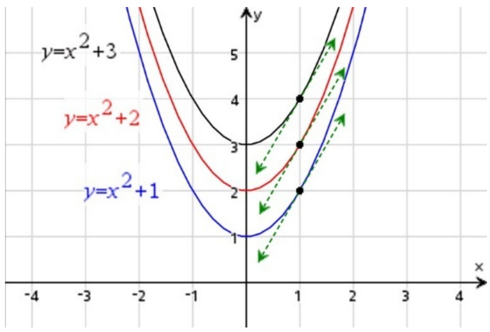
- Press and select the **antiderivative** template.
- Enter the command $\int\left(x^n\right) dx \mid n = \{1, 2, 3, 4\}$ as shown.

Note: The symbol '|' is used to specify that a restriction or condition is to be imposed. This symbol, and the inequality symbols can be accessed via ctrl =.





Visualising families of curves using derivative equations



Consider the family of parabolas with equations $y = x^2 + 1, y = x^2 + 2, y = x^2 + 3$, which all have the same derivative $\frac{dy}{dx} = 2x$. This means that for any x value for which the above rules are defined, the graphs of $y = x^2 + 1, y = x^2 + 2, y = x^2 + 3$ will have the same gradient, as illustrated right. Further, since $\frac{dy}{dx} = 2x$, the value of that gradient will be twice the value of the x value.

Conversely, the family of parabolas for which $\frac{dy}{dx} = 2x$ will have equations of the form $y = x^2 + c$, where c is a real constant. That is, if $\frac{dy}{dx} = 2x$, then $\int 2x\,dx = x^2 + c, c \in R$.

Question

For any parabola for which $\frac{dy}{dx} = 2x$:

(a) Represent $\frac{dy}{dx} = 2x$ on the cartesian plane.

(b) Plot the parabola for which $\frac{dy}{dx} = 2x$, and also passes through the point with coordinates (1,4).

(c) Use calculus to show that the specific parabola obtained in part **(b)** has equation $y = x^2 + 3$.

Solution

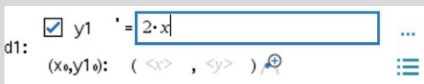
(a) To represent $\frac{dy}{dx} = 2x$, add a **Graphs** page, then:

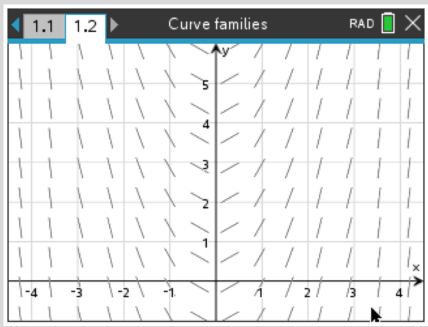
- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values.

  XMin = –4.5 XMax = 4.5 XScale = 1
  YMin = –1 YMax = 6 YScale = 1

- Press menu > **Graph Entry/Edit** > **Diff Eq**. (This allows for graphing curves defined by a derivative rule).
- In the dialog box that follows, enter $\boldsymbol{y1' = 2x}$.

This will plot a series of contours that illustrate where such curves with the property $\frac{dy}{dx} = 2x$ must lie.

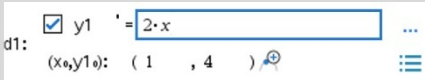




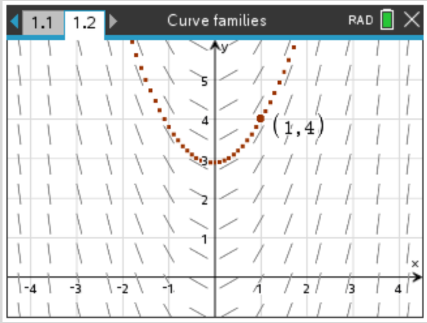
... continued

Solution (continued)

(b) To plot the specific parabola for which $\frac{dy}{dx} = 2x$, and also passes through the point with coordinates (1,4):

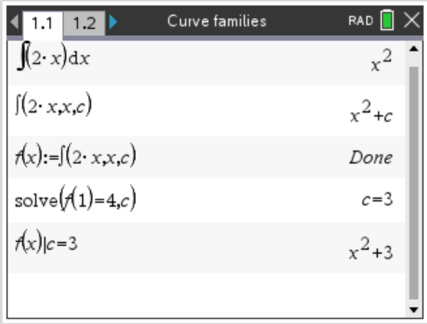


- Press ctrl G and then ▲ to edit the definition for $y1'$.
- In the dialog box that follows, add the coordinates of the known point (1,4) – that is, enter $(x_0, y1_0):(1,4)$.



This will plot a series of contours that illustrate where such curves with the property $\frac{dy}{dx} = 2x$ must lie, as well a set of points for the only parabola which has that property and passes through the point (1,4). The plotted graph appears to have equation $y = x^2 + 3$.

(c) To verify that the specific parabola obtained in part **(b)** has equation $y = x^2 + 3$.



$\frac{dy}{dx} = 2x$, then $y = \int 2x\, dx = x^2 + c \Rightarrow y = x^2 + c$.

At (1,4): $4 = (1)^2 + c \Leftrightarrow c = 3$

So $y = x^2 + 3$.

***Note**: The **Antiderivative** template uses c = 0. The 'integral command' (accessible via [book] I) will perform the same calculation but permits a generalised constant (e.g. using c) to be added. This may be useful for finding the value of c using the coordinates of a known point on the graph (see example right).*

VCE Mathematical Methods Unit 3&4

3.1 Algebra and functions

3.1.1 Further polynomial functions

Analysing key features of an interactive cubic graph

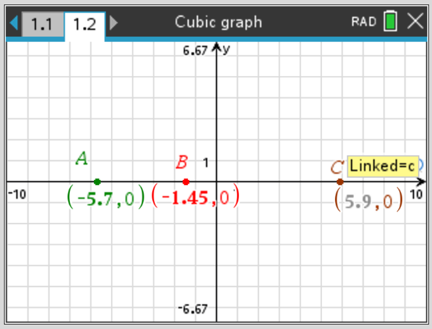
Question

Consider the graph of a cubic function with rule of the form $f(x)=k(x-a)(x-b)(x-c)$, where $a,b,c\in R$ and $k\neq 0$. Create an interactive graph with moveable x-intercepts and analyse key features of the graph, including cases where $k<0$, $a=b$ and $a=b=c$.

Solution

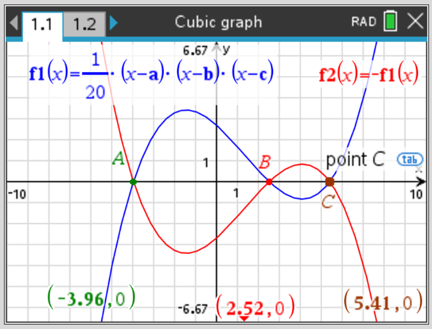
To set up moveable x-intercepts, on a **Graphs** page:

- Press [P] > **Point** and click 3 different points on the x-axis.
- Hover over a point, press [ctrl] [menu] > **Coordinates & Equations**. Repeat for the other two points.
- Hover over a point, press [ctrl] [menu] > **Label**. Enter ***A*** etc.
- Hover over the x-coordinate of A, press [ctrl] [menu] > **Store** and enter ***a***. Repeat for ***B*** and ***C*** with variables ***b*** and ***c***.
- Press [ctrl] ▲ then [ctrl] [C] [ctrl] [V] to clone the page.



To graph positive and negative cubics on page **1.1**:

- Enter $\boldsymbol{f1(x)=1/20(x-a)\cdot(x-b)\cdot(x-c)}$ and $\boldsymbol{f2(x)=-f1(x)}$. Display a graph at a time on page **1.2**.
- Change the values of a, b or c by dragging points A, B or C along the axis, or by editing their x-coordinates.
- Consider a variety of cases, including $a=b$ and $a=b=c$ (repeated factors).



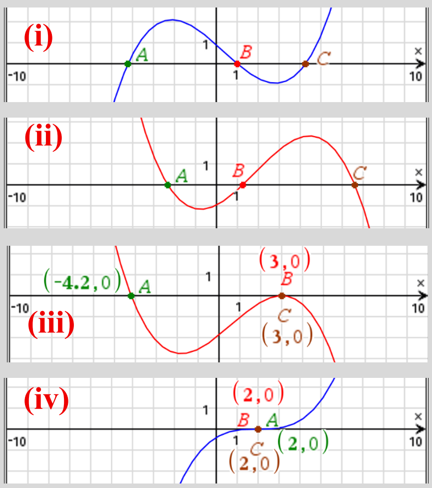
Answer: For $f(x)=k(x-a)(x-b)(x-c)$, $a\neq b\neq c$:

(i) Positive cubic: $x\to-\infty, f(x)\to-\infty;\ x\to\infty, f(x)\to\infty$

(ii) Negative cubic: $x\to-\infty, f(x)\to\infty;\ x\to\infty, f(x)\to-\infty$

(iii) For repeated factor, $f(x)=k(x-a)(x-b)^2$: turning point at $(b,0)$.

(iv) For triple factor, $f(x)=k(x-a)^3$: stationary point of inflection at $(a,0)$.



***Note**: A possible extension is to explore quartic graphs. Add a fourth point and store its x-coordinate as **d**. Explore graphs with four factors, including cases with repeated factors.*

Investigating a property of the zeros of a cubic polynomial function

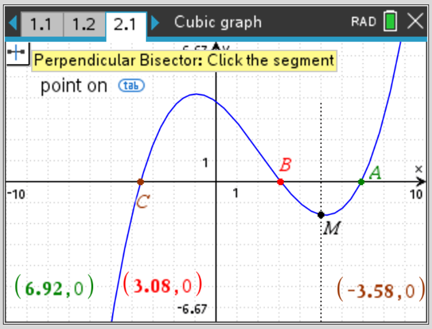
Question

Consider the graph of a cubic function of the form $f(x)=k(x-a)(x-b)(x-c)$.
Use a copy of the document from the previous problem to graphically investigate the tangent to the curve at the x-coordinate which is the midpoint of two zeros of the graph and make and test conjectures about its relationship with the third zero.

Solution

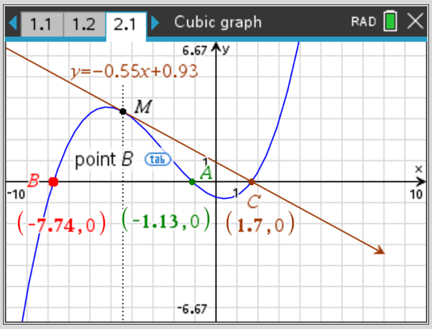
To set up the context, open a copy of the previous problem (or use the instructions in the previous problem to create it), then:

- Press menu > **Geometry** > **Construction** > **Perpendicular Bisector**. Click any two zeros, such as points *A* and *B*.
- Press menu > **Geometry** > **Points & Lines** > **Intersection Point(s)**. Click the graph then the perpendicular bisector.
- Press esc to exit the tool. Hover over the intersection point, press ctrl menu > **Label**. Enter the label ***M***.

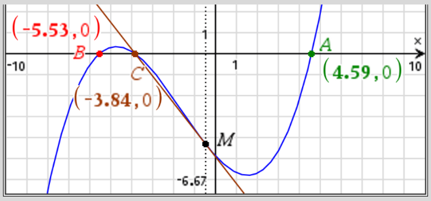


To add a tangent line at point *M*:

- Press menu > **Geometry** > **Points & Lines** > **Tangent**. Click point *M*, then press esc to exit the tool.
- Extend the tangent line by grabbing (grab icon) an end.
- Change the positions of the zeros by dragging the points along the axis or editing their x-coordinates. Observe the relationship between the tangent line and the third zero for a variety of cases, including the trivial case where $a = b$.



Answer: For a cubic graph with three real zeros, the tangent to the curve at the x-coordinate which is at the midpoint of any two zeros always intersects the third zero, regardless of the symmetry, relative positions or spacing between the zeros. In the case where $a = b$, the tangent at $(a, 0)$ has equation $y = 0$ and also intersects point *C*.



To confirm the result, on a **Calculator** page (new **Problem**):

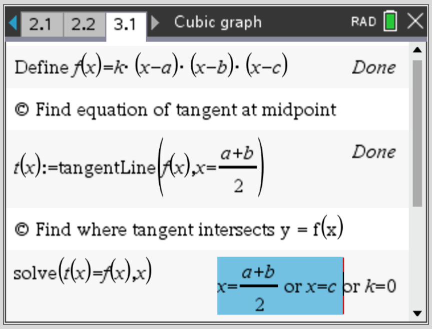
- Enter $\boldsymbol{f(x):=k\cdot(x-a)\cdot(x-b)\cdot(x-c)}$.

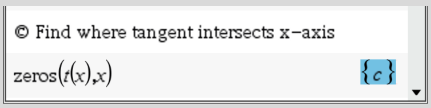
To find the equation of the tangent to $y=f(x)$ at a midpoint:

- Key in $\boldsymbol{t(x):=}$, then press (catalog) 1 T to select **tangentLine** and enter $\boldsymbol{t(x):=\textbf{tangentLine}\left(f(x),x=\frac{a+b}{2}\right)}$

To find the intersection of the tangent with the:

- graph of *f*: Enter $\boldsymbol{\textbf{Solve}(t(x)=f(x),x)}$.
- x-axis: Enter $\boldsymbol{\textbf{Zeros}(t(x),x)}$.





Answer: The tangent invariably intersects the graph of *f* on the x-axis at the point with coordinates $(c, 0)$.

Determining polynomial quotients and remainders

Question

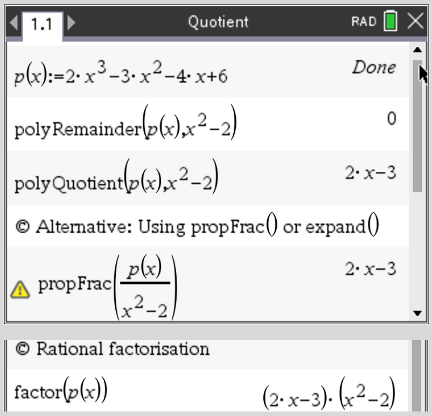
Consider the cubic polynomial $p(x)=2x^3-3x^2-4x+6$. Use the remainder theorem to show that x^2-2 is a quadratic factor of $p(x)$. Hence determine the rational linear factor of $p(x)$.

Solution

To find the remainder and quotient, on a **Calculator** page:

- Enter $\boldsymbol{p(x):=2x^3-3x^2-4x+6}$
- **Approach 1**. Press [book] [1] [P], select **polyRemainder** and enter **polyRemainder**$\boldsymbol{(p(x),x^2-2)}$.
- Similarly, enter **polyQuotient**$\boldsymbol{(p(x),x^2-2)}$
- **Approach 2**. Press [book] [1] [P], select **propFrac** and enter **propFrac**$\boldsymbol{(p(x)/(x^2-2))}$.

Answer: Confirmed: $p(x)=(2x-3)(x^2-2)$



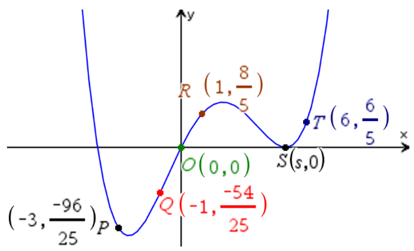
Finding the equation of the graph of a polynomial function from its features

Question

Part of the graph of *f*, a polynomial function of degree four, is shown. The graph contains the points *O*, *P*, *Q*, *R* and *T* with coordinates as shown. *S*(*s*, 0) is a turning point on the *x*-axis.

(i) Determine the general equation of the graph.

(ii) Determine the equation as the product of linear factors.



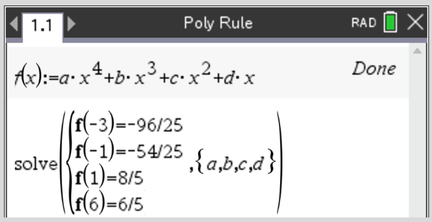
Solution

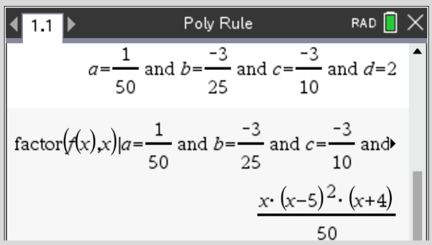
As the graph passes through the origin, the general rule is of the form $f(x)=ax^4+bx^3+cx^2+dx+0$.

To evaluate *a*, *b*, *c* and *d*, on a **Calculator** page:

- Enter $\boldsymbol{f(x):=a\cdot x^4+b\cdot x^3+c\cdot x^2++d\cdot x}$
- Press [menu] > **Algebra** > **Solve System of Equations** > **Solve System of Equations**. In the dialog box enter: Number of equations: **4**, Variables: ***a***, ***b***, ***c***, ***d***.
- On the template, enter the equations $\boldsymbol{f(-3)=-96/25}$ $\boldsymbol{f(-1)=-54/25}$, $\boldsymbol{f(1)=8/5}$ and $\boldsymbol{f(6)=6/5}$.
- Press [menu] > **Algebra** > **Factor**.
- Enter **factor**$\boldsymbol{(f(x),x)|}$**ans**

Answer: $y=\frac{1}{50}x^4-\frac{3}{25}x^3-\frac{3}{10}x^2+2x=\frac{1}{50}x(x+4)(x-5)^2$





3.1.2 Exponential, inverse and logarithmic functions

***Note**: See **Section 1.2.3** for 'inverse functions' activities that do not involve exponentials.*

Introducing Euler's number e through a compound interest example

Question

Suppose that \$1 is invested at an interest rate of 100% per annum.

(a) Calculate the value of the investment after 1 year if interest is compounded:
(i) annually, **(ii)** quarterly, **(iii)** monthly, **(iv)** weekly, **(v)** daily, **(vi)** every second.

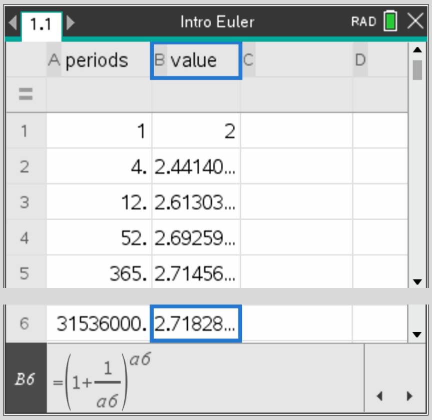
(b) Plot the value against the compounding period and interpret the connection of the results with Euler's number, $e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

Solution

(a) The number of compounding periods is (i) 1, (ii) 4, (iii) 12, (iv) 52, (v) 365, (vi) 365×24×3600. After 1 year, value is given by (i) $(1+1)^1$, (ii) $(1+\frac{1}{4})^4$, (iii) $(1+\frac{1}{12})^{12}$, etc.

To determine the values, on a **Lists & Spreadsheet** page:

- Enter the column headings **periods** and **value**, as shown.
- In column A enter the compounding periods, 1., 4., etc. as shown, with a decimal point at the end of each entry.
- In cell B1 enter the formula, $=(\mathbf{1}+\mathbf{1}/\boldsymbol{a}\mathbf{1})^{a1}$.
- Navigate to cell **B1** and press ctrl menu > **Fill**. Press ▼ to fill the formula down to cell **B6**, then press enter.



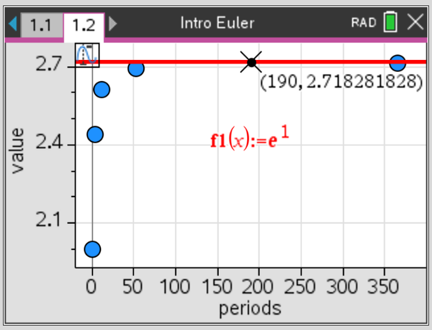
Answer: (i) $(1+1)^1=2$ **(ii)** $(1+\frac{1}{4})^4=2.44...$

(iii) $(1+\frac{1}{12})^{12}=2.61...$ **(iv)** $(1+\frac{1}{52})^{52}=2.69...$

(v) $(1+\frac{1}{365})^{365}=2.71...$ **(vi)** $(1+\frac{1}{31536000})^{31536000}=2.71828...$

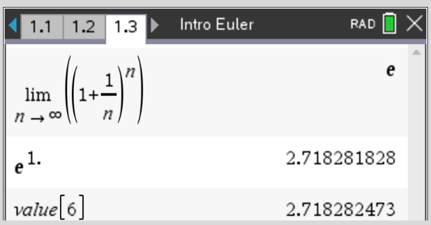
(b) To plot the value against the compounding period, add a **Data & Statistics** page, then:

- Click the horizontal axis to select **periods**, then click the vertical axis to select **value**.



To compare with the value of e, graphically and numerically:

- Press menu > **Analyse** > **Plot Function** and press e^x to enter $f1(x):=e^1$. Then menu > **Analyse** > **Graph Trace**.
- Adjust the window settings to ignore the point associated with compounding every second.
- Add a **Calculator** page, enter e^1 then enter **value[6]**.



Answer: The graph shows that as n increases, the value approaches e. For compounding 'every second' $n\to\infty$ and the value of 2.718282 approximates e to six decimal places.

Investigating why e is the natural exponential base in many modelling contexts

The previous problem established that Euler's number, e, arises naturally from continuous compounding. This problem explores what makes e special among exponential bases.

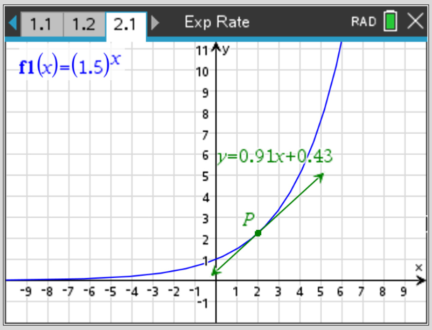
Question

Use dynamic features to graphically investigate the rate of change at any point of exponentials of different bases. Comment on notable features and interpret what is special about the rate of change of $y = e^x$, and why this property is powerful in modelling many growth or decay processes.

Solution

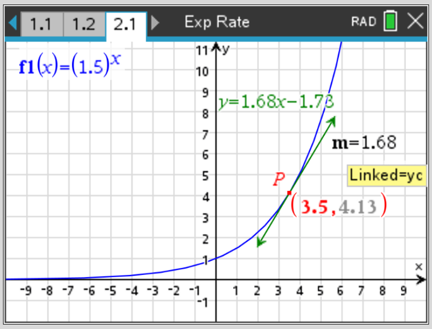
To explore the rate of change of $y = a^x$, on a **Graphs** page:

- Start with a base of $a = 1.5$ by entering $\boldsymbol{f1(x) = 1.5^x}$.
- Adjust the window by clicking and editing the maximum y-axis value to **11.33** and the minimum to **2**, as shown.
- Add a tangent by pressing menu > **Geometry** > **Points and lines** > **Tangent**. Double-click the graph then press esc.
- Hover over the contact point, press ctrl menu > **Label** and enter the label ***P***.



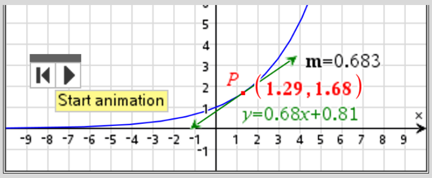
To record the gradient of the tangent at any point $P(x, y)$:

- Press menu > **Geometry** > **Measurement** > **Slope** then double-click the tangent (or click once and press enter).
- Display the coordinates of P by hovering over point P and pressing ctrl menu > **Coordinates & Equations**.
- Store the slope measurement by hovering over it and pressing ctrl menu > **Store**. Enter variable name, ***m***.
- Similarly, store the x-coordinate of point P as variable ***xc*** and the y-coordinate as variable ***yc***.



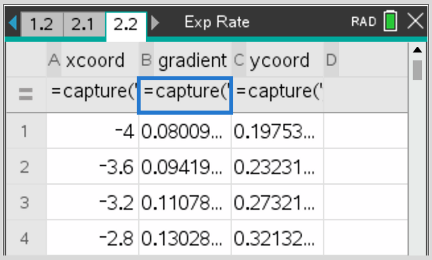
To animate point P with speed 2 (on a scale of 0 to 9):

- Edit the x-coordinate of P to **– 4**.
- Hover over point P, press ctrl menu > **Attributes**, select Unidirectional animation speed and press 2 enter enter.
- Use the control buttons to start/pause/reset the animation.



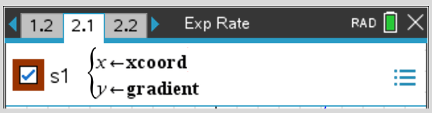
To capture the values of m, xc and yc as the location of point P changes, add a **Lists & Spreadsheet** page, then:

- Enter the column headings as shown.
- Navigate to the column A formula cell, press ctrl menu > **Data Capture** > **Automatic**. Press var and select xc.
- Similarly, capture variables m and yc in columns B and C.
- On page 2.1, start the animation to populate the lists.



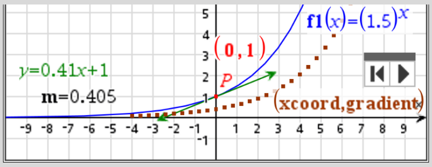
To plot gradient m against the x-coordinate of P, xc, on page 2.1:

- Press menu > **Graph Entry/Edit** > **Scatter Plot**. For **s1**, enter $x \leftarrow$ ***xcoord*** and $y \leftarrow$ ***gradient***.



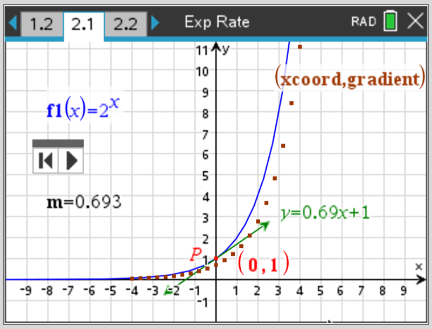
... continued

Solution (continued)



Answer: For $y=(1.5)^x$, the plot indicates that the rate of change follows an exponential pattern that is 'shallower' than the original graph. At $x=0$, the gradient, $m\approx 0.405$.

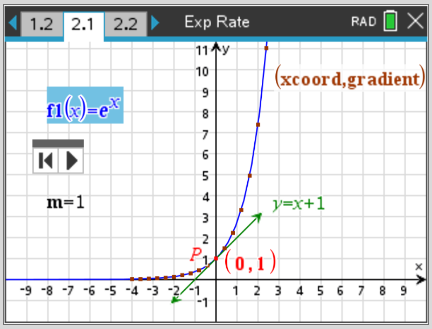
To explore the rate of change of $y=2^x$, on page 2.1:



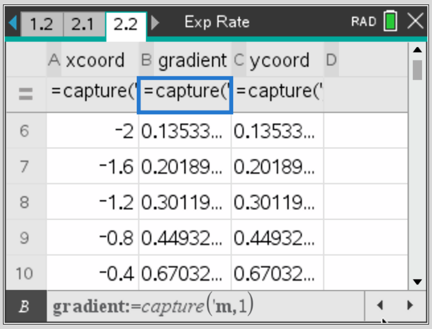
- Press menu > **Graph Entry/Edit** > **Function**.
- Edit $\boldsymbol{f1}$ to $\boldsymbol{f1(x)=2^x}$ and reset the animation.
- On page 2.2 navigate to the column A formula cell. Press ctrl menu > **Clear Data**. Repeat for columns B and C.
- On page 2.1 restart the animation to populate the lists.

Answer: $y=2^x$ grows more quickly than $y=(1.5)^x$. As the function gets larger, *m* also gets larger. At $x=0$, $m\approx 0.693$.

To explore the rate of change of $y=e^x$, on page 2.1:



- Press menu > **Graph Entry/Edit** > **Function**.
- Edit $\boldsymbol{f1}$ to $\boldsymbol{f1(x)=e^x}$ by pressing e^x to input *e*.
- Reset the animation.
- On page 2.2 navigate to the column A formula cell. Press ctrl menu > **Clear Data**. Repeat for columns B and C.
- On page 2.1 restart the animation to populate the lists.

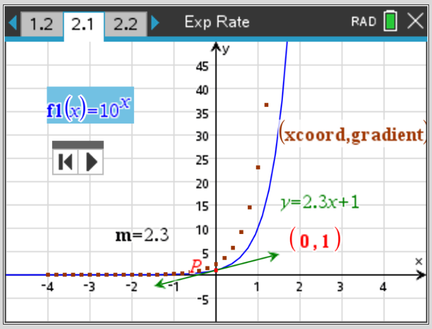


	A xcoord	B gradient	C ycoord	D
=	=capture(	=capture(	=capture(	
6	-2	0.13533...	0.13533...	
7	-1.6	0.20189...	0.20189...	
8	-1.2	0.30119...	0.30119...	
9	-0.8	0.44932...	0.44932...	
10	-0.4	0.67032...	0.67032...	

B gradient:=capture('m,1)

Answer: What makes *e* special among exponential bases is that $y=e^x$ is the only function whose *rate of change at any point is exactly equal to its current value*, as seen on the graph and in the columns B and C values. At $x=0$, $m=y=1$.
This is powerful in modelling because many real-world systems (radioactive decay, interest compounding, spread of disease, atmospheric pressure) evolve so that their *rate of change is proportional to their current size*.

To compare with the rate of change of $y=10^x$, on page 2.1:



- Press menu > **Graph Entry/Edit** > **Function**.
- Edit $\boldsymbol{f1}$ to $\boldsymbol{f1(x)=10^x}$. Reset the animation.
- Adjust the window settings to observe more plot points.
- On page 2.2 navigate to the column A formula cell. Press ctrl menu > **Clear Data**. Repeat for columns B and C.
- On page 2.1, restart the animation to populate the lists.

Answer: For $y=10^x$, $m\neq yc$. At $(0,1)$, $m\approx 2.3$.

In many contexts, exponential functions with base *e* match how the real systems behave. This makes the mathematics of growth or decay *simpler* and more *natural* for these contexts.

Modelling the rate at which air pressure decreases with altitude: Halley's Law

Question

Atmospheric pressure, P hPa (hectopascals), at an altitude of x km above sea, is modelled by the function $P(x) = Ae^{-\frac{x}{h}}, x \geq 0$. On a particular day, weather balloon measurements indicate that $P = 563$ hPa at $x = 5.0$ km and $P = 311$ hPa at $x = 10.0$ km.

(a) Determine the values of A, in hPa correct to the nearest integer, and h, in km correct to two decimal places. Interpret the significance of the parameters A and h.

(b) Find the value of P at an altitude of 18 km, in hPa, correct to the nearest integer.

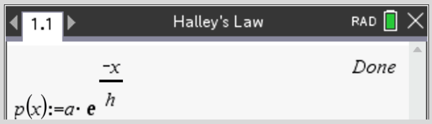
(c) Determine the altitude at which atmospheric pressure is half the value of that at sea level. Give the answer in km, correct to one decimal place.

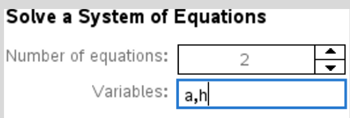
(d) Draw a graph of P and use graphical methods to confirm the results of parts **(b)** and **(c)**.

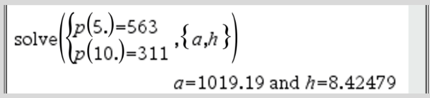
Solution

(a) To find the values of A and h, on a **Calculator** page:

- Enter $\boldsymbol{p(x):=a \cdot e^{-x/h}}$ by pressing [e^x] to input e.
- Press [menu] > **Algebra > Solve System of Equations > Solve System of Equations**. In the dialog box, enter Number of equations: **2**, Variables: ***a***, ***h***.
- Enter equations $\boldsymbol{p(5.)=563}$ and $\boldsymbol{p(10.)=311}$.





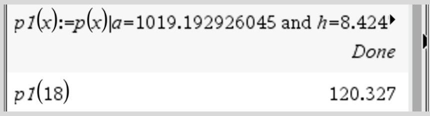


Answer: $A = 1019$ hPa is the pressure at sea level ($x = 0$), $h = 8.42$ km is the scale height. This indicates that the pressure decreases by a factor of $e \approx 2.718$ every 8.42 km.

(b) To update the definition of $P(x)$ and find P at $x = 18$:

- Enter $\boldsymbol{p1(x):=p(x) \mid ans}$, then enter $\boldsymbol{p1(18)}$.

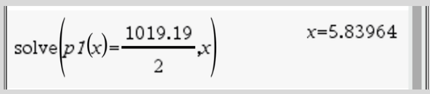
Answer: $P = 120$ hPa at an altitude of 18 km.



(c) To find x such that P is half the sea level value:

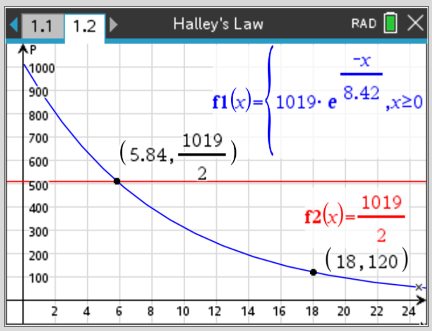
- Press [menu] > **Algebra > Solve** and enter $\boldsymbol{\text{solve}(p1(x)=1019.19/2, x)}$.

Answer: At $x = 5.8$ km, pressure is half the sea level value.



(d) To graph P and find the values, on a **Graphs** page:

- Enter $\boldsymbol{f1(x)=1019e^{-x/8.42} \mid x \geq 0}$, pressing [e^x] to input e.
- Press [menu] **> Window/Zoom > Window Settings** and set XMin = –1 and XMax = 25.
- Press [menu] **> Window/Zoom > Zoom Fit.**
- Press [P] > **Point**. Add two points by clicking the graph at two distinct places, then press [esc] to exit the tool.
- Edit the x-coordinate of one point to 18. Edit the y-coordinate of the other point to **1019/2**.
- Enter $\boldsymbol{f2(x)=1019/2}$ to visualise point of intersection.



Answer: Confirmed: $P(18)=120$ and $P(5.82)=\frac{1019}{2}$.

Reviewing the meaning of a logarithm

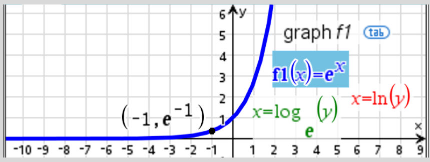
Question

By definition, $2^3 = 8 \Leftrightarrow \log_2(8) = 3$ and $y = a^x \Leftrightarrow x = \log_a(y)$. Use a graphical approach to illustrate the equivalence of $y = e^x$ and $x = \log_e(y)$.

Solution

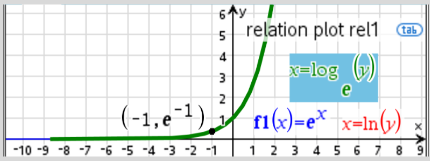
To graph $y = e^x$ and $x = \log_e(y)$, on a **Graphs** page:

- Enter $\boldsymbol{f1(x) = e^x}$, pressing [e^x] to input *e*.
- [menu] > **Graph Entry/Edit** > **Relation**.



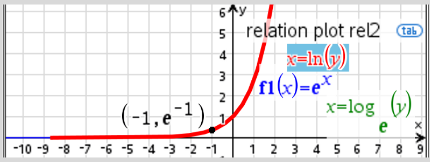
Approach 1:

- Enter $\boldsymbol{x = \log_e(y)}$, by pressing [ctrl] [10^x] (log) for the $\boldsymbol{\log_\square}$ template and [π▸] to select ***e***.



Approach 2:

- Enter $\boldsymbol{x = \ln(y)}$, by pressing [ctrl] [e^x] (ln).
- Hover over the graph and press [tab] to toggle between the function graph ***f*1** and the relation $\boldsymbol{x = \log_e(y)}$.



Answer: The graphs are identical, confirming the equivalence of $y = e^x$ and $x = \log_e(y)$. A *logarithm* is the *exponent*.

Exploring the inverse of exponentials of base *e*

Question

(a) Use a pointwise approach to plot the inverse of the graph of *f*, where $f(x) = e^x, x \in R$.

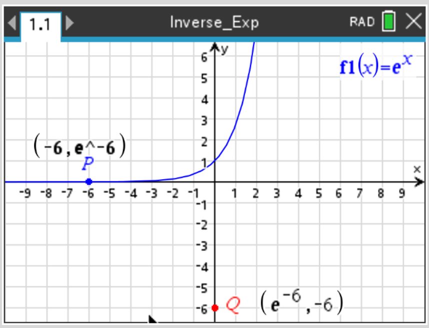
(b) State the rule, domain and range of f^{-1}.

(c) If $g : R \to R,\ g(x) = 3 - e^{2x-1}$, determine the inverse function, g^{-1}.

Solution

(a) To graph $f(x) = e^x$ and $x = \log_e(y)$, open the document from the previous problem (or create it as instructed above):

- Press [P] > **Point**. Click the graph near $x = -6$, press [enter] to locate the point then [esc] to exit the tool. The *x*-coordinate can be edited to exactly **–6** if desired.
- Label this point by hovering over it, pressing [ctrl] [menu] > **Label** and entering the label, ***P***.



To set up the locus of a point *Q* with coordinates (y_c, x_c):

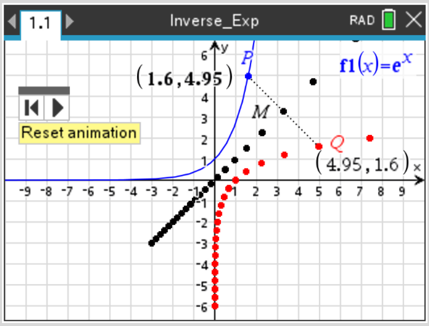
- Hover over the *x*-coordinate of *P*, press [ctrl] [menu] > **Store** and enter variable name, ***xc***.
- Similarly, store the *y*-coordinate of *P*, entering variable ***yc***.
- Press [P] > **Point by Coordinates** and enter $(\boldsymbol{yc}, \boldsymbol{xc})$.
- Label this point ***Q*** by pressing [ctrl] [menu] > **Label**.

... continued

Solution (continued)

To set up a point M, the midpoint of PQ:

- Press [menu] > **Geometry** > **Points & Lines** > **Segment** then click points P and Q. Press [menu] > **Geometry** > **Constructions** > **Midpoint**, click segment PQ then [esc] to exit the tool.
- Label this point $\boldsymbol{M}$ by hovering over the midpoint and pressing [ctrl] [menu] > **Label**.



To animate point P with animation control buttons:

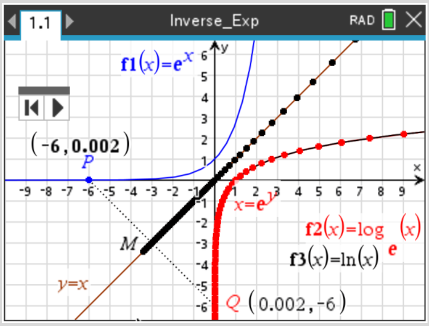
- Hover over point P, press [ctrl] [menu] > **Attributes**, then press ▼ [2] [enter] [enter]. This sets a unidirectional animation speed of 2 (on a scale of 0 to 9).
- Use the control buttons to start/pause/reset the animation.

To obtain a trace of the locus of points Q and M:

- To multi-select points Q and M, click point Q then hover over point M and press [ctrl] [menu] > **Geometry Trace**. Start the animation of P to create a trace of points Q and M.

To graph functions that fit the traces of points Q and M:

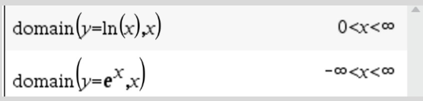
- Press [menu] > **Graph Entry/Edit** > **Relation**. Enter the relations $\boldsymbol{x = e^y}$ (using the [eˣ] key), $\boldsymbol{y = \log_e(x)}$ (using the [log] and [π▸] keys) and $\boldsymbol{y = \ln(x)}$, using ([ln] keys).
- Enter $\boldsymbol{y = x}$ and observe its relationship to point M.



Answer: The graphs with equations $x = e^y$, $y = \log_e(x)$ and $y = \ln(x)$ are identical and all three fit the trace of Q.

The graph of $y = x$ fits the trace of M. The graphs of $y = e^x$, $y = \log_e(x)$, are reflections of each other in the line $y = x$.

(b) For $f^{-1}(x) = \log_e(x)$, press [📖] [1] [D], locate **domain(Expr,Var)** and enter as shown to show that:
dom f^{-1} = ran f = R^+ and ran f^{-1} = dom f = R.



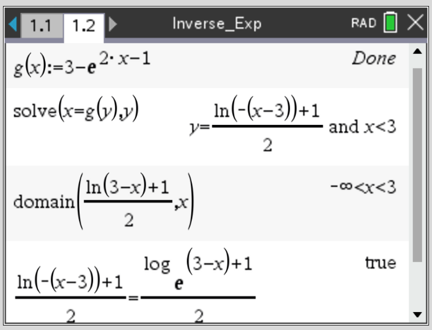
(c) To determine the rule of g^{-1}, on a **Calculator** page:

- Enter $\boldsymbol{g(x) := 3 - e^{2x-1}}$, using the [eˣ] key.
- Press [menu] > **Algebra** > **Solve**.
- Enter $\boldsymbol{\text{solve}(x = g(y), y)}$.

To confirm the domain of g^{-1}:

- Press [📖] [1] [D], select **domain** and enter

$$\textbf{domain}\left(\frac{\ln(3-x)+1}{2}, x\right).$$



Answer: $g^{-1} : (-\infty, 3) \to R$, $g^{-1}(x) = \dfrac{\log_e(3-x)+1}{2}$.

3.1.3 Further circular functions

*Note: See **Section 2.2.2** for additional learning activities related to circular functions, including: (i) Radian measurement (ii) Degree and radian conversions (iii) Unit circle definitions of sine, cosine and tangent (iv) Graphical and analytical solutions of trigonometric equations.*

Graphing functions of the form y = asin(nx), y = acos(nx) and y = tan(nx)

Question

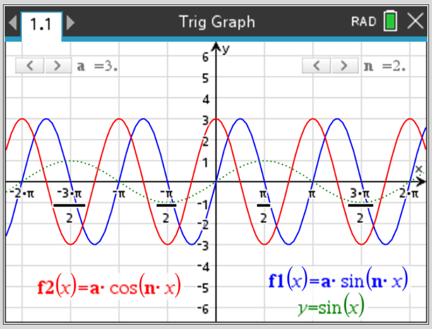
Explore and interpret the effect of changing the parameters a and n for the graphs of $y = a\sin(nx)$ and $y = a\cos(nx)$, and the parameter n for the graph of $y = a\tan(nx)$.

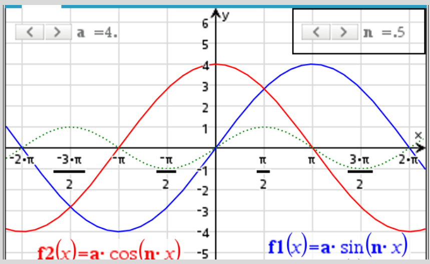
Solution

To graph $y = a\sin(nx)$ and $y = a\cos(nx)$ on a **Graphs** page:

- Set **RAD** mode (click top-right to toggle **RAD/DEG**).
- Enter $\boldsymbol{f1(x) = a \cdot \sin(n \cdot x)}$, $\boldsymbol{f2(x) = a \cdot \cos(n \cdot x)}$
- When prompted to create sliders for ***a*** and ***n***, click **OK**.
- Adjust the slider settings by hovering over a slider and pressing ctrl menu > **Settings**. For each slider, adjust the Step Size: **0.5** and Minimised: ☑, then click **OK**.
- Press menu > **Window/Zoom** > **Window Settings**. In the dialog box that follows, enter the following values.

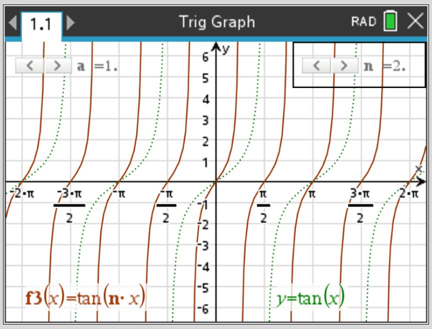
  XMin = $-13\pi/6$ XMax = $13\pi/6$ XScale: $\pi/2$
- Hover over an axis, press ctrl menu > **Attributes** and select **Multiple Labels**.
- Press ctrl menu > **Settings** > **Hide/Show**. Select **Show Lined Grid**.
- Systematically vary the slider values of a and n.

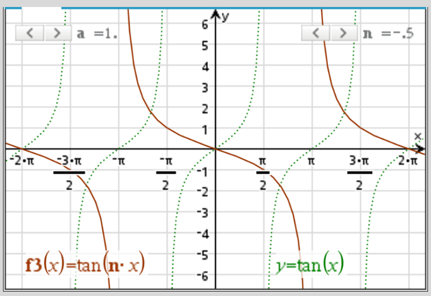




To graph $y = \tan(nx)$, deselect graphs $f1$ and $f2$, then:

- Enter $\boldsymbol{f3(x) = \tan(n \cdot x)}$, pressing trig to select **tan**.
- Observe the effect of varying the slider value of n.





Answer: The parameters a and n are dilations of the graph of $y = \sin(x)$ or $y = \cos(x)$ of scale factors a from the x-axis and $\frac{1}{n}$ from the y-axis, respectively. The magnitude of a determines the amplitude. If $a < 0$ the graph is reflected in the x-axis. The period is given by $\frac{2\pi}{n}$. If $n < 0$ the graph is reflected in the y-axis, illustrating that $\sin(-x) = -\sin(x)$. For $y = \tan(nx)$, n is a dilation of $y = \tan(x)$ by a factor of $\frac{1}{n}$ from the y-axis. The period is given by $\frac{\pi}{n}$. If $n < 0$ the graph is reflected in the y-axis, showing $\tan(-x) = -\tan(x)$.

Exploring translations of sine, cosine and tangent graphs

Question

Explore and interpret the effect of changing the parameters b and c for the graph of $y = af\left(n\left(x - \frac{b\pi}{6}\right)\right) + c$, where f is sin, cos or tan.

Solution

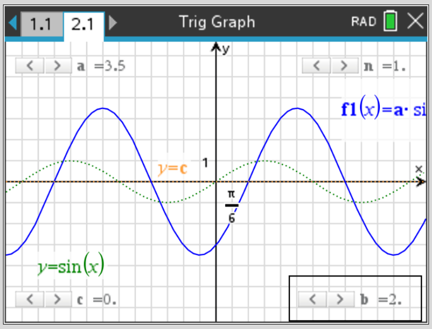
To add the parameters b and c, add a copy of the previous problem:

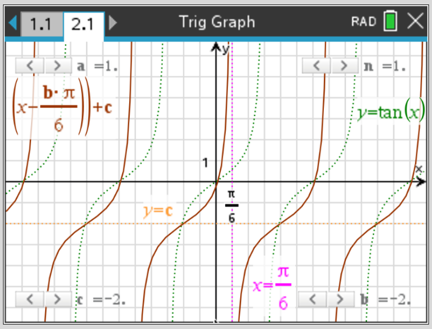
- Press ctrl ▲ then ▲. Press ctrl C ctrl V (cut/paste) enter.
- Deselect $f3$. Enter $\boldsymbol{f1(x) := a \cdot \sin(n \cdot (x - b \cdot \pi / 6)) + c}$.
- When prompted to create sliders for ***b*** and ***c***, click **OK**.
- Adjust slider settings for b to Min.: **–12**, Max.: **12**.
- Edit the $\frac{\pi}{2}$ label on the x-axis to $\frac{\pi}{6}$.
- Systematically vary the values of b and c.

Similarly, to observe the effect on cosine and tangent graphs:

- Deselect $f1$. Enter $\boldsymbol{f2(x) = a \cdot \cos(n \cdot (x - b \cdot \pi / 6)) + c}$.
- Deselect $f2$. Enter $\boldsymbol{f3(x) = \tan(n \cdot (x - b \cdot \pi / 6)) + c}$.

Answer: If $b > 0$, the graph is translated $\frac{b\pi}{6}$ units right. If $b < 0$, graph is translated $\frac{b\pi}{6}$ units left. If $c > 0$, graph is translated c units up. If $c < 0$, graph is translated c units down.





Solving trigonometric equations graphically and non-graphically

Question

Determine the coordinates of the points of intersection of the graphs of the functions with rules $f(t) = 4\sin\left(t - \frac{\pi}{3}\right)$ and $g(t) = -2\cos\left(2\left(t + \frac{\pi}{6}\right)\right)$ for $t \in [-\pi, 2\pi]$, correct to two decimal places.

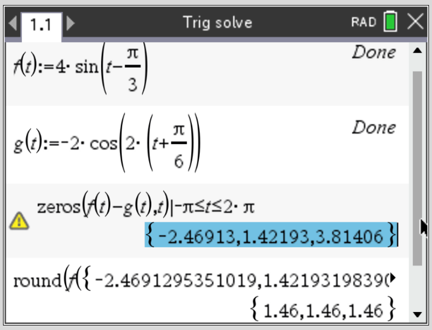
Solution

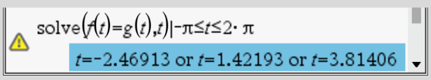
To find coordinates using **Zeros**, on a **Calculator** page:

- Enter $\boldsymbol{f(t) = 4\sin(t - \pi/3)}$, $\boldsymbol{g(t) = -2\cos(2(t + \pi/6))}$
- Press menu > **Algebra** > **Zeros**.
- Enter $\boldsymbol{\text{zeros}(f(t) - g(t), t) | -\pi \le t \le 2\pi}$.
- Press 📖 1 S, select **round** and enter **round(*f*(ans),2)**,

To find 'x-coordinates' using **Solve**, on the **Calculator** page:

- Press menu > **Algebra** > **Solve**. Enter $\boldsymbol{\text{solve}(f(t) = g(t), t) | -\pi \le t \le 2\pi}$.





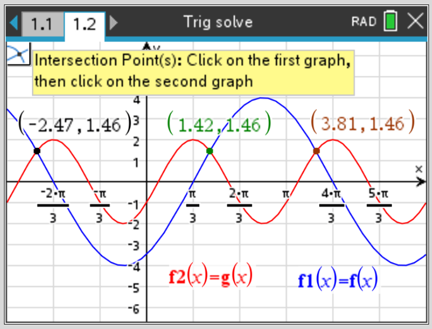
... continued

Solution (continued)

To find coordinates graphically, add a **Graphs** page, then:

- Enter $f1(x)=f(x)$ and $f2(x)=g(x)$.
- Edit x-axis endpoints to $-\pi$, 2π and x-axis tick label to $\pi/3$.
- Hover over x-axis, then press ctrl menu > **Attributes**. Select **Multiple Labels**.
- Press menu > **Geometry** > **Points & Lines** > **Intersection Points**. Click the graphs, then press esc to exit the tool.

Answer: (–2.47, 1.46), (1.42, 1.46) and (3.81, 1.46).



Modelling day length through the year with a cosine function

Question

For Melbourne, the daylight length, L_M hours, t days from the start of the year (so $t=1$ represents 1st January), can be modelled by the function $L_M(t)=a\cos\left(\frac{2\pi}{365}(t-b)\right)+c$.

For the shortest day, L_M = 9.42 hours at t = 172 (21st June, winter solstice).

For the longest day, L_M = 14.57 hours at t = 355 (21st December, summer solstice).

The day length for Reykjavík, Iceland, is modelled by $L_R(t)=7.78\cos\left(\frac{2\pi}{365}(t-172)\right)+12$.

For the following, give the answer correct to two decimal places, unless otherwise stated.

(a) State the statistical range of L_M. Hence determine the values of the parameters a and c.

(b) Determine the value of the parameter b (to the nearest integer) and interpret the result.

(c) Draw the graphs of L_M and L_R for $t\in[1,365]$, showing maximum/minimum values.

(d) For each city, determine the day length on **(i)** 1st January and **(ii)** 30th April (day 120).

(e) For each city, determine the interval over which the day length is less than 10 hours.

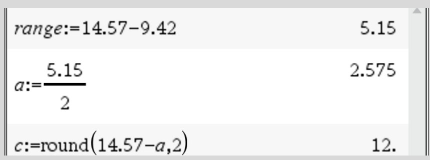
(f) Find t for the equinoxes and interpret the relationship to inflection points of the graphs.

Solution

(a) To determine range, a and c, on a **Calculator** page:

- Enter **range:=14.57 – 9.42**, followed by ***a*:=ans/2**.
- Press 📖 1 S, select **round**. Enter **round(14.57 – *a*,2)**.

Answer: Range: 5.15 hours, a = 2.58 and c = 12.00 (2 d.p.)



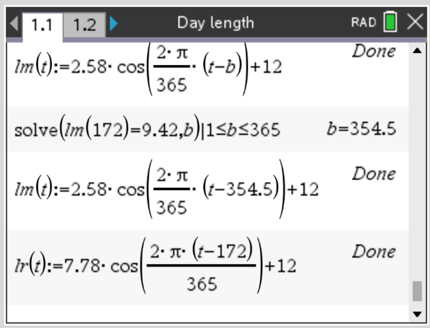
(b) To determine the value of b:

- Enter $lm(t):=2.58\cos(2\pi/365(t-b))+12$.
- Press menu > **Algebra** > **Solve**.
- Enter $\text{solve}(lm(172)=9.42,b)|1\le b\le 365$.

Update L_M, L_R:

- Enter $lm(t):=2.58\cos(2\pi/365(t-354.5))+12$.
- Enter $lr(t):=7.78\cos(2\pi/365(t-172))+12$.

Answer: b = 354.5 is approximate phase shift (21 Dec).



... continued

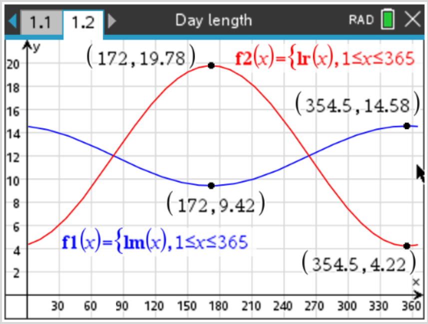
Solution (continued)

(c) To graph L_M and L_R, add a **Graphs** page, then:

- Enter $\boldsymbol{f1(x) = lm(x) \mid 1 \le x \le 365}$.
- Enter $\boldsymbol{f2(x) = lr(x) \mid 1 \le x \le 365}$.
- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values.

  XMin = –20 XMax = 370 XScale = 30
  YMin = –2 YMax = 22 YScale = 2

- Press [menu] > **Analyse Graph** > **Maximum** or **Minimum**.
- Click the graph then click left and right of the max./min.



Answer:

Melbourne: Max 14.58 h, $t \approx 355$; Min 9.42 h, $t = 172$.

Reykjavík: Max 19.78 h, $t = 172$; Min 4.22 h, $t \approx 355$.

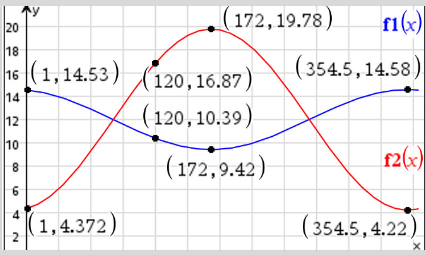
(d) To determine the day length on t = 1, 120, on page 1.1:

- Enter $\boldsymbol{lm(t) \mid t = \{1,120\}}$ then $\boldsymbol{lr(t) \mid t = \{1,120\}}$.

$lm(t)\mid t=\{1,120\}$	$\{14.5296, 10.3864\}$
$lr(t)\mid t=\{1,120\}$	$\{4.37195, 16.8657\}$

To confirm the day length graphically, on page 1.2:

- Press [P] > **Point**. Click each graph twice, then press [esc].
- For each graph, edit the x-coordinates to **0** and **120**.



Answer: 1 Jan: Melbourne: 14.53 h; Reykjavík: 4.37 hours.
30 Apr.: Melbourne: 10.39 hours; Reykjavík: 16.87 hours.

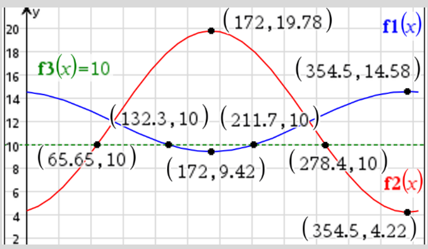
(e) To determine the required interval, on page 1.1:

- Press [menu] > **Algebra** > **Solve**.
- Enter $\boldsymbol{\text{solve}(lm(t) = 10, t) \mid 1 \le t \le 365}$, then enter $\boldsymbol{\text{solve}(lr(t) = 10, t) \mid 1 \le t \le 365}$

$\text{solve}(lm(t)=10,t)\mid 1\le t\le 365$
$t=132.278$ or $t=211.722$
$\text{solve}(lr(t)=10,t)\mid 1\le t\le 365$
$t=65.6469$ or $t=278.353$

To confirm the intervals graphically, on page 1.2:

- Enter $\boldsymbol{f3(x) = 10}$, press [menu] > **Geometry** > **Points & Lines** > **Intersection Points**. Click the graphs, then [esc].



Answer: Melbourne: $L_M \in [133, 211]$ (13 May to 30 July).

Reykjavík: $L_R \in [1, 65] \cup [279, 365]$ (6 Oct to 6 Mar)

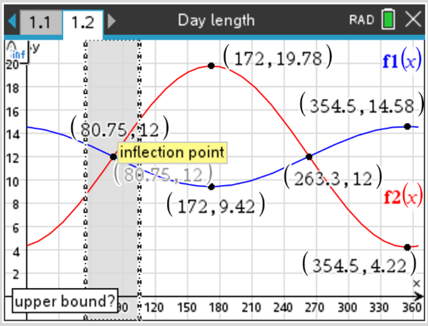
(f) To find t for equinoxes (day = night = 12 h), on page 1.1:

- Press [menu] > **Algebra** > **Zeros**.
- Enter $\boldsymbol{\text{zeros}(lm(t) - 12, t) \mid 1 \le t \le 365}$, then enter $\boldsymbol{\text{zeros}(lr(t) - 12, t) \mid 1 \le t \le 365}$.

$\text{zeros}(lm(t)-12,t)\mid 1\le t\le 365$	$\{80.75, 263.25\}$
$\text{zeros}(lr(t)-12,t)\mid 1\le t\le 365$	$\{80.75, 263.25\}$

... continued

Solution (continued)



To confirm the relationship between equinoxes and the inflection points, on page 1.2:

- Press menu > **Analyse Graph** > **Inflection**. Click graph ***f*** **1** then click the lower and upper bounds for each inflection point. Repeat for graph ***f*** **2**.

Answer: The model predicts that the equinoxes occur on approximately day 81 (March 22) and day 264 (Sept 21). It is an inflection as around that time of the year, the length of day goes from changing more quickly to more slowly, or vice versa.

Note*: It is important for students to recognise that this model contains approximations and roundings meaning the values found may be slightly inaccurate.*

3.1.4 Newton's method for finding numerical roots of a polynomial

Implementing pseudocode for Newton's method in the Calculator application

Question

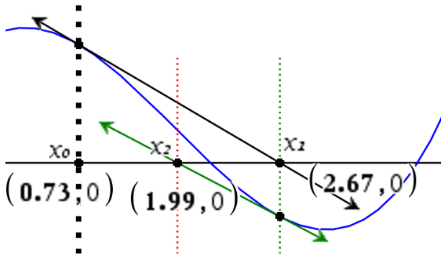
A student writes a simplified version of pseudocode for Newton's method, as shown below.

1. **Define**
define *function* $f(x)$
define *derivative* $f'(x)$
2. **User inputs**
$\boldsymbol{xv} \leftarrow$ starting x value
$\boldsymbol{n} \leftarrow$ max. iterations

3. **For** k from 1 to n

$$xv \leftarrow xv - \frac{f(x)\,|\,x = xv}{f'(x)\,|\,x = xv}$$

print k, xv
end for



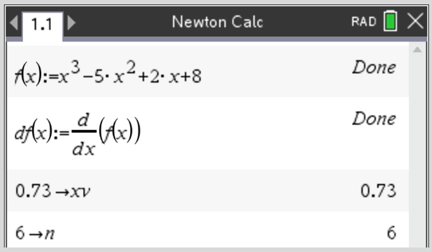
(a) Implement this pseudocode by recursively generating successive results in the Calculator application. Use the inputs $f(x) = x^3 - 5x^2 + 2x + 8$, $xv = 0.73$ and $n = 6$.
Give the result of the sixth iteration correct to six decimal places.
(b) Compare the above result with the exact roots of the function.
(c) Suggest improvements to the pseudocode to make it more robust.

Solution

(a) To assign the user inputs, on a **Calculator** page:

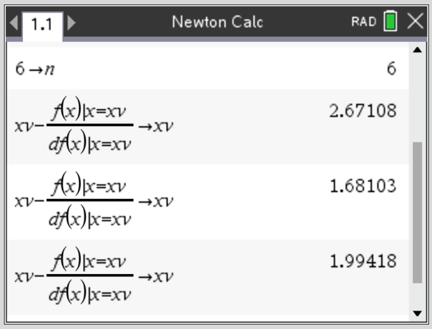
- Enter $\boldsymbol{f(x) := x^3 - 5x^2 + 2x + 8}$.
- Enter $\boldsymbol{df(x) := \frac{d}{dx}(f(x))}$ by pressing [shift] [−] for the derivative template. Alternatively, press [templates] to select it.
- Enter $\boldsymbol{0.73 \rightarrow xv}$ and $\boldsymbol{6 \rightarrow n}$ by pressing [templates] [var] ([sto→]) for the **store** symbol. This has the same effect as **assign**.

Note: Storing the value of ***n*** *is not needed here but is included to simulating the steps in the pseudocode.*



To recursively generate successive approximations:

- Key in $\boldsymbol{xv - \frac{f(x)\,|\,x = xv}{df(x)\,|\,x = xv} \rightarrow xv}$ by pressing [ctrl] [=] to select the **given** symbol, |.
- Press [enter] repeatedly up generate successive iterations.

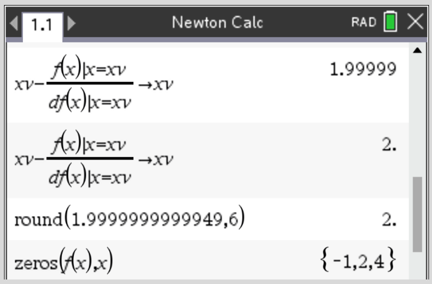


Answer: Successive results rapidly converge towards $x = 2$.

(b) To compare the previous results with the exact roots:

- Press [menu] > **Algebra** > **Zeros** and enter $\boldsymbol{\text{zeros}(f(x), x)}$.

Answer: The roots of f occur at –1, 2 and 4. With an initial guess of 0.73, the algorithm converges to the root $x = 2$.



(c) Answer: Improvements could include: using a *While* loop with a tolerance (to halt execution once a desired accuracy is achieved). Also, a check for division by zero (or near-zero) would be useful.

TEXAS INSTRUMENTS

Using the Programme Editor to implement pseudocode for Newton's method

Question

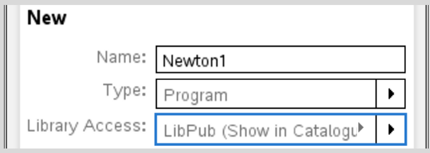
Use the Programme Editor application to implement the pseudocode from the previous problem for Newton's method algorithm, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, as working code. Test the code with the following user inputs and give the result of the final iteration correct to six decimal places.

Use $f(x) = x^3 - 5x^2 + 2x + 8$ **(a)** Starting value 0.73 and $n = 6$ **(b)** Starting value 3.12 and $n = 20$.

Solution

To start coding, in a new **Problem** or a new **Document**:

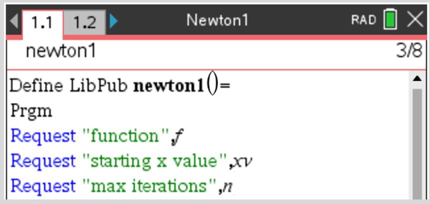
- Select **Add Programme Editor > New**.
- In the dialog box that follows, enter as shown.



The **Program Editor** will follow, ready to accept the code.

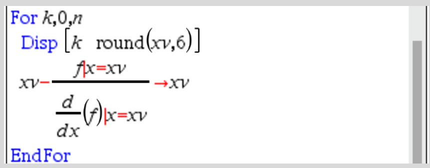
To add code that requests user inputs:

- Press [menu] > **I/O > Request**, then press [?!▸] to select quotation marks **"** and enter **Request "function",** ***f***.
- Similarly, enter **Request "starting *x* value",** ***xv*** and **Request "max iterations",*n***, as shown.



To instruct repetition for ***n*** iterations using a **For** loop:

- Press [menu] > **Control > For … End For** and enter **For *k*,0,*n***, as shown.
- Press [menu] > **I/O > Disp**, then press [template] to select the 1×2 matrix template and enter **Disp [*k* round(*xv*,6)]**.
- Enter the next line as shown by pressing [ctrl][=] to select the **given |** symbol, [⇧shift][−] for the derivative template, and [template][var]([sto→]) for the **store** symbol, →.

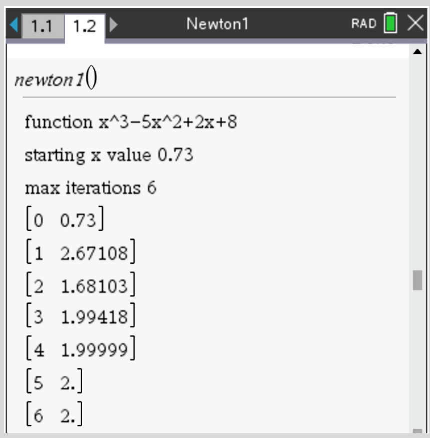


To instruct clearing variables and storing the program:

- Below the **End For** command, press [catalog][1][D], select **DelVar** and enter **DelVar *xv*,*f*,*n***, as shown.
- Press [menu] > **Check Syntax & Store > Check Syntax & Store** (or [ctrl][B]).

To run the program for $f(x) = x^3 - 5x^2 + 2x + 8$, $n = 6$ and starting x values **(a)** 0.73, **(b)** 3.12, on a **Calculator** page:

- Press [ctrl][R] then [enter]. In the dialog boxes that follow, enter the following:
  (a) function: $\mathbf{x^3 - 5x^2 + 2x + 8}$, starting x value (xv): **0.73**, and max iterations (n): **6**
  (b) Repeat for $xv = 3.12$, $n = 20$.



Answer:

(a) With $x_0 = 0.73$, convergence to $x = 2$.

(b) With $x_0 = 3.12$, convergence to $x = 4$.

Implementing pseudocode for Newton's method as working code in Python

Question

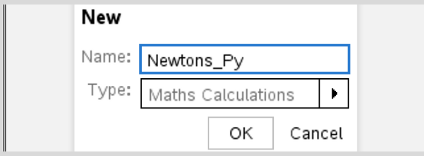
(a) Implement the pseudocode from the previous problem in the Python application.

(b) Modify the code to include: **(i)** a *While* loop to achieve a desired level of accuracy, **(ii)** a check for division by near zero. Test the code using $f(x)=x^3-5x^2+2x+8$ and $f(x)=x^3-9$.

Solution

(a) To start coding, in a new **Document** (or a new **Problem**):

- Select **Add Python > New**.
- In the dialog box that follows, enter as shown.



*Note: The **Python** commands to be used can be accessed by pressing* [menu] *> **Built-ins** then -*

*> **Function** for: '**def**' (define function) and '**return**'.*

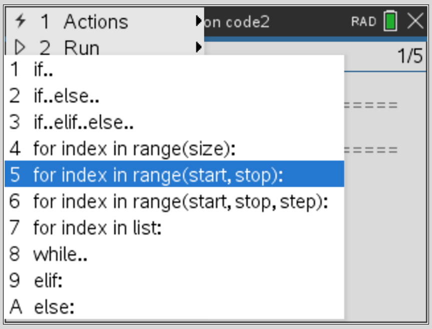
*> **Control** for: '**while**', '**if**' and '**for** index in range(start,stop)'*

*> **Type** for: '**float**', '**int**' and '**round**'*

*> **I/O** for: '**input**' and '**print**'.*

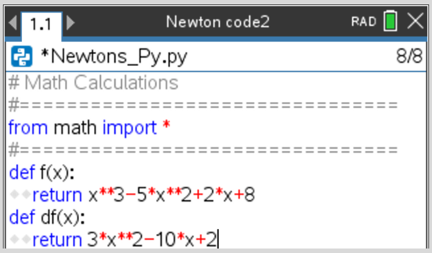
***Text** in quotation marks: press* [?!▸] *to select ".*

***Indentation**: ensure correct indentation. Press* [tab] *to indent.*



To define $f(x)=x^3-5x^2+2x+8$ and $f'(x)$:

- Enter **def *f*(*x*):** then enter **return** $\boldsymbol{x^3-5\times x^2+2\times x+8}$
- Enter **def *df*(*x*):** then enter **return** $\boldsymbol{3\times x^2-5\times x+2}$



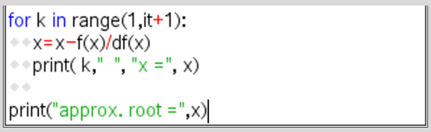
***Note**: Use the* [×] *key for multiplication and* [×] [×] *for exponentiation. Output will appear as * for multiplication, and as ** for exponentiation, as shown right.*

To request user input for the initial guess, ***x0*** and ***iterations***:

- Enter ***x*** **= float(input("*x0*: "))**. For a floating-point value.
- Enter ***it*** **= int(input("iterations: "))**. For an integer value.
- Enter **print(0, " ","*x* =",*x*)** (to display initial value).

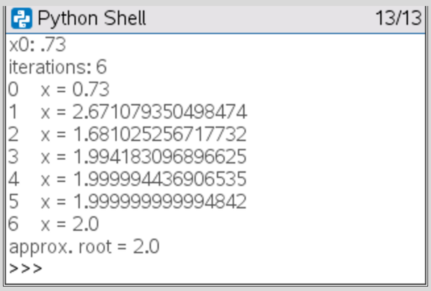
To instruct repetition for ***it*** iterations using a **for** loop (using ***k*** as the index), and display the output of the iterations:

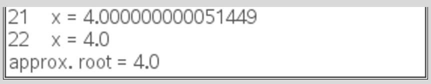
- Enter **for *k* in range(1,*it*+1):**
- Enter $\boldsymbol{x=x-f(x)/df(x)}$ then enter (indents as shown) **print(k, " ","*x* =",*x*)** and **print("approx. root=",*x*)**



To run the program (and to check syntax):

- Press [menu] **> Run > Run** (or [ctrl] [R]).
- In the **Python Shell** page that follows, enter initial guess, e.g. ***x*0: 0.73**, and number of iterations, e.g. **iterations: 6**.





Answer: (i) $x_0=0.73$, $x=2.0$, **(ii)** $x_0=3.12$, $x=4.0$.

... continued

Solution (continued)

(b) (i) To modify the code to include a **while** loop to set desired accuracy, save the document under a different name, then:

- Between lines 9 and 10, insert a tolerance input (accuracy level) by entering ***tol* = float(input("tol: "))**.
- Enter ***k* = 1** after line 12 (i.e. after **print(0, " ",…)**) to start the iteration count.
- In the next line, replace the **for** statement with **while *k* < *it* and *f* (*x*) > tol or *f* (*x*) < – tol:**
- Enter ***k* = *k*+1** after line 16 (as shown) to count iterations.
- Edit the last line to **print("approx. root=",round(*x*,6))**

```
1.1  1.2  Newton code3                 RAD
*Newtons_Py.py                         5/18
def f(x):
    return x**3-5*x**2+2*x+8
def df(x):
    return 3*x**2-10*x+2
x=float(input("x0: "))
tol=float(input("tol: "))
it=int(input("iterations: "))
print( 0,"  ", "x =", x)
k=1
```

```
while k<it and f(x)>tol or  f(x)<-tol:
    x=x-f(x)/df(x)
    print( k,"  ", "x =", x)
    k=k+1
print("approx. root =",round(x,6))
```

*Note: Boolean operators, including **and**, **or** etc. can be typed or selected from* menu *> **Built-ins** > **Ops**.*

To test the code for initial guess, ***x0* = 1.52**, tolerance = **0.000001 = 10^{-6}** (i.e. press **1** EE **–6**) and max. iterations = **50**:

- Press ctrl R. In the **Python Shell** page that follows, enter ***x*0: 1.52**, **tol: 0.000001** (or **1** EE **–6**), **iterations: 50**

```
x0: 1.52
tol: 0.000001
iterations: 50
0   x = 1.52
1   x = 1.998529862174579
2   x = 1.999999641017132
3   x = 1.999999999999979
approx. root = 2.0
```

Answer: The desired accuracy is reached in 3 iterations.
To test the code for initial guess, ***x0* = 3.12**, tolerance = **0.000001 = 10^{-6}** and maximum iterations = **50**:

- Press ctrl R. Enter ***x*0: 3.12**, **tol: 1E–6**, **iterations: 50**

```
17   x = 4.387745972323216
18   x = 4.073618348170211
19   x = 4.003506462963762
20   x = 4.000008573202466
21   x = 4.000000000051449
approx. root = 4.0
```

Answer: The desired accuracy is reached in 21 iterations.

(b) (ii) To modify the code to check for division by near zero (say, $\mathbf{0 \pm 10^{-8}}$), after the **while** statement (i.e. after line 14):

- Enter **if 1E–8 < *df*(*x*) < 1E–8:** (taking note of indentation).
- Enter **print("error at *x*= ", *x*)**), then enter **break**.
- Enter **if –tol <= f(x)<=tol:** (as a new block) after line 20 (as shown), pressing ctrl = (≠≥▸) for <= symbol.
- Indent the last line as shown by pressing tab.

```
def f(x):
    return x**3-9
def df(x):
    return 3*x**2
x=float(input("x0: "))
tol=float(input("tol: "))
it=int(input("iterations: "))
print( 0,"  ", "x =", x)
k=1
```

```
while k<it and f(x)>tol or  f(x)<-tol:
    if -1E-8<df(x)<1E-8:
        print("error at x=",x)
        break
    x=x-f(x)/df(x)
    print( k,"  ", "x =", x)
    k=k+1
if -tol<=f(x)<=tol:
    print("approx. root =",round(x,6))
```

To test the code using $f(x) = x^3 - 9$, with ***x0* = 0**, tolerance = $\mathbf{10^{-6}}$(i.e. press **1** EE **–6**) and max. iterations = **50**:

- Edit lines 6 and 8 to **return** $\mathbf{x^3 - 9}$ and **return** $\mathbf{3 \times x^2}$.
- Press ctrl R. Enter ***x*0: 0**, **tol: 1E–6**, **iterations: 50**

```
x0: 0
tol: 1E-6
iterations: 50
0   x = 0.0
error at x= 0.0
```

Answer: Returns error message warning of unreliable result.
To test using $f(x) = x^3 - 9$, with (i) $x0 = 0.1$, (ii) $x0 = 1.5$, tolerance = $10^{-6}$and max. iterations = 50:

- Press ctrl R. Enter *x*0: 0.1, tol: 1E–6, iterations: 50
- Press ctrl R. Enter *x*0: 1.5, tol: 1E–6, iterations: 50

```
16   x = 2.08009884616336
17   x = 2.080083823160405
approx. root = 2.080084
```

```
x0: 1.5
tol: 1E-6
iterations: 50
0   x = 1.5
1   x = 2.333333333333333
2   x = 2.106575963718821
3   x = 2.080415589606098
4   x = 2.080083875956332
approx. root = 2.080084
```

Answer: $x = 2.080084$. Required accuracy in (i) 17 iterations for $x0 = 0.1$, (ii) 4 iterations for $x0 = 1.5$.

3.2 Combinations of functions

3.2.1 Composite functions

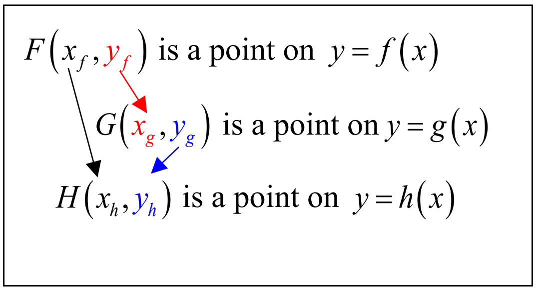
Visualising composition of functions through a coordinate geometry approach

Question

Let $h(x)=g(f(x))=(g\circ f)(x)$.

The diagram illustrates what happens to the coordinates of a point *F* on the graph of *f* when operated on by *g*.

Use a coordinate geometry approach to dynamically explore the graph, rule, domain and range of $h(x)$.



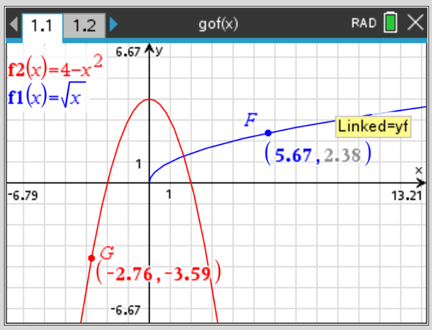
(a) Explore the locus of $h(x)$ if $f(x)=\sqrt{x}, x\ge 0$ and $g(x)=4-x^2, x\in R$.

(b) Find the rule, domain and range of $h(x)=g(f(x))$.

Solution

(a) To set up the composition of *g* and *f*, on a **Graphs** page:

- Enter $\boldsymbol{f1(x)=\sqrt{x}}$ and $\boldsymbol{f2(x)=4-x^2}$.
- Press [P] > **Point**, click graphs $f1$ and $f2$ then [esc].
- Hover over the point on $f1$, press [ctrl] [menu] > **Label** and enter label ***F***. Similarly, label the other point ***G***.
- Hover over the *x*-coordinate of *F*, press [ctrl] [menu] > **Store** and enter ***xf***. Similarly, store the *y*-coordinate as ***yf***. Repeat for point *G*, storing the coordinates as ***xg*** and ***yg***.

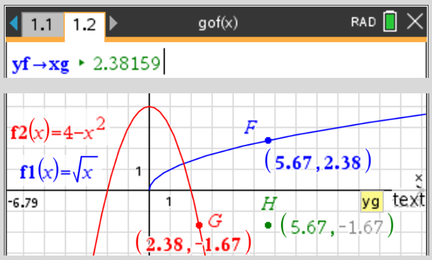


To use the output of *f* as the input of *g*, add a **Notes** page:

- Press [ctrl] [M]. Enter $\boldsymbol{yf\to xg}$ by pressing [ctrl] [var] ([sto→]).

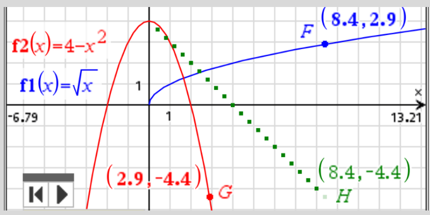
To create a point ***H***(***xf***, ***yg***), on page 1.1:

- Press [P] > **Point by Coordinates**. Enter ***xf*** and ***yg*** as the coordinates then press [esc] to exit the point tool.
- Hover over the point, press [ctrl] [menu] > **Label**. Enter ***H***.



To observe the locus of point *H*:

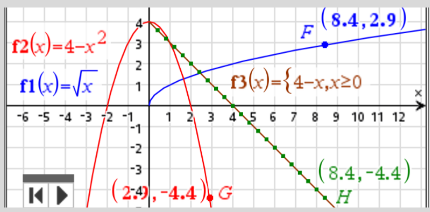
- Hover over point *H*, press [ctrl] [menu] > **Geometry Trace**.
- Drag or animate point *F* along curve. Press [esc] to exit.



(b) To draw a continuous graph for the locus of point *H*:

- Enter $\boldsymbol{f3(x)=4-x\,|\,x\ge 0}$.

Answer: The function with rule $h(x)=4-x$ contains all points of the trace of point *H*. dom *h* = dom *f* = $[0,\infty)$ and ran *h* = $(-\infty,4]$.



Analysing the composition of functions involving e^x and sin(x)

Question

Consider the functions $[-2\pi, 2\pi] \to R,\ f(x) = \sin(x)$ and $R \to R,\ g(x) = e^x$.

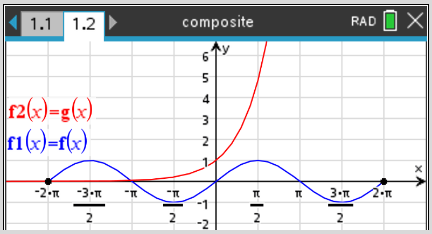
(a) Determine whether the following are define **(i)** $f \circ g$, **(ii)** $g \circ f$, **(iii)** $f \circ f$.

(b) Graph the defined composite function(s) and state the rule, domain and range.

Solution

(a) To set up analysis of f and g, on a **Calculator** page:

- Enter $\boldsymbol{f(x) := \sin(x) \mid -2\pi \le x \le 2\pi}$ and $\boldsymbol{g(x) := e^x}$, using the [trig] and [e^x] keys for **sin** and ***e***.
- Add a **Graphs** page. Enter $\boldsymbol{f1(x) = f(x),\ f2(x) = g(x)}$.
- On the x-axis, click on and edit endpoints to $\boldsymbol{-5\pi/2}$ and $\boldsymbol{5\pi/2}$, and edit the tick label to $\boldsymbol{\pi/2}$, as shown.

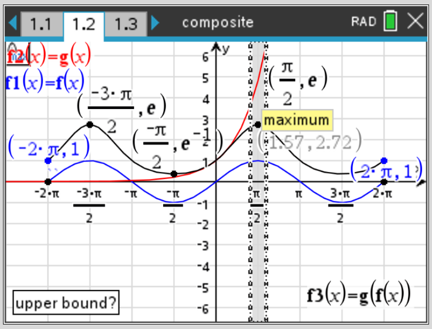


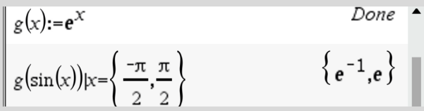
Answer: Defined: $g \circ f$ and $f \circ f$. Not defined: $f \circ g$.

$\text{dom} f = [-2\pi, 2\pi]$	$\text{ran} f = [-1, 1]$	$\text{ran}\, g \not\subset \text{dom} f, \text{ran} f \subseteq \text{dom} f$
$\text{dom}\, g = R$	$\text{ran}\, g = R^+$	$\text{ran} f \subseteq \text{dom}\, g$

(b) To graph $g \circ f$ and $f \circ f$, make a copy of page 1.2:

- Press [ctrl] ▲ then [ctrl] [C] & [ctrl] [V] to copy/paste and [enter].
- On page 1.2, enter $\boldsymbol{f3(x) = g(f(x))}$.
- Press [P] > **Point** and click the graph at 5 different points.
- Edit their x-coordinates to: **−2π, −3π/2, −π/2, π/2, 2π**.
- Press [menu] > **Analyse Graph** > **Maximum/Minimum** to confirm location of max./min. stationary points.



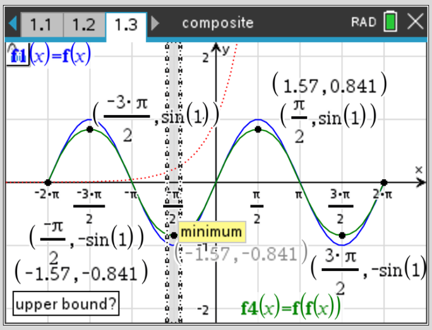


Answer: $(g \circ f)(x) = e^{\sin(x)}$. $\text{dom}\,(g \circ f) = \text{dom} f = [2\pi, 2\pi]$.

$\text{ran}\,(g \circ f) = \left[e^{\sin(-\pi/2)}, e^{\sin(\pi/2)}\right] = \left[e^{-1}, e^{1}\right]$.

To graph $f \circ f$, on page 1.3:

- Enter $\boldsymbol{f4(x) = f(f(x))}$.
- Press [P] > **Point** and click the graph at 4 different points.
- Edit their x-coordinates to: **−3π/2, −π/2, π/2, 3π/2**
- Adjust **Window Settings** as desired to enlarge the graphs.
- Press [menu] > **Analyse Graph** > **Maximum/Minimum** to confirm location of max./min. stationary points.



Answer: $(f \circ f)(x) = \sin(\sin(x))$.

$\text{dom}\,(f \circ f) = \text{dom} f = [-2\pi, 2\pi]$.

$\text{ran}\,(f \circ f) = \left[\sin(\sin(-\pi/2)), \sin(\sin(\pi/2))\right] = [\sin(-1), \sin(1)]$

$\sin(-1) = -\sin(1)$. Range $= [-\sin(1), \sin(1)] \approx [-0.841, 0.841]$.

***Note**: The closeness of the graphs of $y = \sin(x)$ and $y = \sin(\sin(x))$ illustrates that $\sin(x) \approx x$ when x is small.*

3.2.2 Modelling with combined functions

Modelling the waveform of a musical note using addition of ordinates

Question

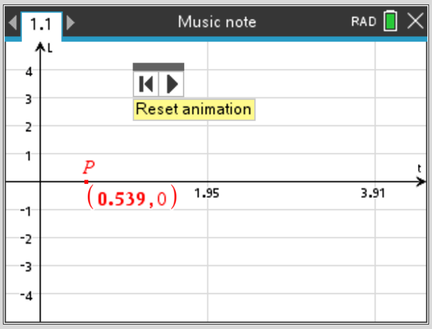
A fundamental musical note is modelled by $L_1(t) = \sin(0.256 \times 2\pi t)$, and the first and second harmonics by $L_2(t) = \sin(2(0.256 \times 2\pi t))$ and $L_3(t) = \sin(3(0.256 \times 2\pi t))$, where L is the relative loudness at time t milliseconds (ms) and $t \geq 0$.
A musical instrument emits a note represented by $L(t) = 3L_1(t) + L_2(t) + L_3(t)$.

(a) Use addition of ordinates to construct the graph of L over one cycle.
(b) Graph a continuous function that contains all points obtained by addition of ordinates.
(c) Find the range of L. Interpret why L and L_1 have the same period but different ranges.

Solution

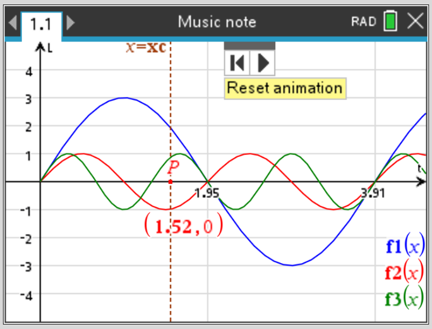
(a) To set up addition of ordinates, on a **Graphs** page:

- Press [P] > **Point**. Click the x-axis, then press [esc].
- Hover over the point, press [ctrl] [menu] > **Label**. Enter ***P***.
- Hover over point P, press [ctrl] [menu] > **Coordinates & Equations**. Hover over the x-coordinate, press [ctrl] [menu] > **Store**. Enter the variable name ***xc***.
- Animate point P by editing the x-coordinate to 0, then hover over P, press [ctrl] [menu] > **Attributes**. Select **Unidirectional animation speed** and press [1] [enter] [enter].



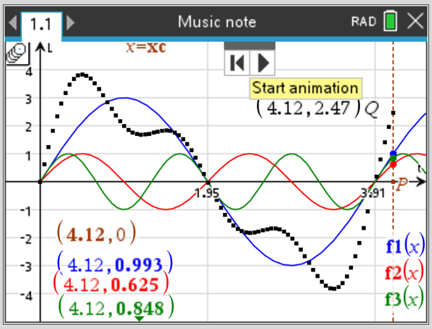
To graph $3L_1, L_2, L_3$ and a vertical line through point P:

- Enter $\boldsymbol{f1(x) = 3\sin(0.256 \cdot 2\pi x) \mid x \geq 0}$.
- Enter $\boldsymbol{f2(x) = \sin(2 \cdot 0.256 \cdot 2\pi x) \mid x \geq 0}$,
- Enter $\boldsymbol{f3(x) = \sin(3 \cdot 0.256 \cdot 2\pi x) \mid x \geq 0}$
- Press [menu] > **Graph Entry/Edit** > **Relation**. Enter ***x*** = ***xc***.
- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values.
  XMin = – 0.4 XMax = 4.5 XScale = 0.5/0.256
  YMin = – 5 YMax = 5 YScale = 1



To add the y-coordinates of $3L_1$, L_2, L_3 at $x = \boldsymbol{xc}$:

- Press [menu] > **Geometry** > **Points & Lines** > **Intersection Point(s)**. Click ***x*** = ***xc*** then a graph. Repeat for each graph.
- Press [esc]. Hover over the y-coordinate of an intersection point, press [ctrl] [menu] > **Store** and enter variable ***y*1**. Repeat for the other intersection points with variables ***y*2** and ***y*3**.
- Press [P] > **Point by Coordinates**. Enter (***xc***, ***y*1+ *y*2+ *y*3**).
- Hover over the point, press [ctrl] [menu] > **Label**. Enter ***Q***.
- Hover over point Q, press [ctrl] [menu] > **Geometry Trace** then start the animation to trace the locus of Q.



... continued

Solution (continued)

(b) To graph a continuous function for the trace of point Q, press ctrl G and then:

- Enter $f4(x) = f1(x) + f2(x) + f3(x)$.

Answer: The graph of $3L_1 + L_2 + L_3$ contains all points traced by Q, obtained by addition of ordinates, $y1 + y2 + y3$.

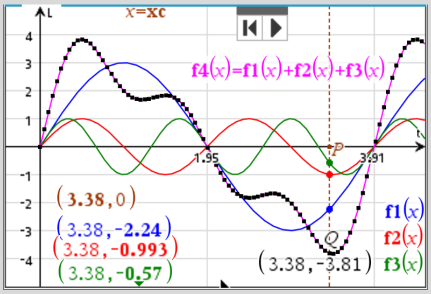
(c) To determine the range of L, add a **Calculator** page, then:

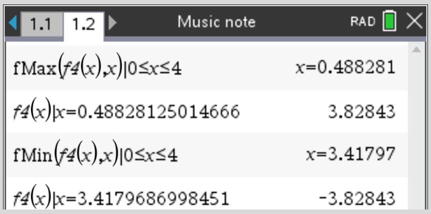
- Press menu > **Calculus** > **Function Maximum**. Enter $\textbf{fMax}(f4(x), x) \mid 0 \le x \le 4$, then $f4(x) \mid \textbf{ans}$. Similarly, enter $\textbf{fMin}(f4(x), x) \mid 0 \le x \le 4$, then $f4(x) \mid \textbf{ans}$.

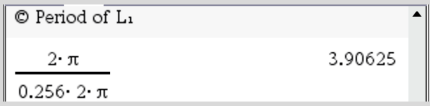
Answer: The range is approximately [3.83, –3.83] (2 d.p.)
To determine the period of L, on a **Calculator** page:

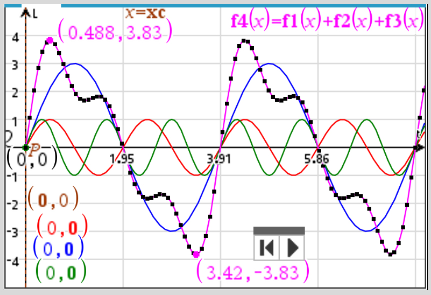
- Enter $2\pi / (0.256 \cdot 2\pi)$

Answer: The waveform, L, and fundamental, L_1, have the same period of approx. 3.91 ms. They share the same period because the harmonics, L_2 and L_3 are exact integer multiples, so the entire superposition 'lines up' every 3.91 ms. However, the harmonics change the **shape** of the repeating cycle. This becomes clearer by plotting multiple cycles, as shown.









Analysing the least upper bound in a modelling context

Question

The population of a cell culture, P, after t hours is modelled by $P(t) = 1200\left(1 + 8e^{-0.45t}\right)^{-1}, t \ge 0$.

(a) Find the initial cell population and the least upper bound, P_U.

(b) Graph P. On the same axes, graph $y = P_U$ and $y = 0.5P_U$.

Solution

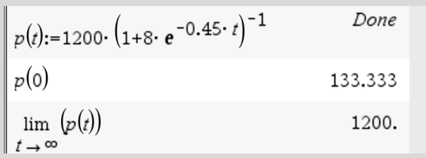
(a) To find the initial P and P_U, on a **Calculator** page:

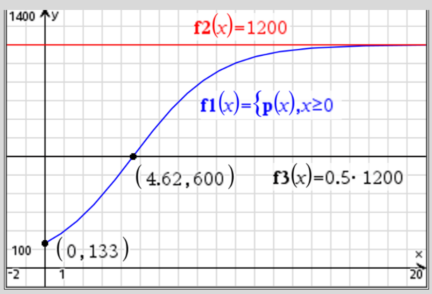
- Enter $\boldsymbol{p(t) := 1200\left(1 + 8e^{-0.45t}\right)^{-1}}$ using e^x, then $\boldsymbol{p(0)}$.
- Press the templates key, select **lim**. Enter $\lim\limits_{t \to \infty}(p(t))$, pressing π▸ for ∞.

(b) To graph P, add a **Graphs** page:

- Enter $\boldsymbol{f1(x) = p(x) \mid x \ge 0,\ f2(x) = 1200,\ f3(x) = 600}$.
- Click and edit the axes endpoints as shown.

Answers: Initial value: 133. $P_U = 1200$, being the population ceiling (or carrying capacity) predicted by the model.





Modelling damped oscillation using the product of two functions

Question

A small object is suspended from a spring and displaced from its equilibrium position. Due to friction, the object undergoes damped oscillations. The displacement s units from the equilibrium position at time t seconds is modelled by $s(t)=6e^{-(t/5)}\cos\left(\frac{\pi}{4}(t-2)\right), t\in[0,20]$.

(a) Graph the function, s, and its exponential envelope.

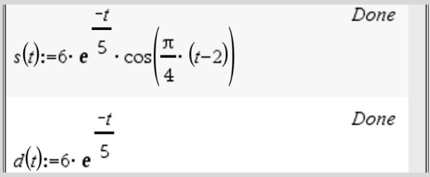
(b) Find the coordinates of the points of intersection of the graph of s and its envelope.

(c) Amplitude is governed by the exponential term, $d(t)=6e^{-(t/5)}$. Find the time, t s, taken for the amplitude to reduce to half its initial value.

Solution

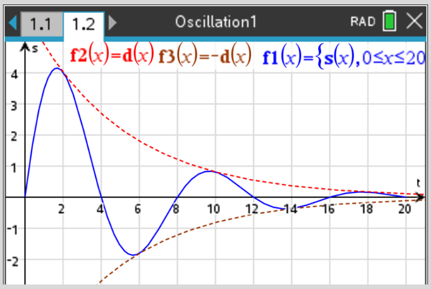
To set up the problem, on a **Calculator** page:

- Enter $s(t):=6e^{-t/5}\cdot\cos(\pi/4\cdot(t-2))$ and $d(t):=6e^{-t/5}$, using the [e^x] and [trig] keys to input ***e*** and **cos**.



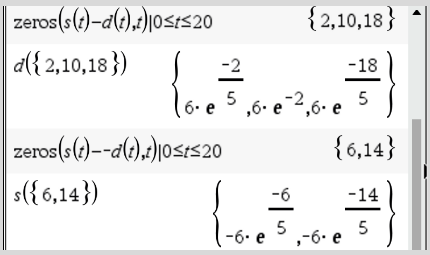
(a) To graph s and its envelope, add a **Graphs** page:

- Enter $f1(x)=s(x)|0\le x\le 20$, $f2(x)=d(x)$ and $f3(x)=-d(x)$.
- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values.
  XMin = –1 XMax = 21 XScale = 2
  YMin = –4 YMax = 5 YScale = 1



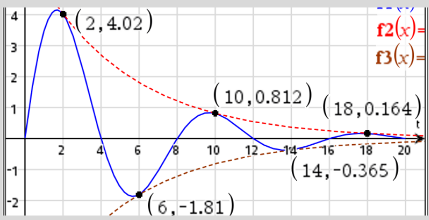
(b) To find the exact coordinates of intersection, on page 1.1:

- Press [menu] > **Algebra** > **Zeros**. For $s(t)\pm d(t)$, enter $\mathbf{zeros}(s(t)-d(t),t)|0\le t\le 20$, then $s(\mathbf{ans})$.
- Enter $\mathbf{zeros}(s(t)-(-d(t)),t)|0\le t\le 20$, as shown.



To find the coordinates graphically, on page 1.2:

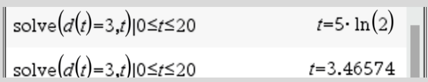
- Press [menu] > **Geometry** > **Points & Lines** > **Intersection Point(s)**. Click graph $f1$ and $f2$, $f1$ and $f3$, then [esc].



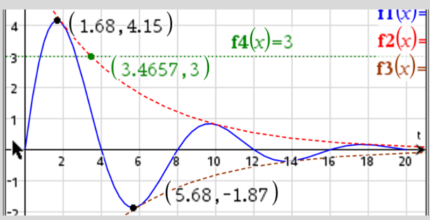
Answer:
$\left(2,6e^{-2/5}\right),\left(6,-6e^{-6/5}\right),\left(10,6e^{-2}\right),\left(14,-6e^{-14/5}\right),\left(18,6e^{-18/5}\right)$

(c) To find when the exponential term is halved, on page 1.1:

- Press [menu] > **Algebra** > **Solve**. Enter $\mathbf{solve}(d(t)=3,t)|0\le t\le 20$.



Answer: The exponential term is halved at $t=5\log_e(2)\approx 3.47$ seconds. Graphically, the point of intersection of $y=6e^{-t/5}$ and $y=3$ is approx. (3.47, 3).



Texas Instruments

3.3 Differentiation

3.3.1 Continuity, limits and differentiability

A function f is continuous at $x = a$ if it satisfies the following conditions:

- $f(a)$ exists,
- $\lim_{x \to a} f(x)$ exists and
- $\lim_{x \to a} f(x) = f(a)$

Investigating the behaviour of a function at x=a

Question

Consider the function f where $f(x) = \dfrac{x^2 + 3x + 2}{x + 1}$.

(a) Determine whether f is continuous for all real values of x?

(b) Plot the graph of f indicating the location of any discontinuities.

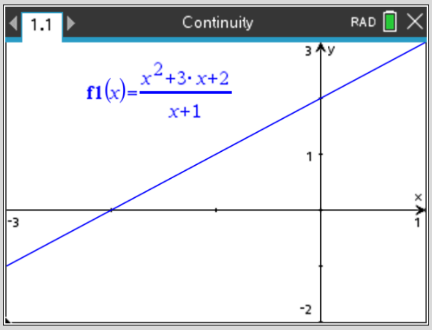
(c) Investigate $\lim_{x \to -1} f(x)$ using a numerical, a graphical and an algebraic approach.

Solution

(a) On a **Graphs** page:

- Enter $\boldsymbol{f1(x) = \dfrac{x^2 + 3x + 2}{x + 1}}$.
- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:

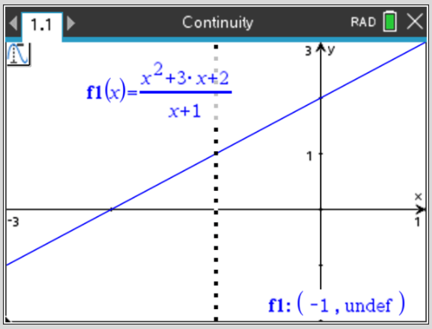
  XMin = –3   XMax = 1   XScale = 1
  YMin = –2   YMax = 3   YScale = 1



***Note**: For the purposes of this investigation, it will be helpful to set the Display Digits to Float. To do this, press menu > **Settings** and set **Display Digits** to **Float**, then press OK.*

- Press menu > **Trace** > **Trace Step** and edit its value to 0.1.
- Press menu > **Trace** > **Graph Trace**.
- Trace to $x = -1$ as shown.

A function value is not obtained from the graph at $x = -1$.



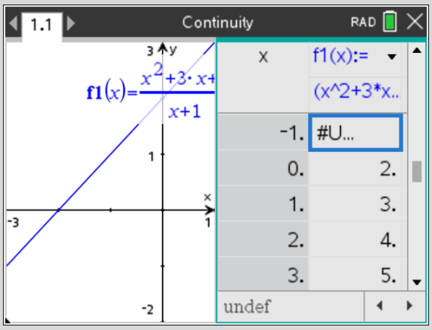
... continued

Solution (continued)

- Press ctrl T to add a table of values.
- Scroll up the table to $x = -1$.

An undefined output appears in the table of values or alternatively on a **Calculator** page.

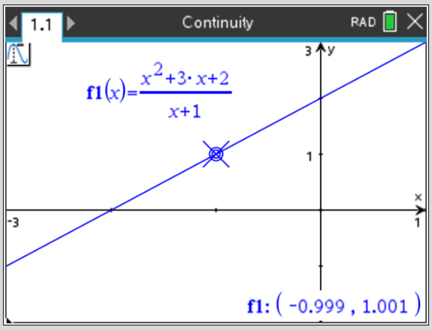
Notes: *Press ctrl tab to toggle between the graph and the table. Press ctrl T to remove and add the table of values.*



Answer: $f(-1)$ is undefined because $f(-1)$ gives the result $\frac{0}{0}$, which is not defined, and so the function is discontinuous at $x = -1$.

(b) Consider the graph of f.

- Press menu > **Trace** > **Trace Step** and edit its value to 0.001.
- Press menu > **Trace** > **Graph Trace**.
- Trace as shown.



Answer: The graph of f is linear with a hole (discontinuity) at $x = -1$.

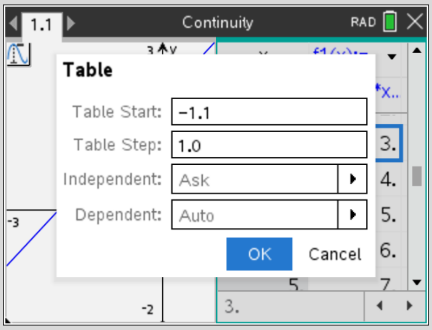
(c) Numerical and graphical approaches:

To investigate the behaviour of f as $x \to -1$, a table of values or a graph can be used.

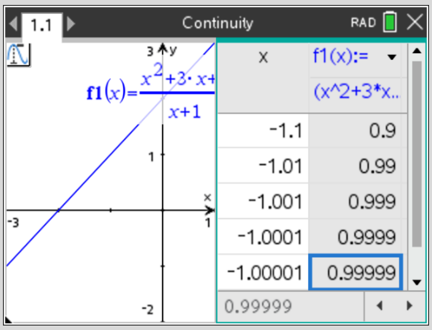
Investigate f as $x \to -1$ from below ($x < -1$) and above ($x > -1$).

From below:

- Press ctrl T to add a table of values.
- Press menu > **Table** > **Edit Table Settings**.
- Set the table as shown.
- Set **Independent** to **Ask** then click OK.



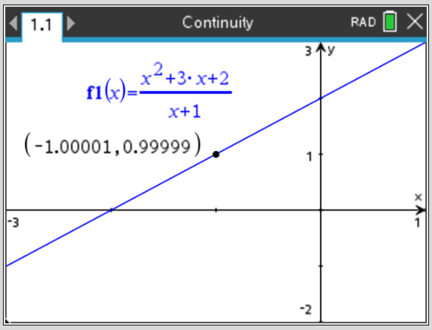
The table will be empty. Enter a sequence of values for x which get closer, but still less than $x = -1$. As the value of f gets closer to 1, this illustrates the limiting behaviour of the function as x approaches -1.



... continued

Solution (continued)

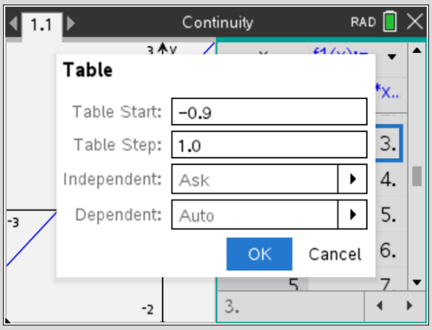
- Press ctrl T to remove the table of values.
- Press menu > **Geometry** > **Points & Lines** > **Point by Coordinates**.
- Enter **–1.00001** for the x-coordinate and press enter.
- Enter $\boldsymbol{f1(-1.00001)}$ for the y-coordinate and press enter.



The table of values and associated graph suggest that as $x \to -1$ from below, $f(x) \to 1$.

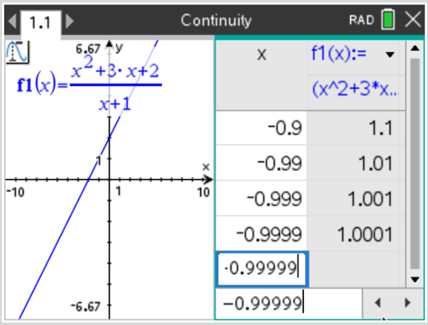
Hence it can be conjectured that $\lim\limits_{x\to -1^-} f(x) = 1$.

Before continuing, delete the point placed on the graph at $x = -1.00001$ by hovering the cursor over the point, pressing ctrl menu and selecting **Delete** from the pop-up menu.

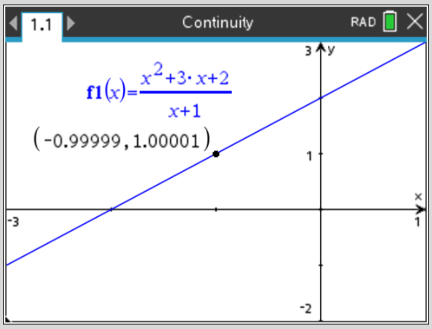


From above:

- Press ctrl T to add a table of values.
- Press menu > **Table** > **Edit Table Settings**.
- Set the table as shown.
- Set **Independent** to **Ask**.



- Press ctrl T to remove the table of values.
- Press menu > **Geometry** > **Points & Lines** > **Point by Coordinates**.
- Enter **–0.99999** for the x-coordinate and press enter.
- Enter $\boldsymbol{f1(-0.99999)}$ for the y-coordinate and press enter.



The table of values and associated graph suggest that as $x \to -1$ from above, $f(x) \to 1$.

Hence it can be conjectured that $\lim\limits_{x\to -1^+} f(x) = 1$.

Answer: $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^+} f(x) = 1$. As the limit is the same from above and below the x-value, $\lim\limits_{x\to -1} f(x) = 1$.

... continued

Solution (continued)

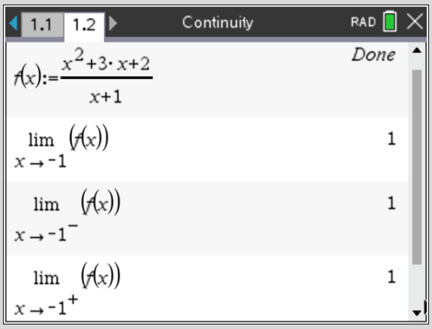
Algebraic approach:

$$\begin{aligned}\lim_{x\to -1} f(x) &= \lim_{x\to -1}\frac{x^2+3x+2}{x+1}\\ &= \lim_{x\to -1}\frac{(x+2)(x+1)}{x+1}\\ &= \lim_{x\to -1}(x+2) \qquad (x\neq -1 \text{ and so } x+1\neq 0)\\ &= -1+2\\ &= 1\end{aligned}$$

Hence $\lim\limits_{x\to -1}\frac{x^2+3x+2}{x+1}=1$.

On a **Calculator** page:

- Enter $f(x):=\frac{x^2+3x+2}{x+1}$.
- Press menu > **Calculus** > **Limit**.
- Enter as shown.



***Note**: TI-Nspire CX II CAS can be used to calculate a limit from below or above $x=a$. A negative direction value indicates a limit calculation from below $x=a$, whereas a positive direction value indicates a limit calculation from above $x=a$.*

Investigating the differentiability of a function at x=a

A function f is not differentiable at $x=a$ if the graph of f:

- is not continuous at $x=a$; or
- has a sharp point (cusp) at $x=a$; or
- has a vertical tangent line at $x=a$.

Question

Consider the function f where $f(x)=x^{\frac{1}{3}}$ and $x\in R$.

Determine whether f is differentiable at $x=0$.

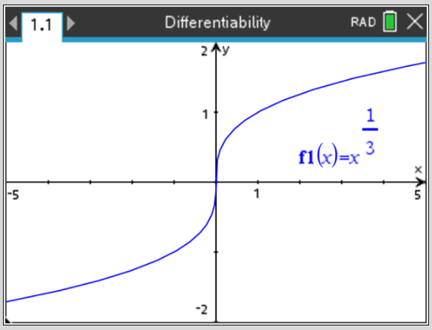
Solution

On a **Graphs** page:

- Enter $f1(x)=x^{\frac{1}{3}}$.
- Press menu > **Window/Zoom** > **Window Settings**.

... continued

Solution (continued)



In the dialog box that follows, enter the following values:

XMin = −5 XMax = 5 XScale = 1
YMin = −2 YMax = 2 YScale = 1

The graph of $y = x^{\frac{1}{3}}$ is a smooth curve for $x \in R$.

There is a vertical tangent to the curve at $(0,0)$.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^{\frac{1}{3}} - 0}{h}$$

$$= \lim_{h \to 0} \frac{1}{h^{\frac{2}{3}}}$$

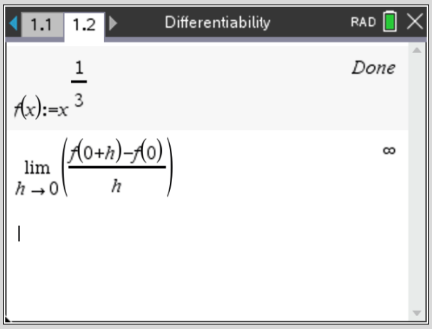
As $h \to 0$, $\frac{1}{h^{\frac{2}{3}}} \to \infty$, so the limit does not exist.

Answer: f is not differentiable at $x = 0$.

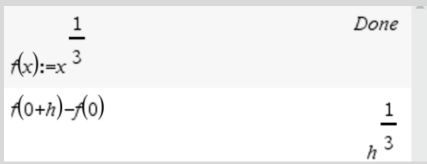
On a **Calculator** page:

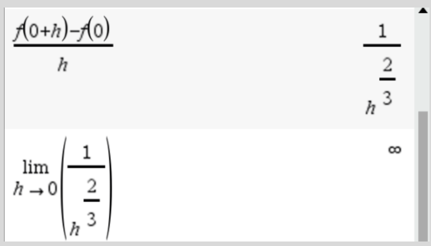
- Enter $\boldsymbol{f(x) := x^{\frac{1}{3}}}$.
- Press menu > **Calculus** > **Limit**.
- Enter as shown.

The above result is confirmed.



***Note**: TI-Nspire CX II CAS can be used to show the required solution steps as shown in the last two screenshots.*





3.3.2 Graphs of derivatives and anti-derivatives

Given the graph of f, the graph of f' has the following properties.

Graph of f	**Graph of f'**
negative gradient	negative (i.e. below the x-axis)
positive gradient	positive (i.e. above the x-axis)
local minimum point	cuts x-axis from negative to positive
local maximum point	cuts x-axis from positive to negative
stationary point of inflection	touches the x-axis
point of maximum gradient	turning point
endpoints (included or excluded)	excluded endpoints
cusp or sharp point	does not exist

Deducing the graph of the derivative function from the graph of a given function

Question

Graph the function $f(x) = x^3 - 3x^2 - 6x + 8$ and its first derivative on the same set of axes.

Find the set of values of x for which $f'(x) > 0$.

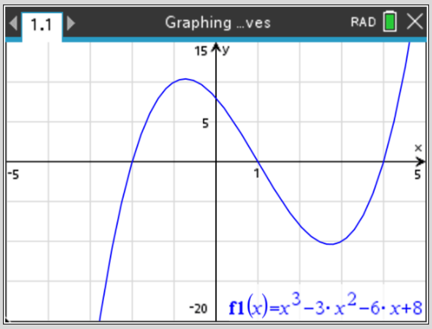
Solution

On a **Graphs** page:

- Enter $\boldsymbol{f1(x) = x^3 - 3x^2 - 6x + 8}$.
- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –5 XMax = 5 XScale = 1
  YMin = –20 YMax = 15 YScale = 5
- Press menu > **View** > **Grid** > **Lined Grid** (if not already visible).
- Press ctrl G to add another graph.
- Press [catalog] 5 and select the **Derivative** template.

Note: *Alternatively, press* shift − *to access the* ***Derivative*** *template.*

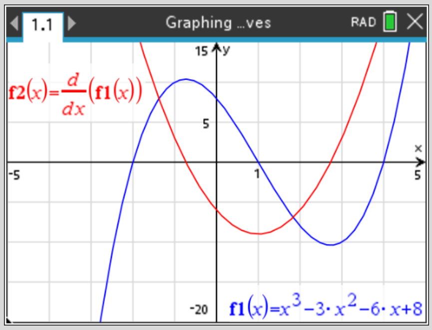
- Enter $\boldsymbol{f2(x) = \frac{d}{dx}(f1(x))}$.



... continued

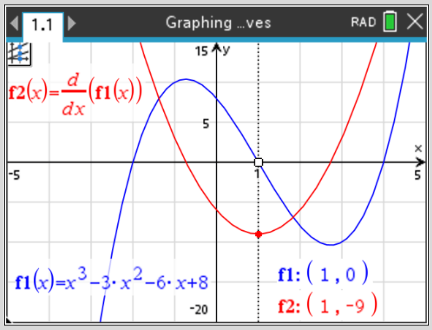
Solution (continued)

The graphs of $y = f(x)$ and its first derivative are now displayed on the same set of axes.



To trace the values of the function and its first derivative, for a given x- value:

- Press menu > **Trace** > **Trace Step** and edit its value to 0.1.
- Press menu > **Trace** > **Trace All**.
- Use ◀ and ▶ to view these values for different values of x (or enter a new x- value directly).

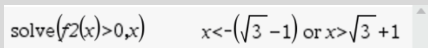


Note: *Use the table on the previous page to link properties of the two graphs.*

For example, intervals where the graph of $y = f(x)$ has positive gradient corresponds to intervals where the graph of $y = f'(x)$ is above the x-axis.

On a **Calculator** page:

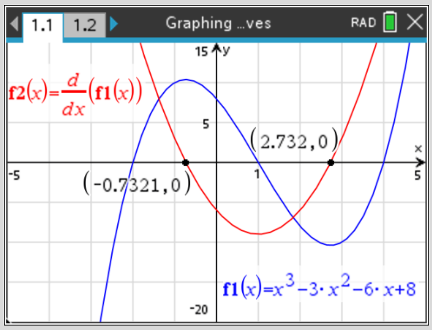
- Press menu > **Algebra** > **Solve**.
- Enter as shown.



Answer: $f'(x) > 0$ for $x < 1 - \sqrt{3}$ or $x > 1 + \sqrt{3}$.

On a **Graphs** page:

- Press menu > **Geometry** > **Points & Lines** > **Intersection Point(s)**.
- Click (press the click key) on the graph of the derivative and click on the x- axis.
- Move the cursor to hover over each intersection point with the x-axis.
- At each intersection point with the x-axis, press ctrl menu > **Coordinates and Equations**.



This leads to stating the required set of values of x in decimal form.

Answer: $f'(x) > 0$ for $x < -0.732$ or $x > 2.732$, correct to three decimal places.

Note: *To change the appearance of a graph, move the cursor to make contact with the graph, press* ctrl menu > ***Attributes***. *See section 1.1.2 for more information about this.*

Deducing the graph of an anti-derivative function from the graph of a given function

When interpreting the graph of f':

- x-axis intercepts correspond to stationary points on the graph of f.
- If the graph of f' is above the x-axis, the graph of f has positive gradient and hence increases as x increases.
- If the graph of f' is below the x-axis, the graph of f has negative gradient and hence decreases as x increases.

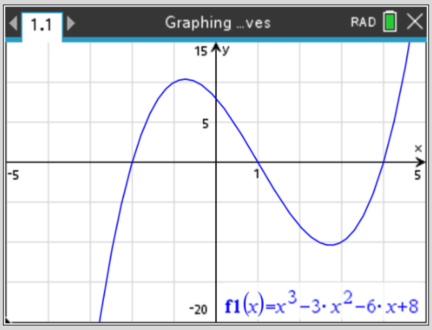
Question

Graph the function $f(x)=x^3-3x^2-6x+8$ and an antiderivative of the function on the same set of axes.

Solution

On a **Graphs** page:

- Enter $\boldsymbol{f1(x)=x^3-3x^2-6x+8}$.
- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –5 XMax = 5 XScale = 1
  YMin = –20 YMax = 15 YScale = 5
- Press [menu] > **View** > **Grid** > **Lined Grid**.
- Press [ctrl] [G] to add another graph.
- Press [📖] [5] and select the **Definite Integral** template.

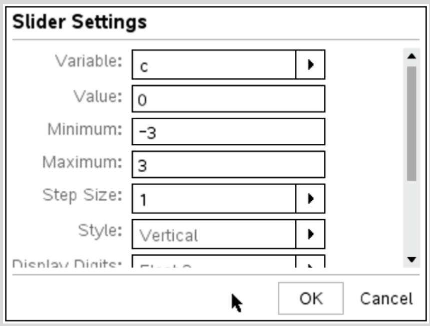


***Note**: Alternatively, press [⇧shift] [+] to access the **Definite Integral** template.*

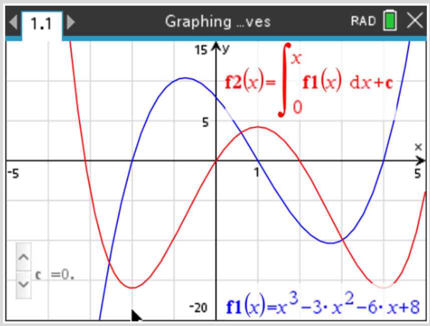
- Enter $\boldsymbol{f2(x)=\int_0^x f1(x)\,dx+c}$.

After entering the function, a prompt will appear to create a slider for ***c***.

- Use the cursor to hover over the slider and press [ctrl] [menu] to access the **Slider Settings**.
- Edit the Slider Settings for ***c*** as shown.
- Check (press [☒]) the **Minimised** box and then click OK.



Note that the graph of the antiderivative has a stationary point at $x=1$. The graph of the original function crosses the x-axis at $x=1$. Using the slider, change the value of c and note that its value does not affect the graph of the original function.



... continued

Solution (continued)

Answer: The antiderivative graphed is $F(x)=\frac{1}{4}x^4-x^3-3x^2+8x$. This equation is based on the general solution $F(x)=\frac{1}{4}x^4-x^3-3x^2+8x+c$ with $c=0$.

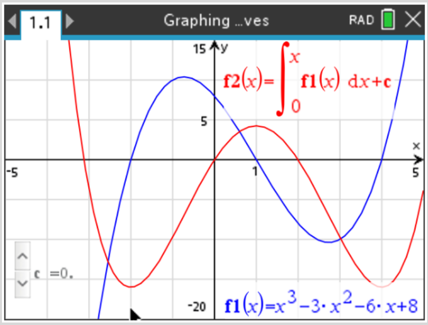
The purpose of adding '$+c$' is to illustrate the formation of a family of curves given by the antiderivative of $f(x)=x^3-3x^2-6x+8$.

When finding the indefinite integral of a function, it always includes an arbitrary constant, the constant of integration.

As a result, there is not a unique antiderivative graph for the graph of a given function.

It may be translated any distance parallel to the y-axis.

Notes**: To animate the family of curves, move the cursor inside the slider box and press* ctrl menu > ***Animate. *To change the appearance of a graph, move the cursor to make contact with the graph and press* ctrl menu > ***Attributes***.



3.3.3 Differentiation

Differentiating from first principles

The derivative, f', of a function f is defined by:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Question

Create a **Notes** page that differentiates functions from first principles.

Solution

To set up a **Notes** page to differentiate functions from first principles:

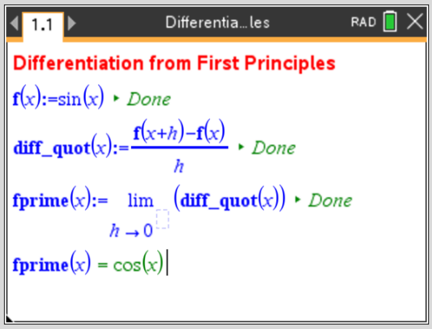
- Enter the title text '**Differentiation from First Principles**' as shown in the screenshot.

***Note**: To edit the text colour, select the text by holding* [shift] *and 'arrow' across the text. Then press* [menu] *> **Format** > **Text colour**.*

- Press [menu] > **Insert** > **Maths Box** (or press [ctrl] [M]) and enter the command $\boldsymbol{f(x):=\sin(x)}$.
- Repeat the last step to enter the following (shown right):

  $\boldsymbol{diff\_quot(x):=\dfrac{f(x+h)-f(x)}{h}}$ and

  $\boldsymbol{fprime(x):=\lim_{h\to 0}(diff\_quot(x))}$.



- Enter $\boldsymbol{fprime(x)}$ as shown.

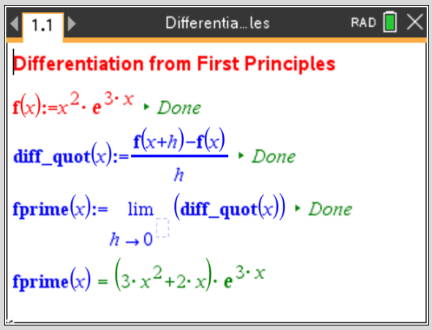
Answer: If $f(x)=\sin(x)$ then $f'(x)=\cos(x)$.

When a change is made to $f(x)$, the page updates and gives $f'(x)$. This shows that the **Maths Boxes** are linked.

For example:

- Enter $\boldsymbol{f(x):=x^2e^{3x}}$.

Answer: If $f(x)=x^2e^{3x}$ then $f'(x)=(3x^2+2x)e^{3x}$.



***Notes**: Entries/objects on a **Notes** page can be rearranged in ways like a word processor. Press* [ctrl] [menu] *> **Maths Box Attributes** to change the attributes of a **Maths Box**.*

3.3.4 Graph sketching and key features

Creating a function graph explorer

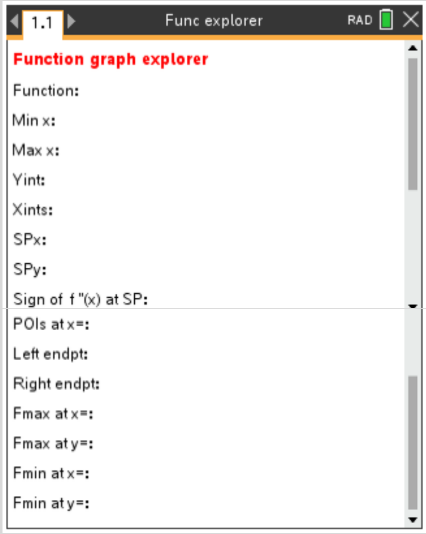
Question

Create a **Notes** page to analyse the key features of the graph of a function, such as axis intercepts, stationary points, and the sign of the second derivative.

Solution

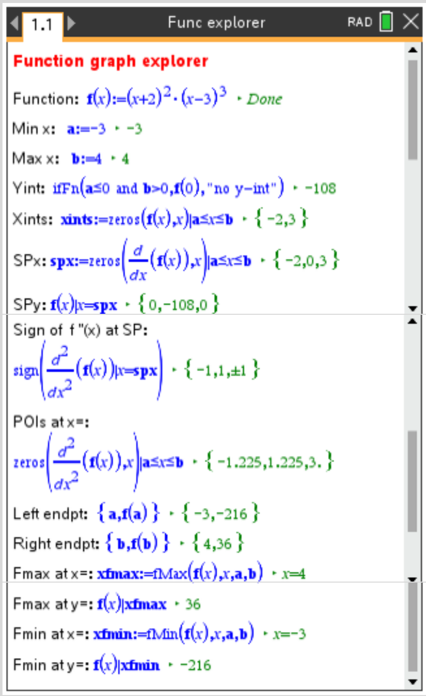
To create a function graph explorer, on a **Notes** page:

- Enter the title text **Function graph explorer** then enter labels for function graph features as follows:
  - Enter **Function:**
  - Enter **Min x:**
  - Enter **Max x:**
  - Enter **Yint:**
  - Enter **Xints:**
  - Enter **SPx:**
  - Enter **SPy:**
  - Enter **Sign of f ''(x) at SP:**
  - Enter **POIs at x=:**
  - Enter **Left endpt:**
  - Enter **Right endpt:**
  - Enter **Fmax at x=:**
  - Enter **Fmax at y=:**
  - Enter **Fmin at x=:**
  - Enter **Fmin at y=:**



Click to the right of each label and press ctrl M to insert a **Maths Box**, then enter the formulas as follows:

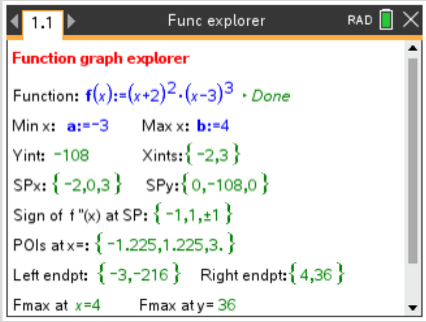
- For **Function**, enter $\boldsymbol{f(x) := (x+2)^2 \cdot (x-3)^3}$
- For **Min x**, enter $\boldsymbol{a := -3}$
- For **Max x**, enter $\boldsymbol{b := 4}$
- For **Yint**, enter $\textbf{iffn}(a \le 0 \textbf{ and } b \le 0, f(0), \textbf{"no y-int"})$
- For **Xints**, enter $\boldsymbol{zeros := \text{zeros}(f(x),x)|a \le x \le b}$
- For **SPx**, enter $\boldsymbol{spx := \text{zeros}(d/dx(f(x)),x)|a \le x \le b}$
- For **SPy**, enter $\boldsymbol{f(x)|x=spx}$
- For **Sign of f ''(x) at SP**, enter $\textbf{sign}\left(\dfrac{d^2}{dx^2}(f(x))|x=spx\right)$
- For **POIs at x**, enter $\textbf{zeros}\left(\dfrac{d^2}{dx^2}(f(x)),x\right)|a \le x \le b$
- For **Left endpt**, enter $\boldsymbol{\{a,f(a)\}}$
- For **Right endpt**, enter $\boldsymbol{\{b,f(b)\}}$
- For **Fmax at x=**, enter $\boldsymbol{xfmax := \text{fmax}(f(x),x,a,b)}$
- For **Fmax at y=**, enter $\boldsymbol{f(x)|xfmax}$
- For **Fmin at x=**, enter $\boldsymbol{xfmin := \text{fmin}(f(x),x,a,b)}$
- For **Fmin at y=**, enter $\boldsymbol{f(x)|xfmin}$



... continued

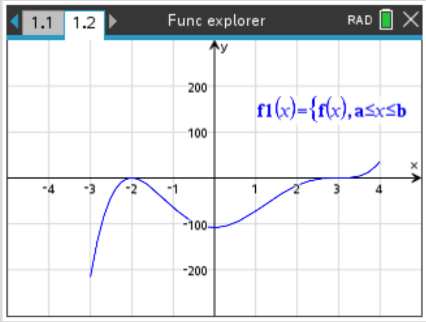
Solution (continued)

Objects on a **Notes** page such as text and maths boxes can be rearranged in ways like working with word processor software. As an example, the maths boxes and their labels can be moved around so that some boxes can be placed on the same line. Maths Box attributes such as display digits and showing/hiding its input/output can be modified by clicking on the relevant Maths Box, then pressing menu > **Maths Box Options** > **Maths Box Attributes**. See the screen right as an example of this Notes page feature.



To visualise the key features of the function graph, add a **Graphs** page and then:

- Enter $f1(x)=f(x)|a\le x\le b$.



You may need to adjust the window settings to get a suitable view of the graph. For the graph here, adjust as follows:

- Press menu > **Window/Zoom** > **Window Settings**.
- In the dialog box that follows, enter the following values:

$\text{XMin}=-5$	$\text{XMax}=5$	$\text{XScale}=1$
$\text{YMin}=-300$	$\text{YMax}=300$	$\text{YScale}=100$

Notes:

(1) The value of sign(0)=0 is neither positive nor negative, however the sign function will return sign(0)=±1, rather than zero, but it has an equivalent meaning in this context.

(2) If the domain is unbounded, the values of a and b can be assigned a = –∞ and b = ∞.

(3) The ***iffn*** *function is used for finding the y-intercept, and helps to handle cases where no y-intercept exists for the function, or within the specified domain restriction.*

3.4 Integration

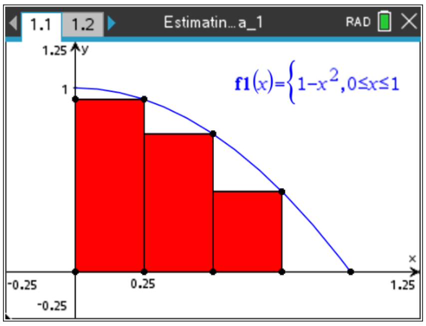
3.4.1 Informal consideration of the definite integral

Using sums to estimate the area under the curve y=f(x)

Question

Consider the region bounded by the curve $f(x)=1-x^2$ and the coordinate axes.

Estimate the area of this region using sums of the form $A\approx\sum_{i=1}^{4}f(x_i)\Delta x$ where $x_1=0.25$, $x_2=0.5$, $x_3=0.75$ and $x_4=1$.



Solution

Let A denote the area of the region, where $A\approx\sum_{i=1}^{4}f(x_i)\Delta x$.

On a **Calculator** page:

- Enter $\boldsymbol{f(x):=1-x^2}$.

The sum of the rectangles (an estimate for A) is given by:

$=f(0.25)\times0.25+f(0.5)\times0.25+f(0.75)\times0.25+f(1)\times0.25$

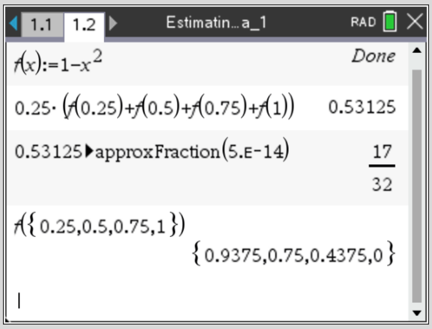
$=0.25\left(f(0.25)+f(0.5)+f(0.75)+f(1)\right)$

$=0.53125$

To represent the estimate as a fraction:

- Press menu > **Number** > **Approximate to Fraction**.

Answer: An estimate for A using these rectangles is 0.53125 $(=\frac{17}{32})$ square units.



***Note**: If intermediate function values are required, enter as shown on the fourth entry line of the screenshot at right.*

Estimating areas: Brief background

The area under the curve of a continuous non-negative function $f(x)$ between $x = a$ and $x = b$ can be approximated with n rectangles and $\Delta x = \dfrac{b-a}{n}$.

The left endpoint approximation, L_n, is given by $L_n = \sum_{i=1}^{n} f\left(a + (i-1)\Delta x\right)\Delta x$.

The right endpoint approximation, R_n, is given by $R_n = \sum_{i=1}^{n} f\left(a + i\Delta x\right)\Delta x$.

In the previous example, a right endpoint approximation, R_4, was used.

In that example, $R_4 = 0.53125 < A$ as the function is decreasing on the interval $[0,1]$.

When the number of rectangles (of equal width) is increased and a smaller value for Δx (rectangles of smaller width) is used, a better approximation to the area A is obtained.

In the next example, the number of rectangles is increased.

The right endpoint approximation, R_n, of the area bounded by the curve $f(x) = 1 - x^2$ and the coordinate axes with n rectangles is $\sum_{i=1}^{n}\left(1 - \left(\frac{i}{n}\right)^2\right)\left(\frac{1}{n}\right)$.

The left endpoint approximation, L_n, of the area bounded by the curve $f(x) = 1 - x^2$ and the coordinate axes with n rectangles is $\sum_{i=1}^{n}\left(1 - \left(\frac{i-1}{n}\right)^2\right)\left(\frac{1}{n}\right)$.

Recognising the definite integral as a limit of sums

Question

Consider the region bounded by the curve $f(x) = 1 - x^2$ and the coordinate axes.

Estimate the area of this region using sums of the form $\sum f(x_i)\Delta x$ where

(a) $x_1 = 0.1$, $x_2 = 0.2$, …, $x_9 = 0.9$ and $x_{10} = 1$.

(b) $x_1 = 0.01$, $x_2 = 0.02$, …, $x_{99} = 0.99$ and $x_{100} = 1$.

(c) $x_1 = 0.001$, $x_2 = 0.002$, …, $x_{999} = 0.999$ and $x_{1000} = 1$.

(d) Find $\int_0^1 \left(1 - x^2\right) dx$ and compare this value with the estimates found in parts **(a)**, **(b)** and **(c)**.

… continued

Solution

(a), **(b)** and **(c)**

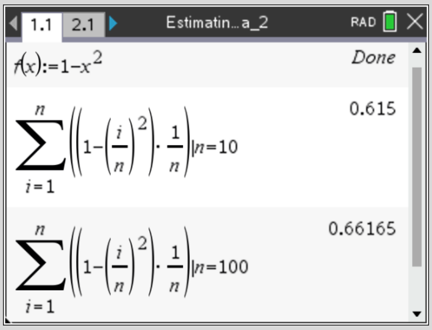
Let A denote the area of the region, where $A \approx \sum_{i=1}^{n} f(x_i)\Delta x$.

On a **Calculator** page:

- Enter $\boldsymbol{f(x):=1-x^2}$.

To calculate this sum:

- Press [menu] **> Calculus** > **Sum**.
- Enter as shown.
- Press [ctrl] [=] to access the 'with' or 'given' symbol '|'.
- Press [ctrl] [enter] to obtain a decimal answer.



Answers: (a) With $n = 10$, $R_{10} = 0.615$.

(b) With $n = 100$, $R_{100} = 0.66165$.

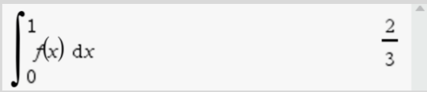
(c) With $n = 1000$, $R_{1000} = 0.6661665$.

Note: *If required, press* [on] ***> Settings > Document Settings****. Change* ***Display Digits*** *to* ***Float****.*

(d) Find $\int_0^1 (1 - x^2)\,dx$.

On a **Calculator** page:

- Press [menu] **> Calculus > Integral**.
- Enter as shown.



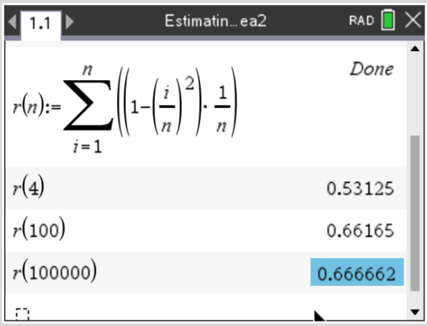
Note: *Press* [shift] [+] *to access the* ***Integral*** *template.*

Answer: $\int_0^1 (1 - x^2)\,dx = \frac{2}{3}$.

As indicated in part **(a)**, $\lim_{n\to\infty} R_n = A = \frac{2}{3}$.

The exact area is equal to the limit as $n \to \infty$ of R_n (or L_n).

Note: *R_n can be assigned as shown in the screenshot at right.*



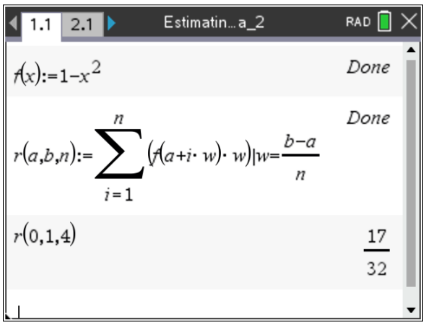
Estimating areas: A general result

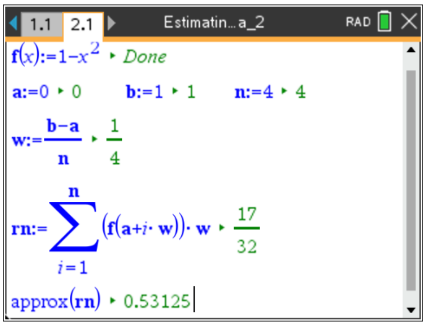
The right endpoint approximation, R_n, can be used to approximate an area under a curve as follows:

Enter $\boldsymbol{r(a,b,n) := \sum_{i=1}^{n} f(a+iw)w \mid w = \frac{b-a}{n}}$.

The **Calculator** page and the corresponding **Notes** page below show that $R_4 = 0.53125$.

Note: Press ctrl M *to insert a* ***Maths Box*** *(alternatively , press* menu ***> Insert >Maths Box****). Press* ctrl [templates] *to access the* ***assign*** *command. Press* [book] 1 A *and scroll down to select* ***approx(****.*





Using the trapezium rule to approximate an area

$$A \approx \frac{x_n - x_0}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right].$$

Question

Use the trapezium rule with four strips to find an approximate value for $\int_1^2 \log_{10} x \, dx$.

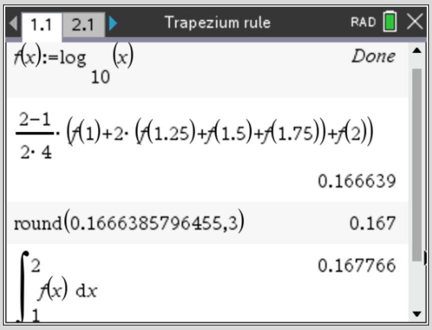
Give your answer correct to three decimal places.

Solution

$$\int_1^2 \log_{10}(x)\,dx \approx \frac{2-1}{2\times 4}\left[f(1) + 2\left(f(1.25) + f(1.5) + f(1.75)\right) + f(2)\right]$$

On a **Calculator** page:

- Enter $\boldsymbol{f(x) := \log_{10}(x)}$.
- Press ctrl 10^x to access log base 10.
- Enter as shown.



Answer: $\int_1^2 \log_{10}(x)\,dx \approx 0.167$, correct to 3 decimal places.

Notes:

(1) The syntax for the ***Round*** *command is* ***round(Value[,Digits])****. To access it, press* menu ***> Number > Number Tools*** *>* ***Round****.*

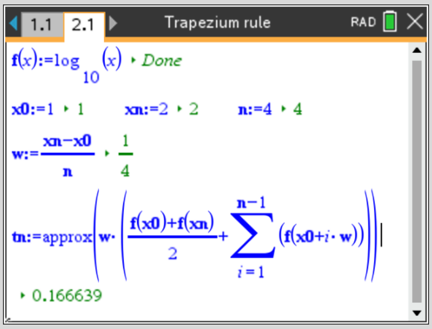
(2) An interesting extension is to consider the accuracy of the trapezium rule.

... continued

Solution (continued)

Alternatively on a **Notes** page:

- Press [ctrl] [M] to insert a **Maths Box**.
- Assign $\boldsymbol{f(x), x0, xn, n, w}$ and $\boldsymbol{tn}$ as shown.
- Press [📖] [1] [A] and scroll down to select **approx(**.
- Press [menu] > **Calculations** > **Calculus** > **Sum** and complete as shown.



The **Notes** page shows an alternative form for the trapezium rule:

$$A \approx w\left[\frac{f(x_0)+f(x_n)}{2}+\sum_{i=1}^{n-1} f(x_i)\right]$$

Answer: $\int_1^2 \log_{10}(x)\,dx \approx 0.167$, correct to three decimal places.

Note*: As this curve is concave down, the trapezium rule underestimates A. Compare the estimate with* $\int_1^2 f(x)\,dx = 0.168$*, correct to three decimal places.* $0.167 < 0.168$.

Exploring the trapezium rule for the effect of the subinterval size

Question

Investigate the effect of increasing the number of subintervals on the accuracy of the trapezium rule for the area bounded by the graph of $f:[0,\pi]\to R,\ f(x)=\sin(x)e^{2x}$ and the x-axis.

(a) Plot the graph of f and find the area bounded by the curve and the x-axis. Give your answer correct to two decimal places.

(b) Explore how the accuracy of the trapezium rule approximation changes as the number of subintervals, n, increases from $n = 2$ to $n = 20$. Hence plot the approximation value against n and interpret key features of this plot.

Solution

(a) To explore the effect of changing the subinterval size:

- Open the TI-Nspire document *Trapezium rule* from the previous problem (or use the instructions to create it).

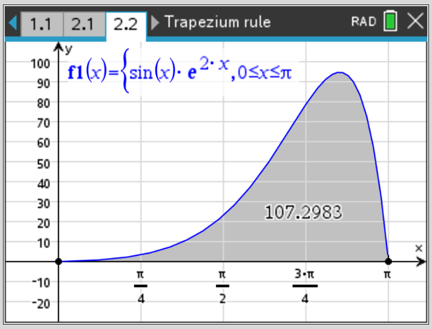
To plot the graph and find the area, add a **Graphs** page, then:

- Enter $\boldsymbol{f1(x)=\sin(x)\times e^{2x} \mid 0\le x\le \pi}$
- Press [menu] > **Window/Zoom** > **Window Settings**
  In the dialog box that follows, enter the following values:
  XMin = −0.5 Xmax = 3.5 XScale = π/4
  YMin = −30 YMax = 110 YScale = 10

Solution (continued)

- Press menu > **Analyse Graph** > **Integral**. Click on the lower bound (intersection point at (0, 0)) then click on the upper bound (intersection point at (π, 0)).
- To increase the precision of the displayed area value, hover over the answer text and press +.

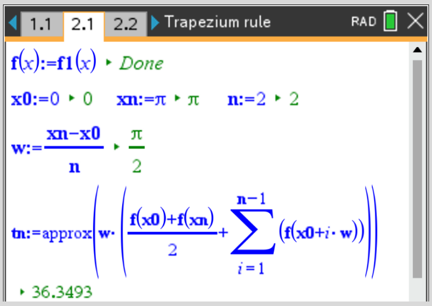
Answer: Area is 107.30 (2 decimal places)



(b) To renew the trapezium rule calculation template:

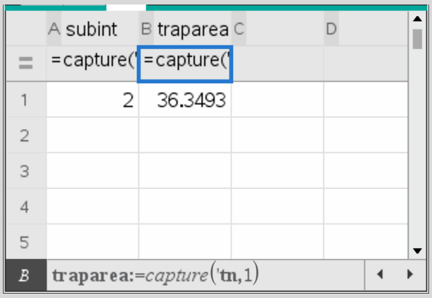
- Navigate to the **Notes** page.
- In first four **Maths Boxes** enter $f(x) := f1(x)$, $x0 := 0$, $xn := \pi$ and $n := 2$, as shown.
- Edit n for increasing values from 2 to 20 and observe the change in the value of tn, the approximate area.

Answer: $n = 2$, $tn \approx 36.35$, …, $n = 20$, $tn \approx 106.20$



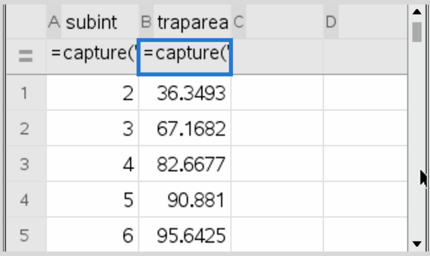
To capture the values of n and tn, edit the 4th **Maths Box** to $n := 2$, add a **Lists & Spreadsheet** page, then:

- Name columns A and B as shown, to declare them as lists.
- Navigate to the column A formula cell.
- Press menu > **Data** > **Data Capture** > **Automatic**. Press var select ***n***, then press enter.
- Similarly, capture ***tn*** in the column B formula cell.



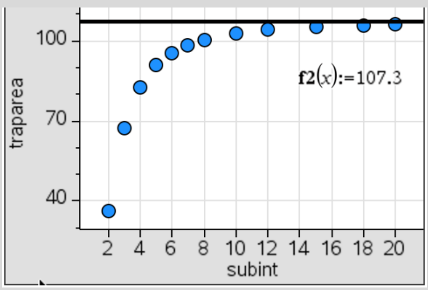
To populate the lists ***subint*** and ***traparea***:

- On the **Notes** page, systematically change the value of ***n***: ***n*****:=3**, ***n*****:=4**, ***n*****:=5**, ***n*****:=6**, then ***n*****:=8**, ***n*****:=10**, …, ***n*****:=20**.



To plot the data in the lists, add a **Data & Statistics** page:

- Press tab and select ***subint*** on the horizontal axis. Press tab and select ***traparea*** on the vertical axis.
- Press menu > **Analyse** > **Plot Function**. In the textbox that follows, enter $f2(x) := 107.30$.



Answer: The accuracy of the approximation increases as the number of subintervals, n, increases. The answer approaches 107.30 when n is large.

***Note**: To reset the spreadsheet lists, select the formula cell for the column and press menu > **Data** > **Clear Data**.*

Using the Programme Editor to implement pseudocode for the trapezium rule

Question

A student writes the following pseudocode for the trapezium rule to approximate $\int_a^b f(x)\,dx$ with the sum of the areas of n trapeziums.

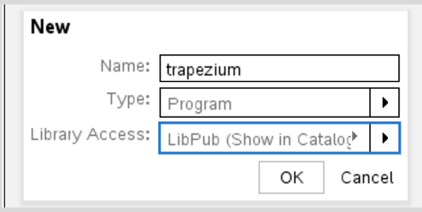
Inputs	Initialise	For k from 1 to n
define *function* $f(x)$ $a \leftarrow$ lowest value, $x \in [a, b]$ $b \leftarrow$ highest value, $x \in [a, b]$ $n \leftarrow$ number of trapeziums	$w \leftarrow (b - a)/n$ $l \leftarrow a$ $r \leftarrow a + w$ $csum \leftarrow 0$	$trap \leftarrow w \times (f(l) + f(r))/2$ $csum \leftarrow trap + csum$ $l \leftarrow l + w$ $r \leftarrow r + w$ **End For** **print** "approx. integral =", *csum*

Implement the pseudocode in the in the Programme Manager, using the inputs $f(x) = \log_{10}(x)$ on the interval [1, 2]. Compare the accuracy of the results for n equal to 4 and 12 trapeziums.

Solution

To start coding, in a new **Document** (or a new **Problem**):

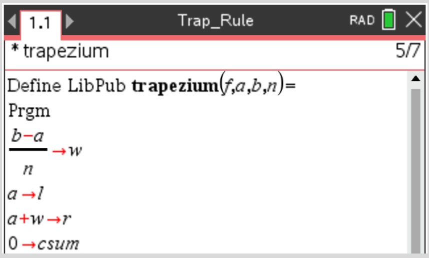
- Select **Add Programme Editor > New**.
- In the dialog box that follows, enter as shown.



The **Program Editor** will follow, ready to accept the code.

To name the inputs ***f*, *a*, *b*** and ***n***, in line 0:

- Enter **trapezium(*f*,*a*,*b*,*n*)=**

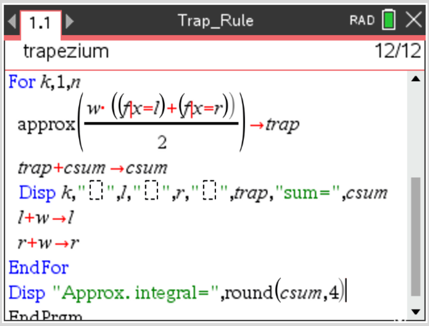


To initialise the left (***l***) and right (***r***) sub-interval boundaries and the cumulative sum of the area of the trapeziums (***csum***):

- Enter **(*b* – *a*)/*n* → *w***, pressing [ctrl] [var] ([sto→]) for **store**, →.
- Enter ***a* → *l***, ***a* + *w* → *r*** and **0 → *csum***, as shown.

To instruct finding the cumulative area for n trapeziums:

- Press [menu] > **Control** > **For …** then enter **For *k*,1,*n*** .
- Enter **approx**$\left(w \times ((f \mid x = l) + (f \mid x = r))/2\right) \rightarrow$ ***trap***, pressing [📖] [1] [A] for **approx** , [ctrl] [=] ([|≠≥▸]) for **given**, |.
- Enter ***trap* + *csum* → *csum***, ***l* + *w* → *l*** and ***r* + *w* → *r***.



To display the progress of the algorithm and the final result:

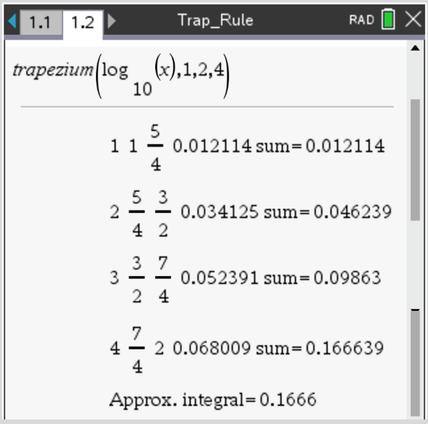
- After line 7, enter **Disp *k*," ",*l*," ",*r*," ",*trap*,"sum= ",*csum*** by pressing [menu] > **I/O** > **Disp** and [?!▸] to select ".
- After line 11, enter **Disp "Approx. integral=",round(*csum*,4)** as shown, pressing [📖] [1] [S] to select **round.**

... continued

Solution (continued)

To test the code for $\int_1^2 \log_{10}(x)\,dx$:

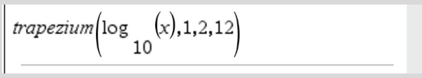
- Press ctrl B then ctrl R to check, store and run program.
- In the **Calculator** page that follows, enter **trapezium($\log_{10}(x)$,1,2,4)** by pressing ctrl 10^x (log).

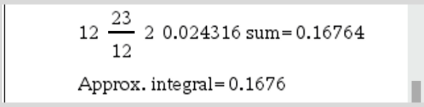


Answer: $\int_1^2 \log_{10}(x)\,dx \approx 0.1666$ using four trapeziums to approximate area under the graph of $y = \log_{10}(x), x \in [1,2]$.

To compare the accuracies using 4 and 12 trapeziums:

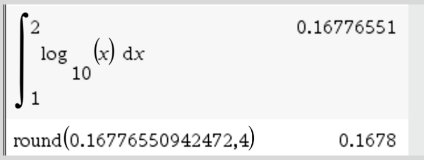
- Press ▲ to top of page and press enter to paste. Edit the n value to 12: **trapezium($\log_{10}(x)$,1,2,12)**.





To validate the results using the definite integral command:

- Press shift +, key in $\int_1^2 \log_{10}(x)\,dx$ then press ctrl enter.



Answer: Integral = 0.1678, correct to 4 decimal places, compared with 0.1666 (n = 4) and 0.1676 (n = 12).

Implementing pseudocode for the trapezium rule in the Python application

Question

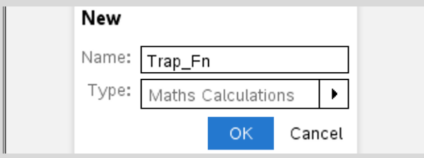
Implement the pseudocode from the previous problem for the trapezium rule in the Python application. Hence find an approximation using n trapeziums, correct to 4 decimal places, for:

(a) $\int_1^2 \log_{10}(x)\,dx$ and **(b)** $\int_0^2 \left(\frac{8}{x^2+4}\right)dx$, where **(i)** n = 4 and **(ii)** n = 12.

Solution

(a) To start coding, in a new **Document** (or a new **Problem**):

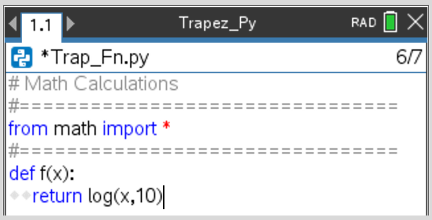
- Select **Add Python** > **New**.
- In the dialog box that follows, enter as shown.



To define $f(x) = \log_{10}(x)$:

- Press menu > **Built-ins** > **Functions** > **def function():**
- Enter **def *f*(*x*):** , then **return log(*x*,10)**, selecting **log(*x*,base)** by pressing menu > **Maths**.

***Note**: Ensure correct indentation, as shown.*



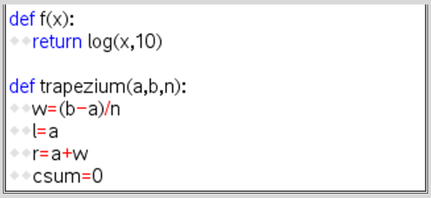
To define a user-defined function **trapezium(*a*,*b*,*n*)**:

- Enter **def trapezium(*a*,*b*,*n*)**, as shown.

Solution (continued)

To initialise the left (***l***) and right (***r***) sub-interval boundaries and the cumulative sum of the area of the trapeziums (***csum***):

- Enter ***w* = (*b* – *a*)/*n***, followed by ***l* = *a***, ***r* = *b*** and ***csum* = 0**, with indentations as shown.



```
def f(x):
  return log(x,10)

def trapezium(a,b,n):
  w=(b-a)/n
  l=a
  r=a+w
  csum=0
```

To instruct finding the cumulative area for *n* trapeziums:

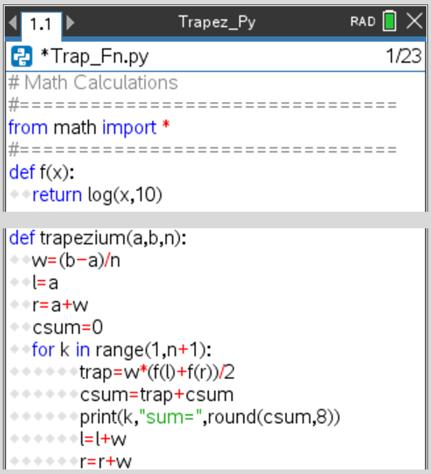
- Press [menu] > **Built-ins** > **Control** > **for index in range(start,stop)**, then enter **for *k* in range(1,*n*+1)**
- Enter $\boldsymbol{trap} = w \times \left(f\left(l\right) + f\left(r\right)\right)/\,\mathbf{2}$ then
- Enter ***csum* = *trap* + *csum***. Ensure indentations as shown.

To display the progress of the algorithm:

- Press [menu] > **Built-ins** > **I/O** > **print()** and enter **print(*k*,"sum= ",round(*csum*,8))** pressing [?!▸] for ".

To update the values of the left and right sub-interval fences:

- Enter ***l* = *l*** + ***w*** and ***r* = *r*** + ***w***, as shown.



```
1.1   Trapez_Py   RAD
*Trap_Fn.py   1/23
# Math Calculations
#================================
from math import *
#================================
def f(x):
  return log(x,10)

def trapezium(a,b,n):
  w=(b-a)/n
  l=a
  r=a+w
  csum=0
  for k in range(1,n+1):
      trap=w*(f(l)+f(r))/2
      csum=trap+csum
      print(k,"sum=",round(csum,8))
      l=l+w
      r=r+w
```

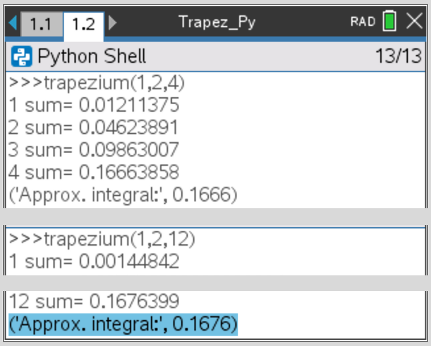
To return the output value of the function **trapezium(*a*,*b*,*n*)**:

- Press [menu] > **Built-ins** > **Functions** > **return** and enter **return "Approx. integral:",round(*csum*,4)**.

```
  return "Approx. integral:",round(csum,4)
```

(a) To test the code for $\int_1^2 \log_{10}\left(x\right)dx$ with $n = 4$, $n = 12$:

- Press [ctrl] [R] to check syntax and run the program.
- **(i)** In the **Python Shell** page that follows, press [var], select **trapezium** and enter **trapezium(1,2,4)**.
- **(ii)** Press [var] > **trapezium** and enter **trapezium(1,2,12)**.



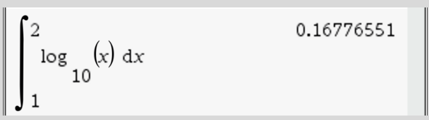
```
1.1  1.2   Trapez_Py   RAD
Python Shell   13/13
>>>trapezium(1,2,4)
1 sum= 0.01211375
2 sum= 0.04623891
3 sum= 0.09863007
4 sum= 0.16663858
('Approx. integral:', 0.1666)

>>>trapezium(1,2,12)
1 sum= 0.00144842

12 sum= 0.1676399
('Approx. integral:', 0.1676)
```

To validate the results on a **Calculator** page:

- Press [⇧shift] [+], key in $\int_1^2 \mathbf{log_{10}}\left(x\right)\boldsymbol{dx}$ then press [ctrl] [enter].

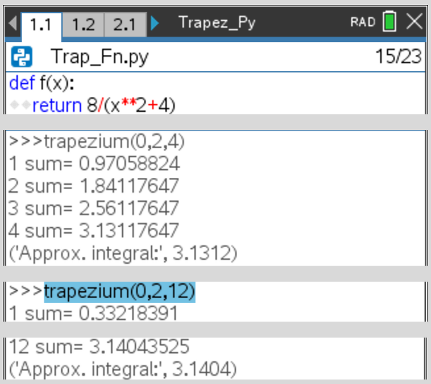


$$\int_1^2 \log_{10}(x)\,dx \qquad 0.16776551$$

Answer: Integral = 0.1678, correct to 4 decimal places, compared with 0.1666 (n = 4) and 0.1676 (n = 12).

(b) To test the code for $\int_0^2 \left(\frac{8}{x^2+4}\right)dx$, with $n = 4$, $n = 12$:

- On page 1.1, edit line 6 to **return** $\mathbf{8}/\left(\boldsymbol{x}^2+\mathbf{4}\right)$, pressing [x²] for the exponent ($\boldsymbol{x}^2$ appears as ***x***2** in the output).
- In the **Python Shell** page press [var], select **trapezium** and enter **(i) trapezium(0,2,4)** then **(ii) trapezium(0,2,12)**.



```
1.1  1.2  2.1   Trapez_Py   RAD
Trap_Fn.py   15/23
def f(x):
  return 8/(x**2+4)

>>>trapezium(0,2,4)
1 sum= 0.97058824
2 sum= 1.84117647
3 sum= 2.56117647
4 sum= 3.13117647
('Approx. integral:', 3.1312)

>>>trapezium(0,2,12)
1 sum= 0.33218391

12 sum= 3.14043525
('Approx. integral:', 3.1404)
```

Answer: Exact integral $= \pi \approx 3.1416$. Trapezium rule gives values of (i) 3.1312 with n = 4, (ii) 3.1404 with n = 12.

3.4.2 Properties of anti-derivatives and definite integrals

Fundamental theorem of calculus: Let f be a continuous function on the interval $[a,b]$.

Let F be an anti-derivative of f on $[a,b]$ so that $F'(x)=f(x)$.

Then $\int_a^b f(x)\,dx=\int_a^b F'(x)\,dx=F(b)-F(a)$.

If f is continuous on $[a,b]$ and $c\in[a,b]$, then note the following properties:

$\int_a^a f(x)\,dx=0$ $\qquad$ $\int_a^b f(x)\,dx=-\int_b^a f(x)\,dx$ $\qquad$ $\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$

Investigating properties of definite integrals

Question

(a) Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{4}} \sin(x)\,dx$ and $\int_k^k \sqrt{x}\,dx$.

(b) Given that $F'(x)=f(x)$, evaluate $\int_a^b f(x)\,dx$.

Comment on the result of evaluating a definite integral for which the upper and lower terminals are the same.

(c) Evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \cos(x)\,dx$ and $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos(x)\,dx$.

(d) Evaluate $\int_0^2 e^x\,dx$ and $\int_2^0 e^x\,dx$.

(e) Given that $F'(x)=f(x)$, evaluate $\int_a^b f(x)\,dx$ and $\int_b^a f(x)\,dx$.

Comment on the effect of interchanging the terminals on the result when a definite integral is evaluated.

(f) Evaluate $\int_0^{\frac{\pi}{2}} \sin(3x)\,dx+\int_{\frac{\pi}{2}}^{\pi} \sin(3x)\,dx$ and $\int_0^{\pi} \sin(3x)\,dx$.

(g) Evaluate $\int_1^3 \frac{1}{x}\,dx+\int_3^5 \frac{1}{x}\,dx$ and $\int_1^5 \frac{1}{x}\,dx$.

(h) Given that $F'(x)=f(x)$, evaluate $\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$ and $\int_a^b f(x)\,dx$.

Comment on the result.

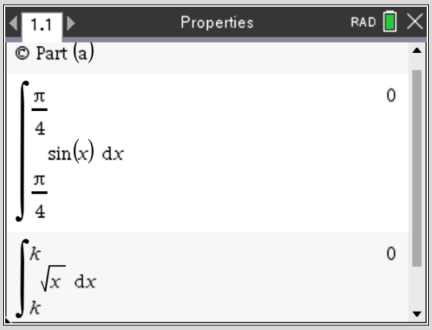
... continued

Solution

Enter as shown on a **Calculator** page:

- Press [menu] > **Calculus** > **Integral**.

Note: *Alternatively, press* [⇧shift] [+] *to access the* ***Integral*** *template. To add a comment to a* ***Calculator*** *page, press* [menu] *> Actions > Insert Comment.*



Answers:

(a) $\int_{\frac{\pi}{4}}^{\frac{\pi}{4}} \sin(x)\,dx = 0$ and $\int_k^k \sqrt{x}\,dx = 0$

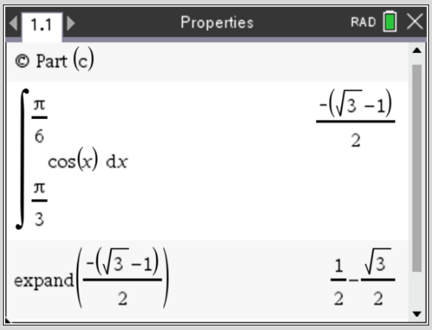
(b) $\int_a^a f(x)\,dx = [F(x)]_a^a$ since $F'(x) = f(x)$

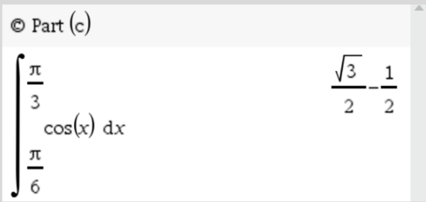
$= F(a) - F(a)$

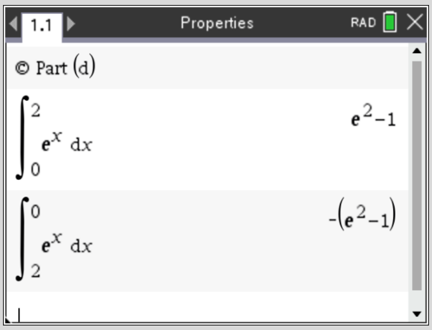
$= 0$

The result is always zero.

(c) $\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \cos(x)\,dx = \frac{1}{2} - \frac{\sqrt{3}}{2}$ and $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos(x)\,dx = \frac{\sqrt{3}}{2} - \frac{1}{2}$







(d) $\int_0^2 e^x\,dx = e^2 - 1$ and $\int_2^0 e^x\,dx = 1 - e^2$

Note: *Students require algebraic insight to recognise equivalent expressions. They need to be able to calculate these definite integrals without the use of CAS.*

... continued

Solution (continued)

(e) $\int_a^b f(x)\,dx = \left[F(x)\right]_a^b$ since $F'(x) = f(x)$

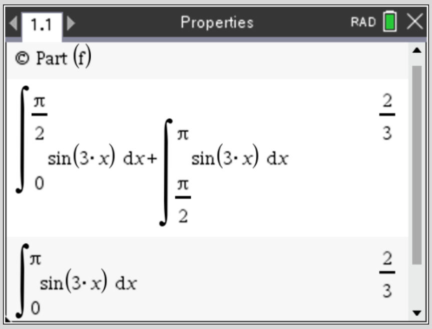
$= F(b) - F(a)$

$\int_b^a f(x)\,dx = \left[F(x)\right]_b^a$

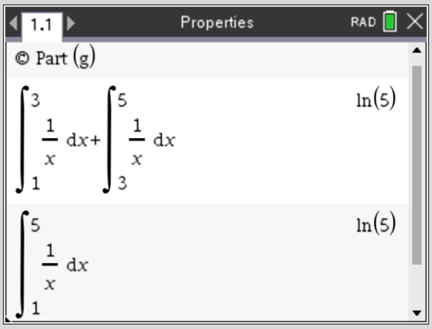
$= F(a) - F(b)$

The sign of the result is reversed.

(f) $\int_0^{\frac{\pi}{2}} \sin(3x)\,dx + \int_{\frac{\pi}{2}}^{\pi} \sin(3x)\,dx = \frac{2}{3}$ and $\int_0^{\pi} \sin(3x)\,dx = \frac{2}{3}$



(g) $\int_1^3 \frac{1}{x}\,dx + \int_3^5 \frac{1}{x}\,dx = \log_e(5)$ and $\int_1^5 \frac{1}{x}\,dx = \log_e(5)$



(h) $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \left[F(x)\right]_a^c + \left[F(x)\right]_c^b$ $\quad$ (as $F'(x) = f(x)$)

$= F(c) - F(a) + F(b) - F(c)$

$= F(b) - F(a)$

$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b$

$= F(b) - F(a)$

So $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

3.4.3 Applications of integration

If the graph of $y = f(x)$ lies above the x- axis, then the area of the region bound by the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b) - F(a)$.

If the graph of f is symmetric about the y- axis, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

Finding the area under the curve y=f(x) between x=a and x=b if f(x)>0 over this interval

Question

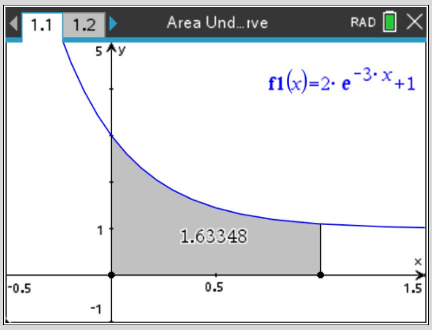
Find the area of the region enclosed by the curve $y = 2e^{-3x} + 1$, the coordinate axes and the line $x = 1$. Give your answer correct to three decimal places.

Solution

The required region lies entirely above the x- axis and can therefore be calculated using a single definite integral.

On a **Graphs** page:

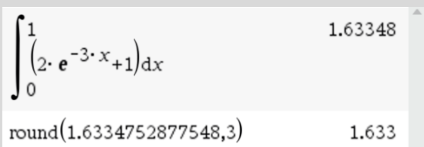
- Enter $\boldsymbol{f1(x) = 2e^{-3x} + 1}$.
- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –0.5 XMax = 1.5 XScale = 0.5
  YMin = –1 YMax = 5 YScale = 1
- Press menu > **Analyse Graph** > **Integral**.
- Enter 0 for the lower bound and press enter.
- Enter 1 for the upper bound and press enter.



Answer: $\int_0^1 (2e^{-3x} + 1)\,dx = 1.633$, correct to three decimal places.

Alternatively on a **Calculator** page:

- Press menu > **Calculus** > **Integral**.
- Enter as shown.
- Press ctrl enter to obtain a decimal Answer:



Answer: $\int_0^1 (2e^{-3x} + 1)\,dx = 1.633$, as before.

***Note**: The answer can be rounded to three decimal places by entering **round(ans,3)**. Remember to press eˣ and not E. Press ⇧shift + to access the **Integral** template.*

Determining the area of a region between two curves

Consider two curves with equations $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$ for all $x \in [a,b]$.

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b \big(f(x) - g(x)\big)\,dx$$

***Note**: This applies even if the curves are partly above and below the x–axis, or entirely below.*

Question

Find the area of the region bounded by the graphs of the functions $f(x) = \sin(x)\cos(2x)$ and $g(x) = \cos(x)$ over the interval $x \in \left[-\frac{3\pi}{4}, \frac{\pi}{2}\right]$. Give your answer correct to two decimal places.

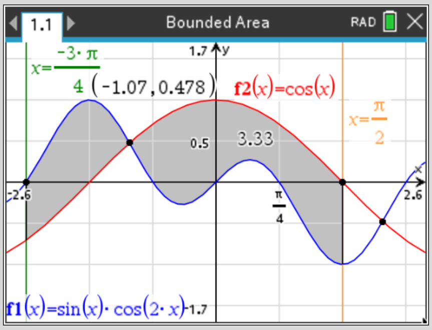
Solution

On a **Graphs** page:

- Enter $\boldsymbol{f1(x) = \sin(x)\cos(2x)}$ and $\boldsymbol{f2(x) = \cos(x)}$.
- Press menu > **Graph Entry/Edit** > **Relation**.
- Enter $\boldsymbol{x = -\frac{3\pi}{4}}$ and $\boldsymbol{x = \frac{\pi}{2}}$.
- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:

  XMin = –2.6 XMax = 2.6 XScale $= \frac{\pi}{4}$

  YMin = –1.7 YMax = 1.7 YScale = 0.5
- Press menu > **Analyse Graph** > **Bounded Area**.
- Click on **Graph $\boldsymbol{f1}$** and then on **Graph $\boldsymbol{f2}$**.
- Click on the lower bound point at $\left(-\frac{3\pi}{4}, 0\right)$ and then on the upper bound point at $\left(\frac{\pi}{2}, 0\right)$.



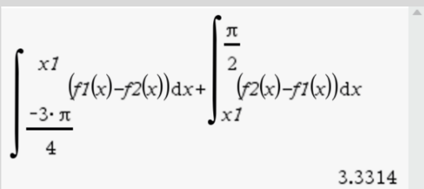
Answer: The total bounded area is 3.33, correct to two decimal places.

***Note**: To find the coordinates of multiple intersection points simultaneously, press menu **> Geometry > Points & Lines >** **Intersection Point(s)**. Click on **Graph f1** and then on **Graph f2**.*

- Hover over the *x*-coordinate of the leftmost intersection point, then press ctrl menu > **Store** and store as ***x*1**.

***Note**: Press var to access assigned/stored variables.*

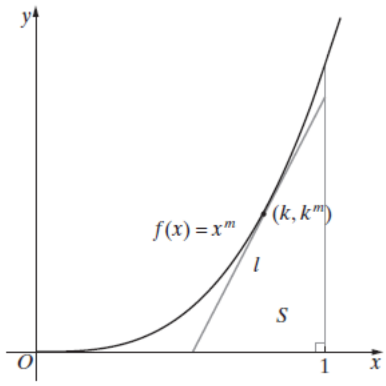
On a **Calculator** page, enter as shown to verify the result.



Applying calculus to the analysis of power functions

Question

Let l be the tangent to the graph of the function $f(x) = x^m$ at a movable point (k, k^m) where $0 \le k \le 1$ and $m > 1$.



Let S be the area of the triangular region bounded by the line l, the x-axis and the vertical line $x = 1$.

(a) Find the equation of l in terms of k and m.

(b) Express S in the form of a definite integral.

(c) Hence evaluate S for $m = 2, 3, 4$ and 5.

(d) Find an expression for S in terms of k and m.

Let $S_{\max}$ be the maximum value of S.

(e) Find the value of k that maximises S and hence determine $S_{\max}$.

(f) Find $\lim\limits_{m \to \infty} S_{\max}$.

Solution

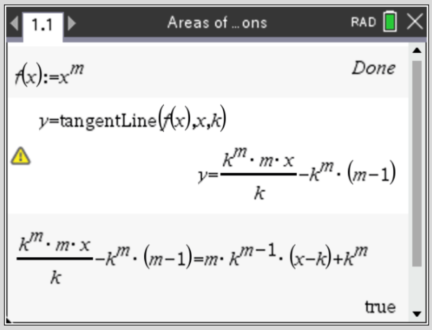
(a) Find the equation of l in terms of k and m.

Using $y = f'(k)(x-k) + f(k)$ with $f(k) = k^m$ and $f'(k) = mk^{m-1}$, the equation of l is $y = mk^{m-1}(x-k) + k^m$.

On a **Calculator** page:

- Enter $\boldsymbol{f(x) := x^m}$.
- Press menu > **Calculus** > **Tangent Line**.
- Enter as shown.

The equivalence of both forms of the equation can be verified as shown.



Answer: $y = mk^{m-1}(x-k) + k^m$.

Note*: Students require algebraic insight to recognise equivalent expressions. They need to be able to do this without the use of CAS.*

... continued

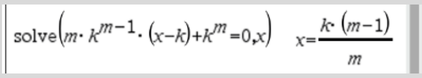
Solution (continued)

(b) Express S in the form of a definite integral.

On a **Calculator** page:

- Press [menu] > **Algebra** > **Solve**.

Enter as shown.



Solving $mk^{m-1}(x-k)+k^{m}=0$ for x gives $x=\dfrac{k(m-1)}{m}$.

The x-intercept of l is at $x=\dfrac{k(m-1)}{m}$.

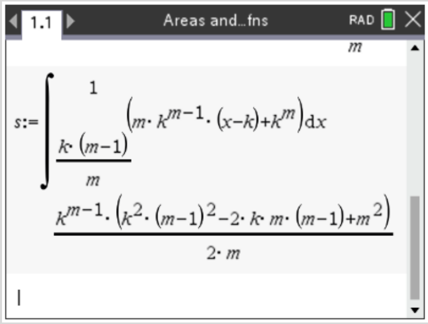
Answer: $S=\displaystyle\int_{\frac{k(m-1)}{m}}^{1}\left(mk^{m-1}(x-k)+k^{m}\right)dx$.

(c) Hence evaluate S for $m=2,3,4$ and 5.

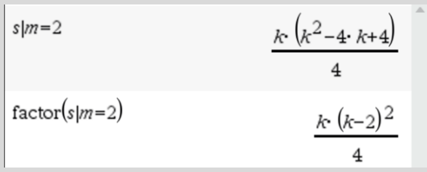
On a **Calculator** page:

- Press [ctrl] [⁼⁼{⁼] to access the **assign** symbol.
- Press [menu] > **Calculus** > **Integral**.
- Enter as shown.

***Note**: Alternatively, press [⇧shift] [+] to access the **Integral** template.*

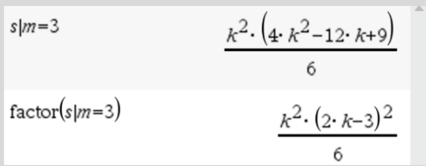


- For $m=2$, enter as shown.
- Press [ctrl] [=] to access the 'with' or 'given' symbol '|'.
- Press [menu] > **Algebra** > **Factor**.

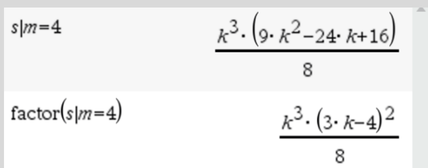


Answers:

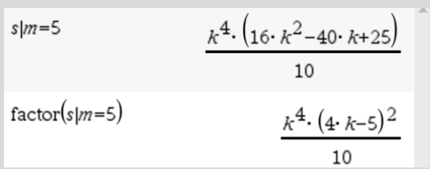
For $m=2$, $S=\dfrac{k}{4}(k-2)^{2}$.



For $m=3$, $S=\dfrac{k^{2}}{6}(2k-3)^{2}$.



For $m=4$, $S=\dfrac{k^{3}}{6}(3k-4)^{2}$.



For $m=5$, $S=\dfrac{k^{4}}{10}(4k-5)^{2}$.

... continued

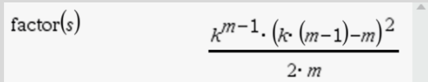
Solution (continued)

Parts **(d)**, **(e)** and **(f)**.

(d) Find an expression for S in terms of k and m.

On a **Calculator** page:

- Press [menu] > **Algebra** > **Factor**.



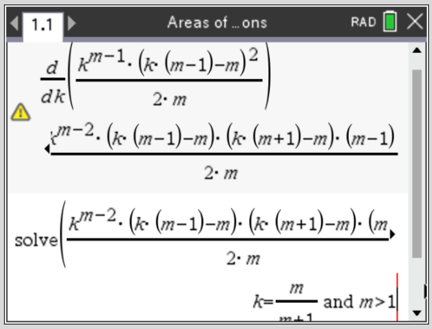
Answer: $S = \frac{k^{m-1}}{2m}\left(\left(m-1\right)k-m\right)^2$ (or equivalent).

(e) Let $S_{\max}$ be the maximum value of S.

Given $S = \frac{k^{m-1}}{2m}\left(\left(m-1\right)k-m\right)^2$:

$$\frac{dS}{dk} = \frac{\left(m-1\right)k^{m-2}}{2m}\left(k\left(m-1\right)-m\right)\left(k\left(m+1\right)-m\right).$$

Solving $\frac{dS}{dk} = 0$ for k with $0 \le k \le 1$ and $m > 1$ gives $k = \frac{m}{m+1}$ (as shown right).



Answer: $k = \frac{m}{m+1}$ gives the maximum value of S.

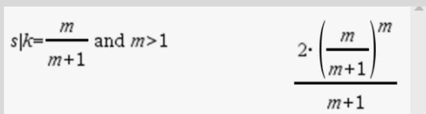
Here are some of the calculator steps needed for the above derivative and solve calculations:

- Press [menu] > **Calculus** > **Derivative**.
- Press [menu] > **Algebra** > **Solve**.
- Press [ctrl][=] to access the '$\le$', '$>$' and '|' symbols.
- Enter as shown adding the conditions $0 \le k \le 1$ and $m > 1$.

***Note**: Alternatively, press [⇧shift][−] to access the **Derivative** template.*

- Perform the substitution into S as shown.

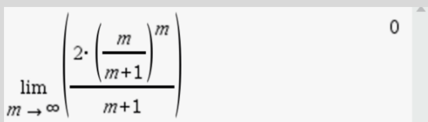
Substituting $k = \frac{m}{m+1}$ into S gives



$$S_{\max} = \frac{2}{m+1}\left(\frac{m}{m+1}\right)^m = \frac{2m^m}{\left(m+1\right)^{m+1}}.$$

Answer: $S_{\max} = \frac{2m^m}{\left(m+1\right)^{m+1}}$ (or equivalent).

(f) Find $\lim_{m\to\infty} S_{\max}$.



- Press [menu] > **Calculus** > **Limit**.
- Press [π▸] to access the '∞' symbol.
- Enter as shown.

Answer: $\lim_{m\to\infty} S_{\max} = 0$.

Creating an average value of a function widget

Create a widget to calculate the average value of a function.

If f is a continuous function on the interval $[a,b]$, then the average (or mean) value of f on $[a,b]$ is given by $f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$.

Question

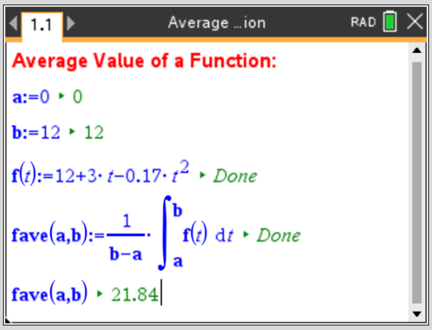
The air temperature, W°C, in a particular suburb during a 12-hour period is modelled by $W = 12 + 3t - 0.17t^2$, where $0 \le t \le 12$ and t is measured in hours.

Find the average temperature during the entire 12-hour period.

Solution

To set up a widget to answer the above (and similar questions), create a **New Document**, then on a **Notes** page:

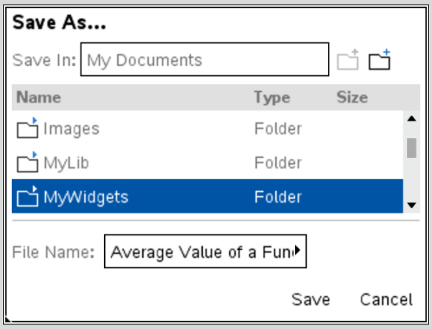
- Enter the widget title text '**Average Value of a Function**' as shown in the screenshot.
- Press [menu] > **Insert** > **Maths Box** (or press [ctrl] [M]) and enter the command $\boldsymbol{a := 0}$.
- Repeat the last step to enter the following (shown right): $\boldsymbol{b := 12}$, $\boldsymbol{f(t) := 12 + 3t - 0.17t^2}$ and $\boldsymbol{fave(a,b) := \frac{1}{b-a}\int_a^b f(t)\,dt}$.



- Enter $\boldsymbol{fave(a,b)}$ as shown.

Answer: 21.8°C, correct to the nearest tenth of a degree.

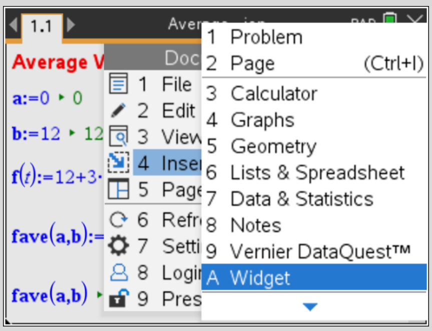
This document can be saved as a widget into the **MyWidgets** folder and opened in any document as a **Widget**.



- Press [doc▾] > **File** > **Save As** and select the **MyWidget** folder.
- Save the widget in this folder as '**Average Value of a Function**'.

To open a saved widget:

- Press [doc▾] > **Insert** > **Widget**. Select the widget you wish to use.
- Alternatively, open a **New** document or press [ctrl] [doc▾] in an existing document and select **Add Widget**. Select the widget you wish to use.



*Note: Entries/objects on a **Notes** page can be rearranged in ways like a word processor.*

3.5 Discrete random variables

3.5.1 General discrete random variables

Finding the mean and variance of a discrete random variable

Question

A discrete random variable X has the following probability distribution:

x	0	10	20	30	40	50
$\Pr(X = x)$	0.04	0.20	0.35	0.25	0.15	0.01

(a) Calculate the value of $\mathrm{E}(X)$ and $\mathrm{Var}(X)$, and the bounds within which 95% of the values of X are expected to lie.

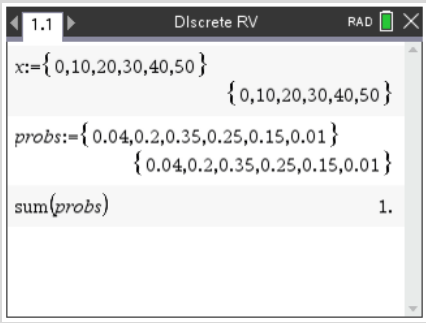
(b) Create a graph of the distribution of X.

Solution

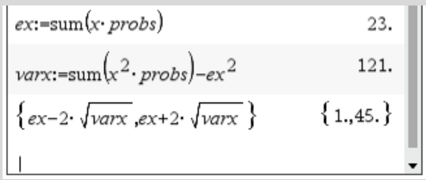
On a **Calculator** page:

(a) Enter the values of X as follows:

- Enter ***x*:={0,10,20,30,40}**.
- Enter ***probs*:={0.04,0.2,0.35,0.25,0.15,0.01}**.
- Enter **sum(*probs*)** to check the values in *probs* sum to 1.



- To define E(X), enter ***ex*:=sum(*x*·*probs*)**
- To define Var(X), enter ***varx*:=sum(*x*²·*probs*)–*ex*²**
- To calculate the estimated bounds for 95% of the values of X, enter the command $\left\{ex-2\sqrt{varx}, ex+2\sqrt{varx}\right\}$.



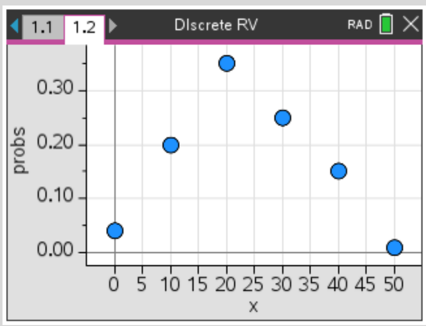
Answer:

It is estimated that 95% of the values of X will be between 1 and 45 (approximately).

(b) Add a **Data & Statistics** page:

- Press [tab] and then select ***x*** for the horizontal axis.
- Press [tab] and then select ***probs*** for the vertical axis.

A plot of the distribution of X will be displayed.



***Note**: If the variables **x** and **probs** are modified to include a different set of values, the plot will be automatically updated, but the viewing window is not updated. Modify the window boundaries to display the new plot, which can be done by pressing* [menu] ***> Window/Zoom > Zoom – Data**.*

Constructing a Notes template to analyse a discrete random variable

It may be convenient to construct a template file for analysing a discrete variable X, including consideration of the mean and variance of linear combinations of X (i.e. $aX + b$).

Question

A discrete random variable X has the following probability distribution:

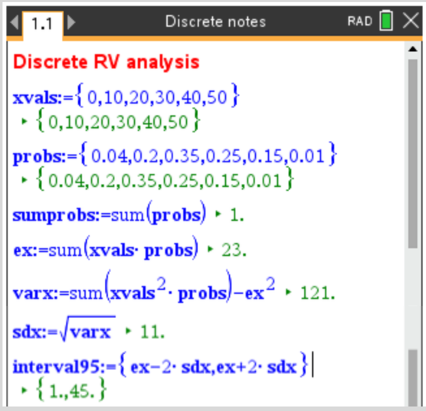
x	0	10	20	30	40	50
$\Pr(X = x)$	0.04	0.20	0.35	0.25	0.15	0.01

Construct a **Notes** page to analyse this discrete random variable where its distribution is given (uses example from previous page).

Solution

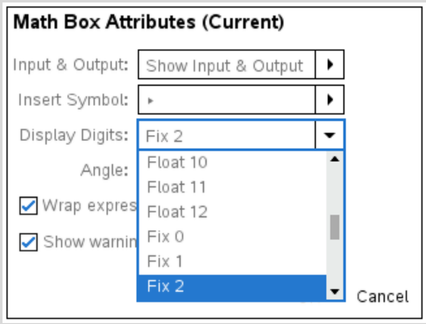
To construct a **Notes** template for assisting with discrete random variable tasks, on a **Notes** page:

- Enter the title text "**Discrete RV** "
- For each of the following, on separate lines, press ctrl M to insert a **Maths Box**, and then:
  - Enter ***xvals*:={0,10,20,30,40,50}**.
  - Enter ***probs*:={0.04,0.20,0.35,0.25,0.15,0.01}**.
  - Enter ***sumprobs*:=sum(*probs*)**.
  - Enter ***ex*:=sum(*xvals*×*probs*)**.
  - Enter ***varx*:= sum(*xvals*²×*probs*)–*ex*²**.
  - Enter ***sdx*:=sqrt(*varx*)**
  - Enter ***interval*95:={*ex*–2*sdx*,*ex*+2*sdx*)**.

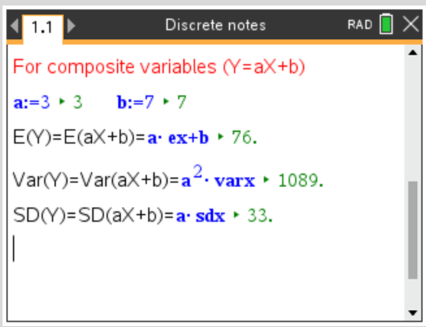


The display precision of any of **Maths Boxes** can be changed in the following manner:

- Click on the relevant **Maths Box**.
- Press menu > **Maths Box Options** > **Maths Box Attributes**.
- Set the **Display Digits** to suit the display precision required (an example is shown right).



***Note**: If needed, calculations for the expected value, variance and standard deviation of the composite variable Y = aX + b can be added to the bottom of the **Notes** page, although formal treatment of the centre and spread of composite variables is no longer in the Mathematical Methods course. See an example on the screen right, which combines text and Maths boxes.*



3.5.2 Binomial distributions

Solving binomial random variable problems

Question

A salesperson has a 30% probability of making a sale to each customer who enters the store. Each sale is independent of all other sales. Find the:

(a) probability that the number of sales exceeds 20 on a day when 40 customers enter the store.
(b) mean & standard deviation of the number of sales on a day when 40 customers enter the store.
(c) an interval within which we expect that the number of sales will lie on 95% of days, when 40 customers enter the store each day.
(d) minimum number of customers who would have to enter the store to have at least a 90% chance or more of making at least one sale.

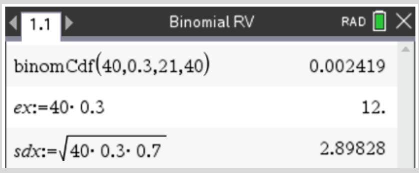
Solution

***Note**: For parts **(a)** to **(c)**, X is the number of sales on a day when 40 customers enter the store:* $X \sim Bi(n = 40, p = 0.3)$.

On a **Calculator** page:

(a) $\Pr(X > 20)$ can be found using the following command:

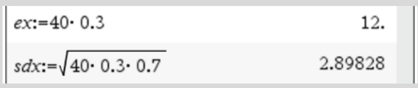
- Press menu > **Probability > Distributions > Binomial Pdf** and enter the command **binomcdf(40,0.3,21,40)**.



Answer: $\Pr(X > 20) = 0.0024$.

(b) The mean number of sales can be found using the formula $E(X) = np$, and the standard deviation can be found using the formula $SD(X) = \sqrt{np(1-p)}$

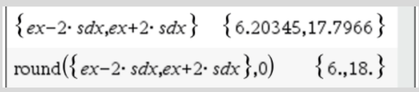
- Enter **ex:=40×0.3**
- Enter **sdx:=$\sqrt{40 \times 0.3 \times 0.7}$**



Answer: $E(X) = 12; SD(X) = 2.90$

(c) Assuming that the distribution of sales is reasonably symmetric about the mean sales (see plot shown on the screen right), it can be estimated that on 95% of days when 40 customers enter the store, the number of sales will be on the interval $[E(X) - 2 \times SD(X), E(X) + 2 \times SD(X)]$.

- Enter **{ex–2×sdx,ex+2×sdx}.**



Answer: The interval is approximately [6,18] after rounding.

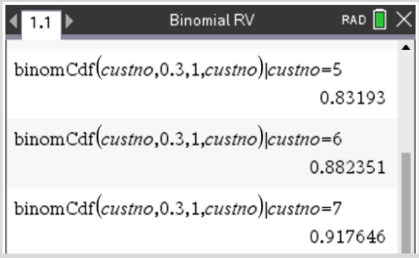
Solution (continued)

(d) To find the minimum number of customers who would have to enter the store for there to be at least a 90% chance of at least one sale, let $X \sim Bi\left(custno, p = 0.3\right)$ and $P(X \geq 1) \geq 0.9$.

To find the minimum number of customers by …

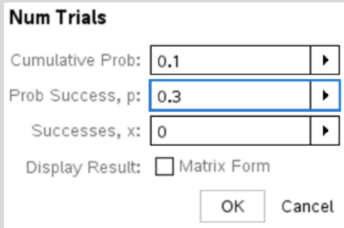
The Guess, check and improve method:

- Enter **binomcdf(custno,0.3,1,custno)|custno=5**
- Increment **custno** until the probability exceeds 0.9



*The **InvBinomN** command method:*

There is an inverse binomial command that will return *n,* the required number of trials. To use it, we need to re-express the desired result as an equivalent inequality with the direction of the inequality reversed. That is, note the following are equivalent statements: $\Pr(X \geq 1) \geq 0.9 \Leftrightarrow \Pr(X = 0) \leq 0.1$.

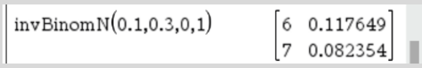


The **invBinomN** command requires the second form (i.e. $\Pr(X = 0) \leq 0.1$).

- Press menu > **Probability** > **Distributions** > **Inverse Binomial N**.
- Enter **Cumulative Prob = 0.1**.
- Enter **Prob Success, *p* = 0.3**.
- Enter **Successes, *x* = 0**.

This gives the answer $n = 7$.

***Note**: When using the **Inverse Binomial N** command via the dialog box, checking the **Matrix Form** box will give the values of* $\Pr(X = 0)$ *for n = 6 and n = 7. This confirms that n = 7 is the minimum n value for which* $\Pr(X = 0) \leq 0.1$ *(and hence* $\Pr(X \geq 1) \geq 0.9$*).*



Visualising the distribution of a binomial random variable

Question

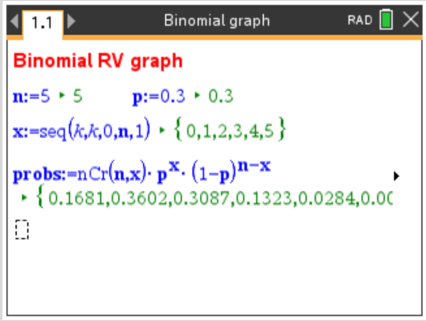
Create a **Notes** page to list and plot the distribution of a binomial random variable.

Solution

On a **Notes** page, enter the title text "**Binomial RV graph** "

Press ctrl M to insert a **Maths Box**, and then:

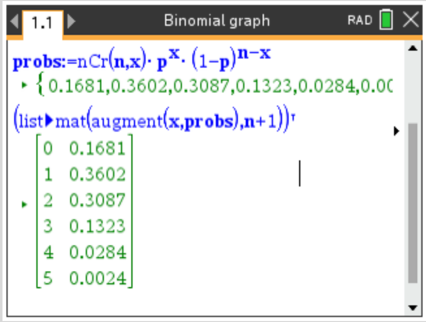
- Enter $n{:=}5$
- Enter $p{:=}0.3$
- Enter $x{:=}\text{seq}(k,k,0,n,1)$
- Enter $probs{:=}\text{nCr}(n,x)\times p^x\times(1-p)^{n-x}$



To create a two-column table of the probabilities for each value of x, add a **Maths Box** and then:

- Press (and then press menu > **Calculations** > **Statistics, List Operations** > **Convert List to Matrix**.
- Press menu > **Calculations** > **Statistics** > **List Operations** > **Augment**.
- Complete as follows: **(list▸mat(augment(*x*,*probs*),*n*+1))^T**

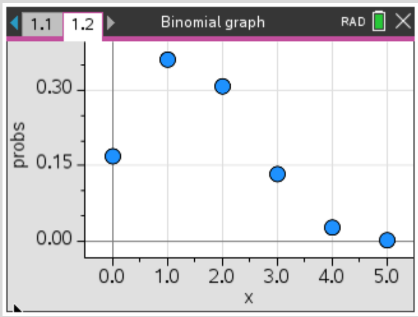
***Note**: To make the probabilities display as a two-column rather than two-row matrix, the **transpose** symbol is added at the end ("^{T}" is found via ctrl 📖).*



To graph the distribution, add a **Data & Statistics** page then:

- Press tab and select ***x*** as the variable to be displayed on the horizontal axis.
- Press tab and select ***probs*** as the variable to be displayed on the vertical axis.

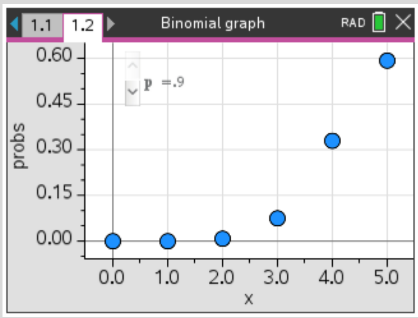
If necessary, to improve the visibility of the plotted points in the window, press menu > **Window/Zoom** > **Zoom–Data**.



To add a slider to view the effect of varying the value of p on the distribution of the binomial random variable:

- Press menu > **Actions** > **Insert Slider**.
- In the **Slider Settings** dialog box that follows, enter the following values:
  Variable = ***p*** Value = **0.3**
  Minimum = **0.1** Maximum = **0.9**
  Step Size = **0.1** Style = **Vertical**
- Check the **Minimised** box and then click **OK** to save these slider settings and return to the **Data & Statistics** page.
- Click the arrow keys to change the value of a within the setting constraints.

***Note**: To move the slider position so the plot is not obscured, hover the cursor over the slider then press ctrl menu and select **Move** to relocate the slider on the page.*



Constructing a Notes template to analyse a binomial random variable

A **Notes** page is a convenient tool for analysing tasks modelled by a binomial random variable. This page can be used as required to answer exam style questions.

Question

Sam is attempting an exam consisting of 40 multiple-choice questions. Each question has four possible answer options (i.e. options A to D). He has not prepared well for this exam, and states that he intends to select his answers to each question by randomly choosing a letter either A, B, C or D. Let X be the number of questions out of the 40 multiple-choice questions that he guesses correctly

(a) What is the chance that he correctly guesses 20 of the 40 questions?

(b) What is the chance that he correctly guesses at least half of the 40 questions?

(c) What is the expected number of questions he guesses correctly?

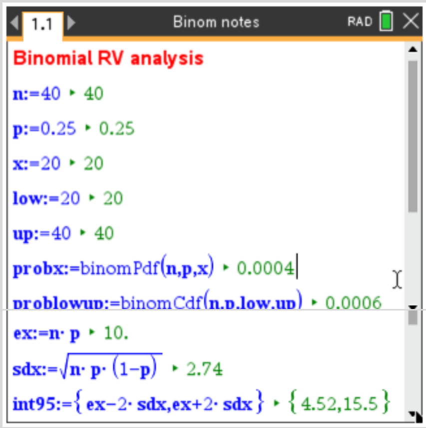
(d) Find an interval within which we would expect 95% of values of X to fall.

(e) Suppose the exam did not consist of 40 multiple-choice questions. How many multiple-choice questions must an exam contain for Sam (using the guessing strategy) to have greater than a 90% chance of correctly guessing at least 10 questions?

Solution

To construct a Notes template for assisting with Binomial random variable tasks, on a **Notes** page:

- Enter the title text "**Binomial RV analysis**"
- For each of the following, on separate lines, press [ctrl] [M] to insert a **Maths Box**, and then:
  - Enter ***n*:=40**.
  - Enter ***p*:=0.25**.
  - Enter ***x*:=20**.
  - Enter ***low*:=20**.
  - Enter ***up*:=40**.
  - Enter ***probx*:=binompdf(*n*,*p*,*x*)**.
  - Enter ***problowup*:=binomcdf(*n*,*p*,*low*,*up*)** .
  - Enter ***ex*:=*n*×*p***.
  - Enter ***sdx*:=sqrt(*n*×*p*× (1–*p*))**.
  - Enter ***int*95:={*ex*–2*sdx*,*ex*+2*sdx*)**.



For the two probability calculations (for ***probx*** and ***problowup***), change the display precision so that they are rounded to four decimal places in the following way:

- Click on the **Maths Box** for ***probx***
- Press [menu] > **Maths Box Options** > **Maths Box Attributes**.
- Set **Display Digits** to **Fix 4** (for 4 decimal places).
- Repeat this for ***problowup***.

*Note: A **Notes** page is like a word processor, in the sense that the text and **Maths Boxes** can be moved around by using the [del], [enter] and [space] keys. For example, the screen at the top of the next page shows a more compact version of the **Notes** page.*

The display precision for the standard deviation (***sdx***) and the 95% intervals (***int*95**) can also be changed to show 3 significant figures (as shown in the screenshots).

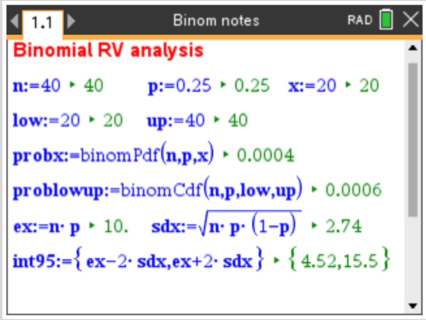
We are now able to answer parts **(a)** to **(d)** – see next page.

... continued

Solution (continued)

Answers to part (a) to (d):
Let X be the number of multiple-choice questions Sam guesses correctly in the 40 question exam.

$$X \sim Bi(n = 40, p = 0.25).$$



Using the Notes page, the following answers can be found.

(a) $\Pr(X = 20) = 0.0004$.

(b) $\Pr(20 \le X \le 40) = 0.0006$.

(c) $E(X) = np = 40 \times 0.25 = 10$ questions correct.

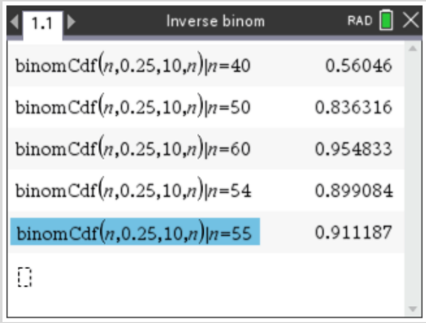
(d) The 95% confidence interval for X is approximately between 4 and 16 correct out of 40.

(e) Let X be the number of multiple-choice questions Sam guesses correctly in an exam with n questions.

$$X \sim Bi(n = ?, p = 0.25)$$

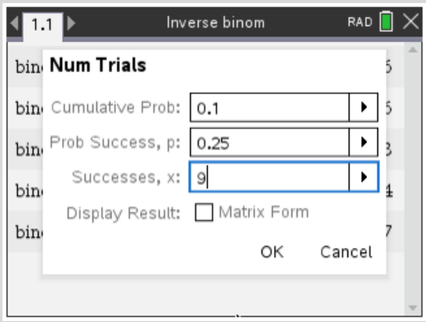
Now find the smallest value of n such that $\Pr(X \ge 10) > 0.9$.

Approach 1: Guess, check and improve
Try possible n values to determine the smallest value of n for which $\Pr(X \ge 10) > 0.9$. To do this, on a **Calculator** page:



- Press [menu] > **Probability** > **Distributions** > **Binomial Cdf …**
- Enter the values ***n*** = ***n***, ***p*** = **0.25**, ***Lower Bound*** = **10** and ***Upper Bound*** = ***n***, then press [enter].
- Copy and paste the previous command and add the condition that ***n*** = **40**, (i.e. **binomCdf(*n*,0.25,10,*n*)|*n*=40**).
- Try different n values to determine that $n = 55$ is the least number of questions for there to be a greater than 90% chance of Sam correctly answering at least 10 questions.

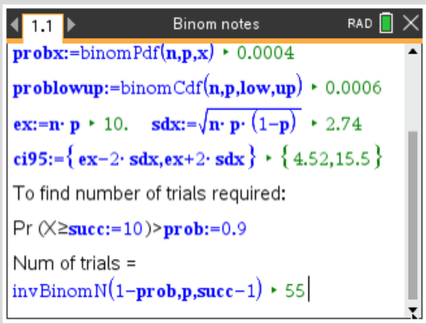
Approach 2: The Inverse Binomial N command
There is an inverse binomial command that will return n, the required number of trials. To use it, we need to re-express the desired result as an equivalent inequality with the direction of the inequality reversed. That is, note the following are equivalent statements: $\Pr(X \ge 10) > 0.9 \Leftrightarrow \Pr(X \le 9) < 0.1$.



The **invBinomN** command requires the second form (i.e. $\Pr(X \le 9) < 0.1$).

- Press [menu] > **Probability** > **Distributions** > **Inverse Binomial N …** and complete the command **invBinomN(0.1,0.25,9).**

This gives the answer $n = 55$.



The screen right shows how calculations using the **invBinomN** command could be added to the Notes page to help answer part **(e)** of the above problem.

3.6 Continuous random variables

3.6.1 General continuous random variables

The probability distribution of a continuous random variable X is a function $f(x)$, called a probability density function, such that $f(x) \geq 0$ for $x \in R$ and $\int f(x)\,dx = 1$.

As probability is given by the area under the curve $y = f(x)$ and the sum of probabilities is 1, then the total area under the curve $y = f(x)$ must be 1.

Performing calculations involving probability density functions

Question

A continuous random variable X has probability density function f defined by

$$f(x) = \begin{cases} kx(2-x), & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}, \text{ where } k \text{ is a real number.}$$

(a) Find the value of k.

(b) Plot the curve $y = f(x)$ and verify graphically that $\int_0^2 f(x)\,dx = 1$.

(c) Find $\Pr(1.2 \leq X \leq 2)$.

(d) Find $\mathrm{E}(X)$.

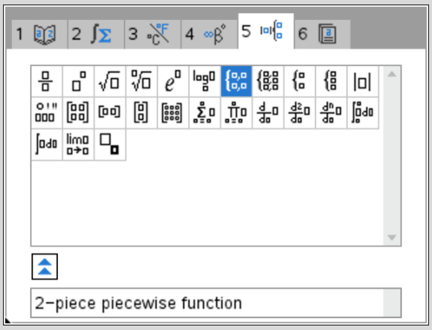
(e) Find $\mathrm{var}(X)$.

Solution

On a **Calculator** page:

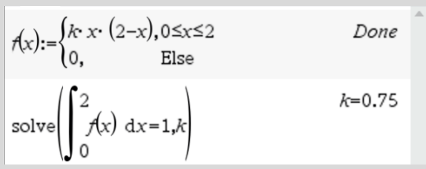
- Press ctrl [templates] to access the **assign** command.
- Press [catalog] 5 to access the **2D symbols** template and select the **2-piece Piecewise Function** template.
- Enter as shown.

***Note**: If the fourth field in the template is left blank, the term **Else** is automatically inserted.*



(a) An example of 'nesting' steps.

- Press menu > **Algebra** > **Solve**.
- Press menu > **Calculus** > **Integral**.
- Enter as shown.



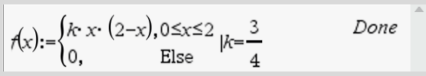
Answer: Solving $\int_0^2 f(x)\,dx = 1$ for k gives $k = 0.75\left(=\frac{3}{4}\right)$.

***Note**: Alternatively, press shift + to access the **Integral** template.*

... continued

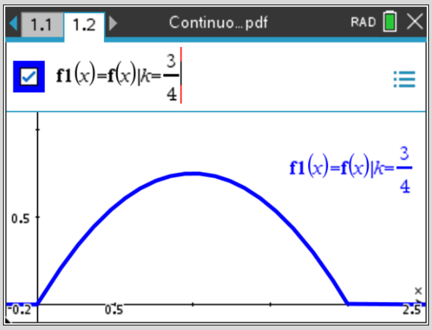
Solution (continued)

Now reassign $f(x)$ with $k=\frac{3}{4}$. (press [ctrl] [=] to access the '|' symbol).



(b) On a **Graphs** page:

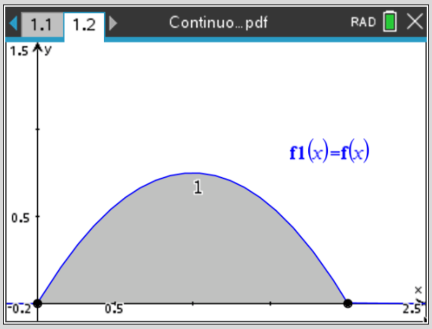
- Enter $\boldsymbol{f1(x)=f(x)\,|\,k=\frac{3}{4}}$.
- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –0.2 XMax = 2.5 XScale = 0.5
  YMin = –0.1 YMax = 1.5 YScale = 0.5



Answer: The curve $y=f(x)$ lies on or above the x- axis, so the requirement that $f(x)$ is non-negative is satisfied.

Check that $\int_0^2 f(x)\,dx=1$.

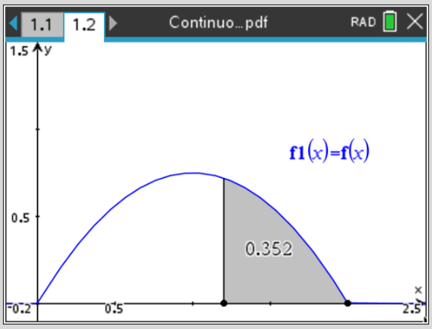
- Go back to the **Calculator** page (press [ctrl] [◀]) where $f(x)$ was originally assigned and reassign it with $k=\frac{3}{4}$. (press [ctrl] [=] to access the '|' symbol).
- Go back to the Graphs page (press [ctrl] [▶]) and re-plot the curve.
- Press [menu] > **Analyse Graph** > **Integral**.
- Enter 0 for the lower bound and press [enter].
- Enter 2 for the upper bound and press [enter].



Answer: As expected, the area is 1.

(c) On the same **Graphs** page that was used for part **(b)**.

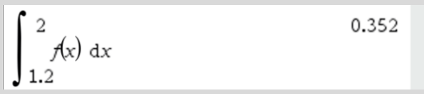
- Press [ctrl] [esc] to remove the previous area calculation.
- Press [menu] > **Analyse Graph** > **Integral**.
- Enter 1.2 for the lower bound and press [enter].
- Enter 2 for the upper bound and press [enter].



Answer: $\Pr(1.2\le X\le 2)=\int_{1.2}^{2} f(x)\,dx=0.352$.

Alternatively on a **Calculator** page:

- Press [menu] > **Calculus** > **Integral**.
- Enter as shown.



Answer: $\Pr(1.2\le X\le 2)=0.352$, as before.

***Note**: Alternatively, press [⇧shift] [+] to access the **Integral** template.*

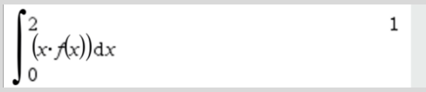
... continued

Solution (continued)

(d) $\mathrm{E}(X)=\int_{0}^{2} xf(x)\,dx$

On a **Calculator** page:

- Press menu > **Calculus** > **Integral**.
- Enter as shown.



Answer: $\mathrm{E}(X)=\int_{0}^{2} xf(x)\,dx=1$.

Note: *Alternatively, press* ⇧shift + *to access the* ***Integral*** *template.*

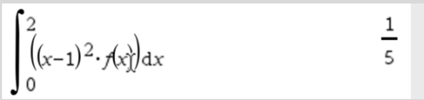
The curve $y=f(x)$ for $0\le x\le 2$ is part of a parabola which is symmetrical about $x=1$.

For a symmetrical pdf, the mean is the x-value corresponding to the line of symmetry.

(e) Using $\operatorname{var}(X)=\int_{0}^{2}(x-1)^{2} f(x)\,dx$

On a **Calculator** page:

- Press menu > **Calculus** > **Integral**.
- Enter as shown.



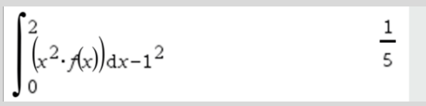
Note: *Alternatively, press* ⇧shift + *to access the* ***Integral*** *template.*

Answer: $\operatorname{var}(X)=\frac{1}{5}(=0.2)$.

Alternatively, $\operatorname{var}(X)=\int_{0}^{2} x^{2} f(x)-1^{2}$:

- Press menu > **Calculus** > **Integral**.
- Enter as shown.

Answer: $\operatorname{var}(X)=\frac{1}{5}(=0.2)$.



3.6.2 Normal distributions

If X is a normally distributed random variable with mean μ and standard deviation σ, then the probability density function of X is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

A random variable Z with the standard normal distribution has probability density function given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}.$$

The standard normal distribution has mean 0 and standard deviation 1.

Recognising features of the graph of the normal distribution probability density function

Question

Explore how changing the values of μ and σ affect the behaviour of the normal probability density curve.

Solution

The parameters used in this example are ***mu*** and ***sigma***.

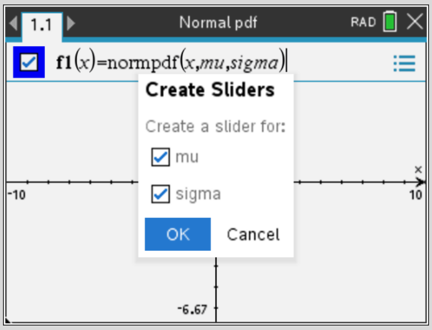
On a **Graphs** page:

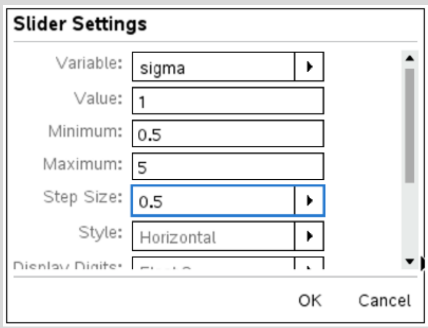
- Enter $\boldsymbol{f1(x) = \text{normPdf}(x, mu, sigma)}$.
- Either access **normPdf(** from the **Catalog** (press `1` `N` and scroll down) or type the command.

***Note**: If typing the command, there is no need to type an uppercase P.*

After entering the function, a prompt will appear to create sliders for ***mu*** and ***sigma***.

- Use the cursor to hover over a slider and press `ctrl` `menu` to access the slider commands.
- Grab each slider and move them both up to the top left-hand corner.
- The slider settings for ***mu*** do not need to be changed here.
- Press `ctrl` `menu` and change the settings as shown for ***sigma*** (standard deviation is always non-negative).
- Press `menu` > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –10 XMax = 10 XScale = 1
  YMin = –0.2 YMax = 1 YScale = 1





... continued

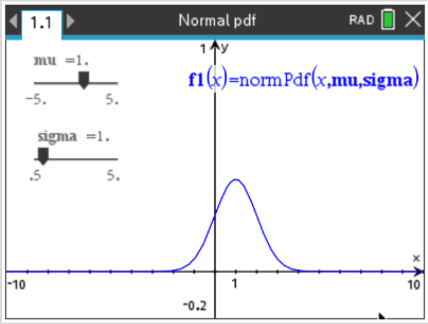
Solution (continued)

- Starting with $\boldsymbol{mu}=0$ and $\boldsymbol{sigma}=1$, vary the value of ***mu*** (the value of ***sigma*** is kept constant) by grabbing the slider for ***mu*** and moving it left and right.

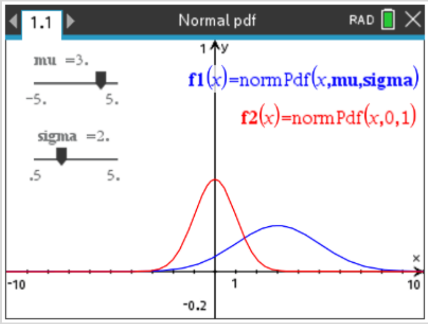
What do you notice about the behaviour of the curve?

- Now repeat for ***sigma*** (the value of ***mu*** is kept constant).

What do you notice about the behaviour of the curve?



- For comparison with the standard normal curve, enter $\boldsymbol{f2(x)=\text{normPdf}(x,0,1)}$.



Answers:

1. Changing μ (keeping σ constant) translates the curve horizontally left and right.
2. Increasing σ (keeping μ constant) causes the curve to be stretched horizontally and compressed vertically. In other words, increasing σ produces a flatter and wider bell-shaped curve.
3. Decreasing σ (keeping μ constant) causes the curve to be compressed horizontally and stretched vertically. In other words, decreasing σ produces a taller and narrower bell-shaped curve.

***Note**: The value of the parameters can also be entered directly, and the sliders minimised (from **Slider Settings**) if desired.*

Calculating probabilities and quantiles associated with a normal distribution

Question

Given that X has a normal distribution with mean 75 and standard deviation 15, find

(a) $\Pr(48 \le X \le 100)$.

(b) $\Pr(X < 60)$.

(c) $\Pr(X > 70)$.

(d) the value of c if $\Pr(X < c) = 0.8$.

(e) the value of d if $\Pr(X \ge d) = 0.745$.

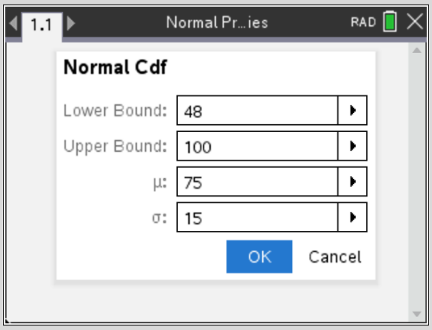
In parts **(a)**, **(b)** and **(c)**, give your answers correct to four decimal places and in parts **(d)** and **(e)**, give your answers correct to two decimal places.

Solution

Use the **Normal Cdf** command to calculate probabilities.

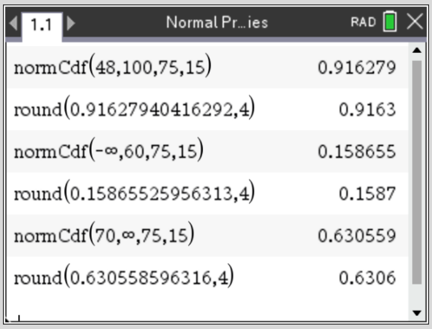
For parts **(a)**, **(b)** and **(c)** on a **Calculator** page:

- Press [menu] > **Probability** > **Distributions** > **Normal Cdf**.
- For part **(a)**, complete the Normal Cdf dialog box as shown.



(a) Answer: $\Pr(48 \le X \le 100) = 0.9163$, correct to four decimal places.

- For part **(b)**, enter **normCdf** $(-\infty, 60, 75, 15)$.



Note: Press [π▸] *to access '∞'.*

(b) Answer: $\Pr(X \le 60) = 0.1587$, correct to four decimal places.

- For part **(c)**, enter **normCdf** $(70, \infty, 75, 15)$.

(c) Answer: $\Pr(X > 70) = 0.6306$, correct to four decimal places.

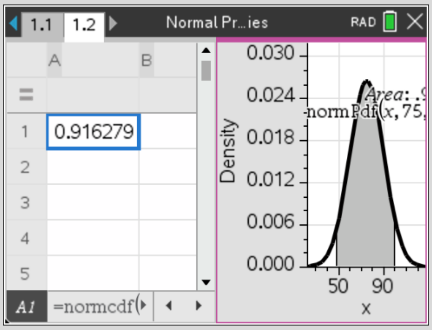
*Note: The answers to parts **(a), (b)** and **(c)** can be rounded to four decimal places by typing **round(ans,4)**.*

Normal distribution calculations can also be performed on a **Lists & Spreadsheet** page.

Accompanying the calculation is the normal pdf curve with the area under the curve shaded (representing probability).

For example, to compete part **(a)**, on a **Lists & Spreadsheet** page:

- Press [menu] > **Statistics** > **Distributions** > **Normal Cdf**.
- Complete the Normal Cdf dialog box and check (press [click]) the **Shade area** box.



(a) Answer: $\Pr(48 \le X \le 100) = 0.9163$ as before.

... continued

Solution (continued)

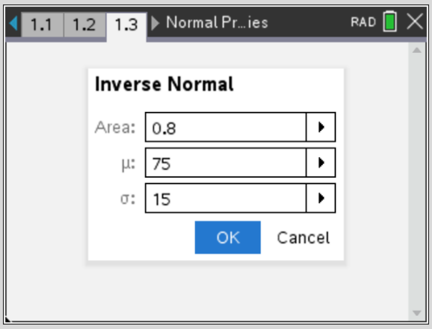
(d) and **(e)**:

The **Inverse Normal** command calculates the value of x for a given probability p where $\Pr(X < x) = p$.

The command and its syntax are $\mathbf{invNorm}(p, \mu, \sigma)$ or $\mathbf{invNorm}(p)$ if $\mu = 0$ and $\sigma = 1$.

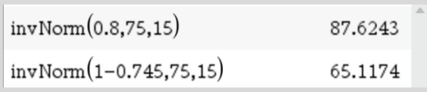
For parts **(d)** and **(e)** on a **Calculator** page:

- Press [menu] > **Probability** > **Distributions** > **Inverse Normal**.
- For part **(d)**, complete the **Inverse Normal** dialog box.



(d) Answer: If $\Pr(X < c) = 0.8$, then $c = 87.62$, correct to two decimal places.

- For part **(e)**, enter $\mathbf{invNorm(1-0.745, 75, 15)}$.



(e) Answer: If $\Pr(X \geq d) = 0.745$, then $d = 65.12$, correct to two decimal places.

***Note**: It is often helpful to sketch a labelled diagram before attempting a normal probability question. A **Notes** page can be set up to solve questions involving the normal distribution.*

Calculating the standard deviation of a normal distribution

Question

The random variable X has a normal distribution with mean 32 and standard deviation σ.

Given that $\Pr(X > 36.8) = 0.3$, find σ.

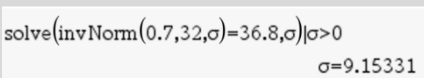
Give your answer correct to two decimal places.

Solution

This example shows how commands can be 'nested'.

Use of **Inverse Normal** on a **Calculator** page:

- Press [menu] > **Algebra** > **Solve**.
- Press [menu] > **Probability** > **Distributions** > **Inverse Normal**.
- Press [📖] [4] and scroll down to select σ.
- Press [ctrl] [=] to access the symbols '|' and '>'.
- Complete as shown.



Answer: $\sigma = 9.15$, correct to two decimal places.

... continued

Solution (continued)

Use of **Normal Cdf** on a **Calculator** page:

- Press menu > **Algebra** > **Solve**.
- Press menu > **Probability** > **Distributions** > **Normal Cdf**.
- Press π▸ to access the '∞' symbol.
- Press 📖 4 and scroll down to select σ.
- Press ctrl = to access the symbols '|' and '>'.
- Complete as shown.

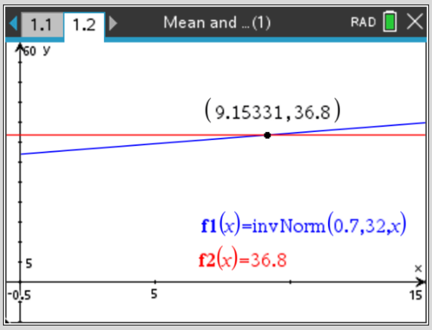
Answer: $\sigma = 9.15$ as before.

Note: *A **Notes** page can be set up to solve such equations.*

Use of **Inverse Normal** on a **Graphs** page:

- Enter $f1(x) = \textbf{invnorm}(0.7, 32, x)$ and $f2(x) = 36.8$.
- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –0.5 XMax = 15 XScale = 5
  YMin = –10 YMax = 60 YScale = 5

Note: *To set an initial window, showing the point of intersection, press* menu *> **Window/Zoom** > **Zoom Fit**.*



To find the point of intersection:

- Press menu > **Geometry** > **Points & Lines** > **Intersection Point(s)**.
- Click on each line.

Answer: $\sigma = 9.15$ as before.

Use of **Normal Cdf** on a **Graphs** page:

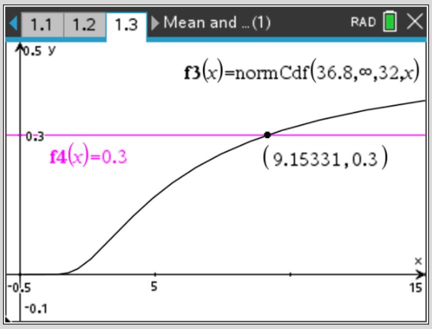
- Enter $f3(x) = \textbf{normCdf}(36.8, \infty, 32, x)$ and $f4(x) = 0.3$.

Note: *Press* π▸ *to access the '∞' symbol.*

- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –0.5 XMax = 15 XScale = 5
  YMin = –0.1 YMax = 0.5 YScale = 0.3

To find the point of intersection:

- Press menu > **Geometry** > **Points & Lines** > **Intersection Point(s)**. Click on each line.



Answer: $\sigma = 9.15$ as before.

Calculating the mean and standard deviation of a normal distribution

Question

The random variable X has a normal distribution with mean μ and standard deviation σ.

Given that $\Pr(X<30)=0.4$ and $\Pr(X<55)=0.9$ find μ and σ.

Give your answers correct to one decimal place.

Solution

Transform from X to Z and form two equations:

$$\Pr(Z<z)=0.4 \Rightarrow \frac{30-\mu}{\sigma}=\text{invNorm}(0.4)(=-0.253...) \quad (1)$$

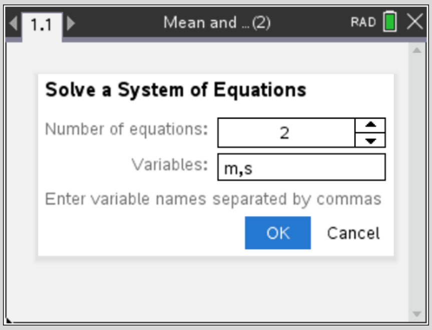
$$\Pr(Z<z)=0.9 \Rightarrow \frac{55-\mu}{\sigma}=\text{invNorm}(0.9)(=1.28...) \quad (2)$$

For example, $\text{invNorm}(0.4)=-0.253...$ means that $\Pr(Z<-0.253...)=0.4$.

On a **Calculator** page:

- Press [menu] > **Algebra** > **Solve System of Equations** > **Solve System of Equations**.

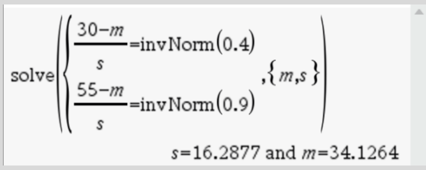
*Note: The **Solve System of Linear Equations** command can be used if both equations are expressed without the denominator σ.*



- Complete the dialog box using m to represent the mean and s to represent the standard deviation.

*Note: The **Solve System of Linear Equations** template does not accept σ.*

- Complete the **Solve System of Linear Equations** template, using [menu] > **Probability** > **Distributions** > **Inverse Normal**.



Answer: $\mu=34.1$ and $\sigma=16.3$, correct to one decimal place.

*Note: A **Notes** page can be set up to solve such systems of equations.*

3.7 Statistical inference for sample proportions

3.7.1 Random sampling

Sampling from a binomial distribution

Question

Thirty students sit a test consisting of 25 multiple-choice questions.

Each question has four possible answers.

Each of the 30 students guess the answer to each question.

Random variable X denotes the number of correct answers obtained by a student.

(a) Use the random binomial (**randBin**) command to simulate this situation.

(b) Determine $\bar{x}$, the sample mean number of correct answers obtained from your simulation. Give the value of $\bar{x}$ correct to two decimal places. Compare the value for $\bar{x}$ to the theoretical mean number of correct answers (expected value).

(c) Repeat part **(b)** for 100 students and then for 1000 students.

(d) Plot a histogram for the simulation of 1000 students sitting the test.

Solution

(a) The syntax for the **Random Binomial** command is **randBin**$(n, p, [\#\text{Trials}])$.

There are 25 multiple-choice questions so $n = 25$.

Each question has 4 alternatives so the probability of obtaining a correct answer (success) , p, is 0.25.

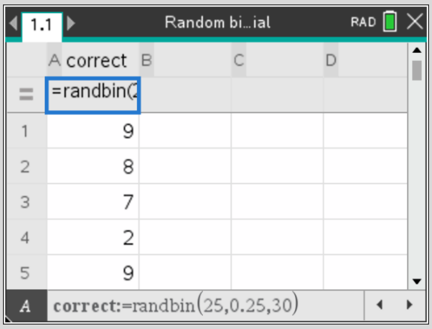
There are 30 'students' so the number of trials is 30.

If required to seed the *TI-Nspire CX II CAS* on a **Calculator** page.

- Press menu > **Probability** > **Random** > **Seed**.
- For example, enter your mobile phone number without the leading zero.

On a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the variable ***correct***.
- In the column A formula cell, press menu > **Data** > **Random** > **Binomial** (for randbin(*n*,*p*,*trials*))
- Enter **randbin(25,0.25,30)** as shown.



The results of the first five 'students' are shown.

Student 1 obtained 9 correct answers.

Student 2 obtained 8 correct answers.

Student 3 obtained 7 correct answers. And so on.

***Note**: The reader's results will be different.*

... continued

Solution (continued)

***Notes**:*
Press ctrl 1 to go to the last entry in a column.
Press ctrl 7 to go to the first entry in a column.
Press ctrl 3 to go down a page
Press ctrl 9 to go up a page.
*To go to a specific cell, press ctrl G and type in the cell reference. The **randBin** command can also be used on a Calculator page.*

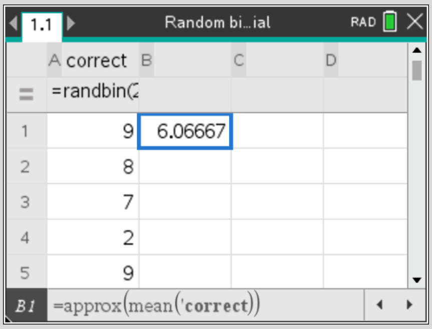
(b) Two approaches on a **Lists & Spreadsheet** page are shown.

Approach 1: Cell formulas

- In cell B1, press = 📖 1 A, scroll down and select **approx(**.

***Note**: The **approximate** command ensures a decimal output.*

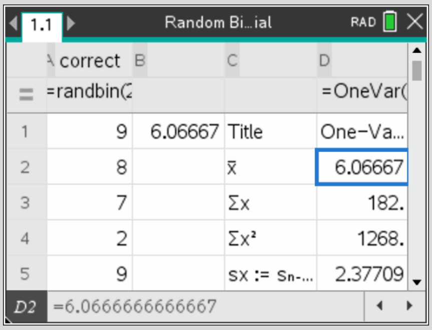
- Press menu > **Data** > **List Maths** > **Mean**.
- Press var and select ***correct***.



Answer: For this simulation, $\bar{x} = 6.07$, correct to two decimal places. Compare to $\mu = 25 \times 0.25 = 6.25$.

Approach 2: Inbuilt statistical functionality

- Press menu > **Statistics** > **Stat Calculations** > **One-Variable Statistics**.
- The number of lists is 1.
- In the X1 list field, press ▸ and select ***correct***.
- Press tab until the 1st Result Column is activated and change it to c[].



Answer: As before, $\bar{x} = 6.07$ and $\mu = 6.25$.

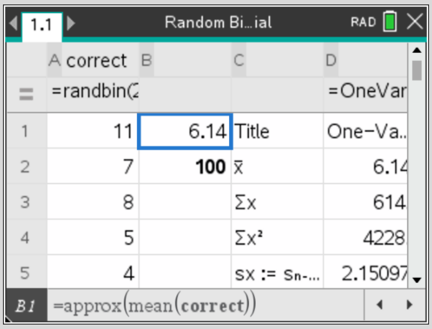
(c) There are now 100 'students' so change the number of trials to 100.

On the same **Lists & Spreadsheet** page:

- In cell B2, enter ***trials*** **:= 100**.
- In the column A formula cell, change 30 to ***trials***.

***Note**: The number of trials can be changed directly in the cell formula without assigning a variable.*

The results of the first five students are now seen.



Answer: For this simulation, $\bar{x} = 6.14$, correct to two decimal places. Compare to $\mu = 6.25$.

... continued

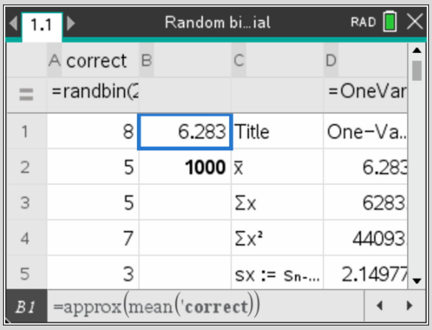
Solution (continued)

Now change the number of 'students' (trials) to 1000.

- In cell B2, change ***trials*** to 1000.

Answer: For this simulation, $\bar{x} = 6.28$, correct to two decimal places. Compare to $\mu = 6.25$.

As the number of 'students' increases, it appears that $\bar{x}$ is getting closer to μ.



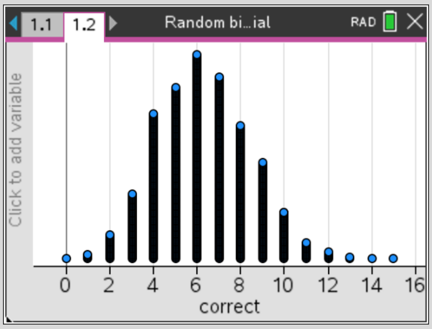
***Note**: To perform another simulation, press* ctrl R *while in the **Lists & Spreadsheet** page.*

(d) On a **Data & Statistics** page:

- Press tab to activate **Click to add variable** underneath the horizontal axis and select ***correct***.

***Note**: The default plot is a dot plot.*

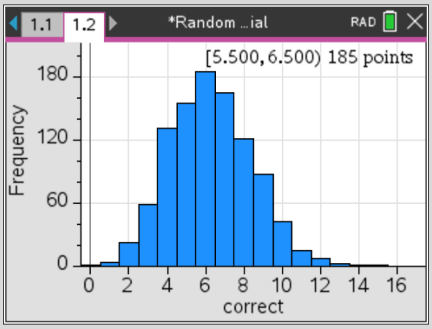
In this simulation, the modal number of correct answers is 6 and the dot plot is approximately normal in shape.



To obtain a histogram:

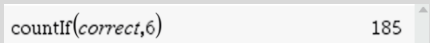
- Press menu > **Plot Type** > **Histogram**.
- Press ctrl menu > **Bin Settings** > **Equal Bin Width**.
- Change **Width** to **1** and **Alignment** to **0**.

Moving the cursor over the histogram will confirm the number of correct answers obtained.



Answer: In this simulation, 185 'students' obtained 6 correct answers.

On a **Calculator** page:

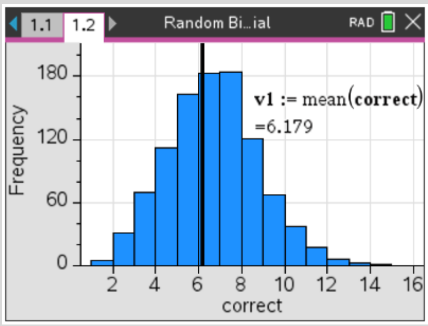


The **countIf** command also shows that 185 'students' obtained 6 correct answers.

***Note**: The **countIf** command (located in the **Catalog**) counts the frequency of a particular outcome. The syntax is **countIf(List,Criteria)**.*

Vertical lines representing $\bar{x}$ and μ can be plotted on the same **Data & Statistics** page as the histogram.

- Press menu > **Analyse** > **Plot Value**.
- Enter $\mathbf{v1 := mean}(\textit{correct})$.
- Press var and select **correct**.
- Press menu > **Analyse** > **Plot Value**.
- Enter $\mathbf{v2 := 6.25}$.



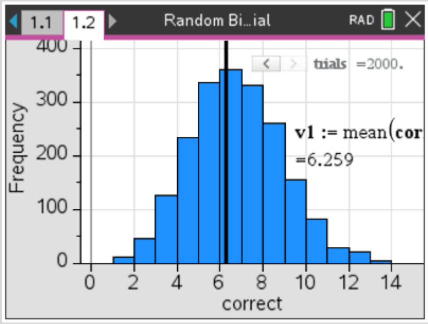
... continued

Solution (continued)

A slider that controls the number of *trials* can be added.

- Press [menu] > **Actions** > **Insert Slider**.

***Note**: In the **Slider Settings**, change **Display Digits** to **Auto** and check (press [click]) the **Minimised** box. To adjust the window when increasing the number of trials, press [menu] > **Window/Zoom** > **Zoom-Data**.*



3.7.2 Sample proportions

Understanding the concept of the sample proportion as a random variable

Question

Assume that 67% of people in Victoria have brown eyes.

(a) Generate a distribution of sample proportions when 200 samples of size 10 ($n = 10$) are randomly selected from the population. Display the distribution on a dot plot and on a histogram. Comment on the distribution generated.

(b) Repeat part **(a)** for 200 random samples of size 30 ($n = 30$). Display the distribution on a dot plot and on a histogram. What do you notice when the sample size is increased?

(c) Explore how changing the sample size, population proportion or the number of samples affects the distribution of sample proportions.

Solution

(a) Generate a distribution on a **Lists & Spreadsheet** page.

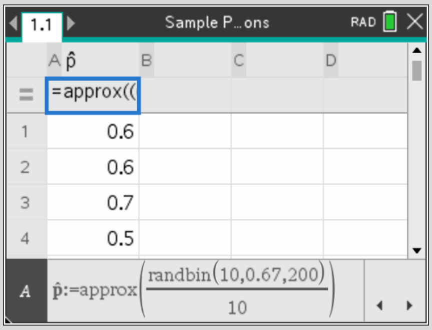
- In the column A heading cell, enter the variable $\hat{p}$.

Note: Press [📖] [4] to access the symbol $\hat{p}$.

- In the column A formula cell, press [📖] [1] [A], scroll down and select **approx(**.
- Press [menu] > **Data** > **Random** > **Binomial**.

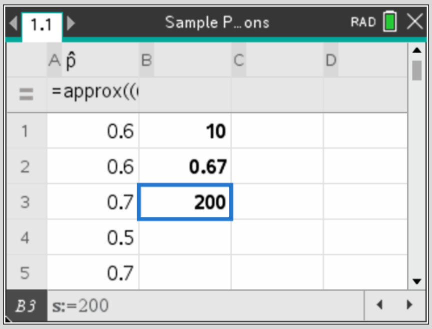
***Note**: To seed TI-Nspire CX II CAS, press* [menu] > ***Probability** > **Random** > **Seed**. The **approximate** command ensures a decimal output.*

- Enter 10 for the sample size, 0.67 for the population proportion and 200 for the number of samples.



To control the sample size, change the population proportion or change the number of samples.

- In cell B1, enter $\boldsymbol{n := 10}$.
- In the column A formula cell, change both occurrences of 10 to $\boldsymbol{n}$.
- In cell B2, enter $\boldsymbol{p := 0.67}$.
- In the column A formula cell, change 0.67 to $\boldsymbol{p}$.
- In cell B3, enter $\boldsymbol{s := 200}$.
- In the column A formula cell, change 200 to $\boldsymbol{s}$.



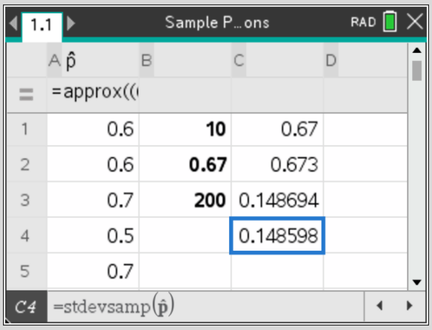
The sample statistics are approximately equal to the population parameters.

***Note**: When defining n, p, and s, you maye be asked to clarify whether each letter is a **column reference** or a **variable reference**. Choose **variable reference**.*

... continued

Solution (continued)

- In cell C1, enter the cell formula '$= p$'.
- In cell C2, enter the cell formula '$=\textbf{mean}(\hat{p})$'.
- In cell C3, enter the cell formula '$=\sqrt{\dfrac{p\cdot(1-p)}{n}}$'.
- In cell C4, press [menu] > **Data** > **List Maths** > **Sample Standard Deviation**. Enter the cell formula '$=\textbf{stdevsamp}(\hat{p})$'.

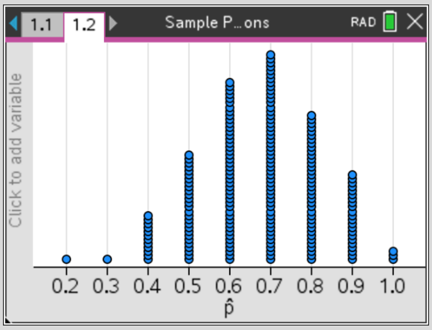


The sample proportion $\hat{P}$ is a random variable whose values vary between samples.

Display the distribution on a **Data & Statistics** page.

- Press [tab] to activate **Click to add variable** underneath the horizontal axis and select $\hat{p}$.

The default plot is a dot plot.

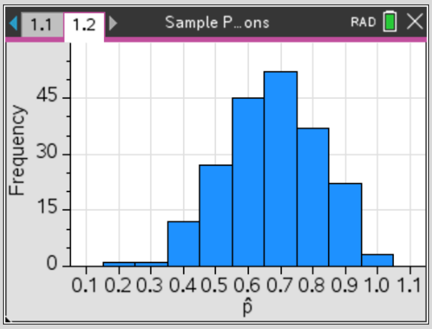


Answer: For small sample sizes, a skewed (negative) distribution results because the population proportion used is not 0.5. We would not expect many samples to have no people with brown eyes or even only one or two people with brown eyes. The distribution is centred roughly around 0.7 (the mode is 0.7) and has values ranging from 0.2 to 1.

To obtain a histogram:

- Press [menu] > **Plot Type** > **Histogram**.
- Press [ctrl] [menu] > **Bin Settings** > **Equal Bin Width**.
- Change **Width** to 0.1 and **Alignment** to 0.05.
- Press [menu] > **Window/Zoom** > **Zoom-Data**.

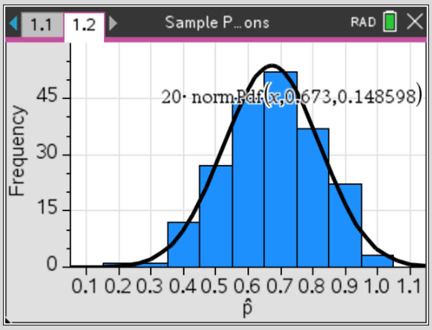
For this simulation, 52 samples had $\hat{p} = 0.7$.



To fit a normal curve to the distribution:

- Press [menu] > **Analyse** > **Show Normal PDF**.

The calculated normal pdf based on the data set shows the mean and standard deviation of the sample proportions.



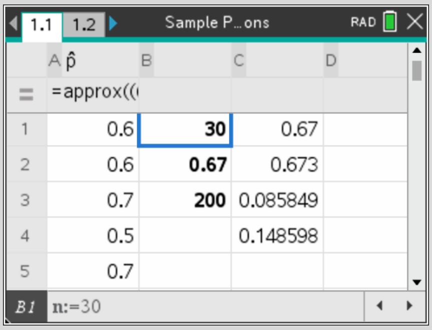
... continued

Solution (continued)

(b) Repeat part **(a)** with a larger sample size of $n = 30$.

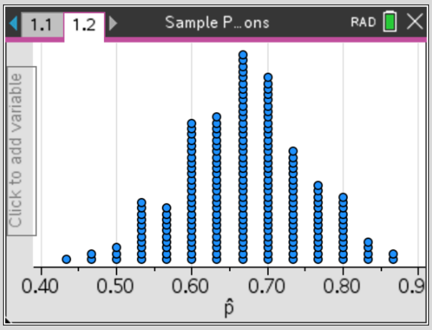
Go back to the **Lists & Spreadsheet** page.

- In cell B1, change ***n*** to 30.



Go back to the **Data & Statistics** page.

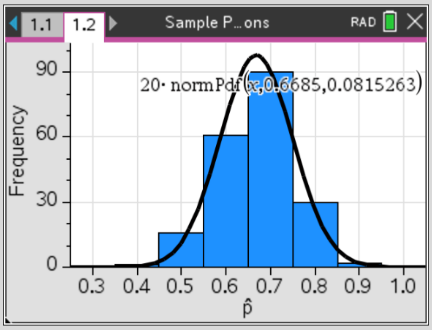
- Press [menu] > **Window/Zoom** > **Zoom-Data** to adjust the window.
- Press [menu] > **Plot Type** > **Dot Plot**.



Answer: As the sample size increases, the distribution of $\hat{P}$ becomes more symmetric and more closely centred around 0.7.

Generate the histogram and superimpose the normal pdf curve over it (as shown on the previous page).

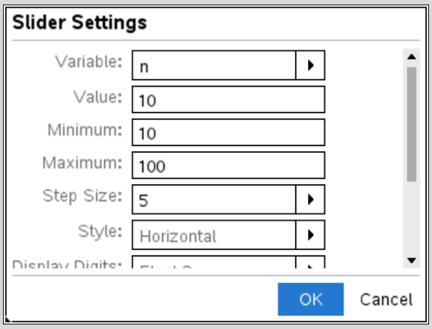
Despite variations, the distribution of $\hat{P}$ tends to behave predictably in terms of shape, centre and spread.



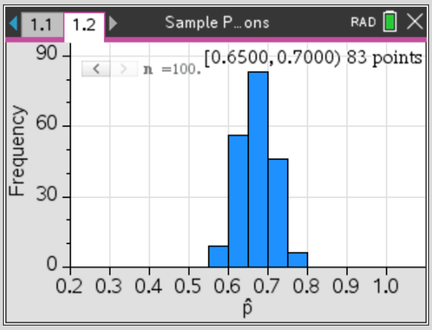
(c) To examine the distribution of $\hat{P}$ more closely, use a slider to increase the sample size, n.

On the **Data & Statistics** page:

- Press [menu] > **Actions** > **Insert Slider**.
- Set the slider as shown and check (press [click]) the **Minimised** box.
- Move the slider to the top left-hand corner and click on it to increase the sample size.
- Press [menu] > **Window/Zoom** > **Window Settings** > **Zoom-Data** to adjust the window.
- Press [ctrl] [menu] > **Bin Settings** > **Equal Bin Width**.
- Change **Width** to 0.05.



Answer: As the sample size increases, the distribution of $\hat{P}$ where $\hat{P} = \dfrac{X}{n}$ becomes more normal.



... continued

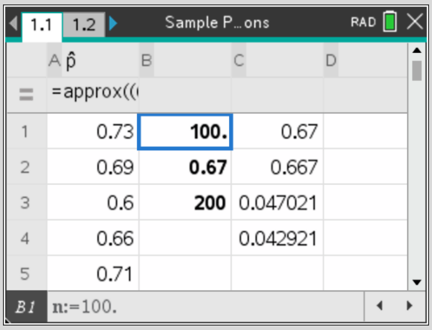
Solution (continued)

When the sample size is large enough, the distribution of a binomial random variable X is well approximated by the normal distribution.

Hence, for large sample sizes, note the approximate normality of $\hat{P}$, the distribution of sample proportions.

For $n = 100$, the sample standard deviation is very close to the population standard deviation and the sample mean of proportions is very close to the population proportion.

When the sample size, n, is large, $\hat{P}$ has an approximately normal distribution with mean p (the population proportion) and standard deviation $\sqrt{\frac{p(1-p)}{n}}$.



Calculating confidence intervals for sample proportions

Question

A survey found that 350 out of 500 people have brown eyes.

Calculate and compare approximate 90%, 95% and 99% confidence intervals for the proportion of people in Victoria who have brown eyes. Give your answers correct to three decimal places.

Solution

The approximate confidence interval $\left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$ is an interval estimate for p, the population proportion where z is the appropriate quantile for the standard normal distribution.

For example, use $\hat{p}$ from a sample to calculate an interval that we are 95% certain contains p (unknown).

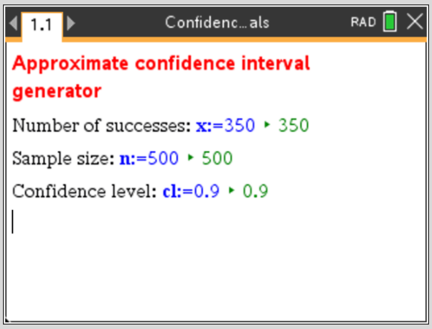
Store the values of the coordinates as follows:

On a **Notes** page:

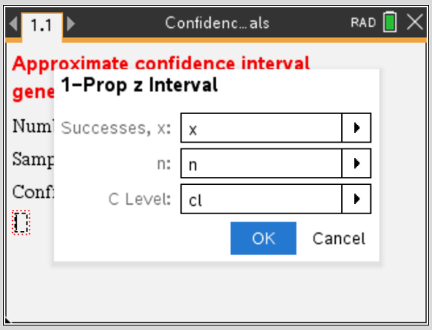
- Enter the text as shown.

*Note: On a **Notes** page, it is useful to add surrounding text that defines or explains what each **Maths Box** represents.*

- Press ctrl menu to insert a **Maths Box**.
- Enter $\boldsymbol{x := 350}$, $\boldsymbol{n := 500}$ and $\boldsymbol{cl := 0.9}$.



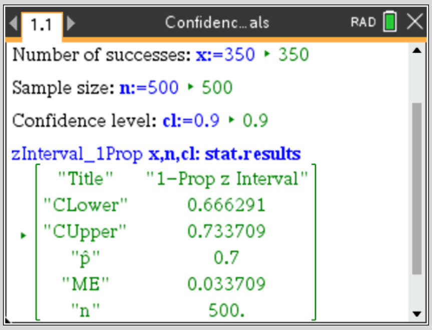
- Press menu > **Calculations** > **Statistics** > **Confidence Intervals** > **1-Prop z Interval**.
- Complete the **Confidence Interval** template as shown.



... continued

Solution (continued)

Answer: The CLower and CUpper values give the 90% approximate confidence interval (0.666, 0.734), correct to three decimal places.



To determine the approximate 95% and 99% confidence intervals:

- Change the confidence level to 0.95.

The 95% approximate confidence interval is $(0.660, 0.740)$, correct to three decimal places.

- Change the confidence level to 0.99.

The 99% approximate confidence interval is $(0.647, 0.753)$, correct to three decimal places.

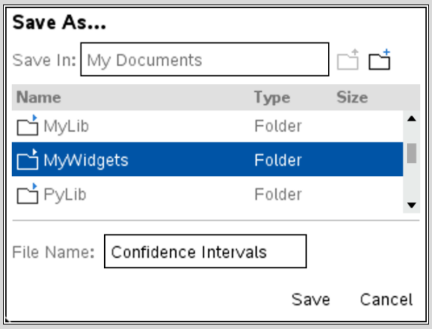
Being more confident means a wider interval is required.

There are variations in confidence intervals between samples and most, but not all, confidence intervals contain *p*.

***Note**: 'ME' in the output stands for margin of error.*
Press [📖] [4] to access the symbol $\hat{p}$.

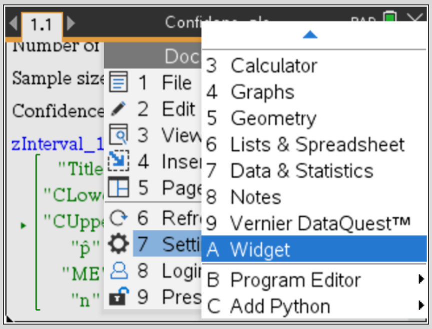
This document can be saved as a widget into the **MyWidgets** folder and opened in any document as a **Widget**.

- Press [doc▾] > **File** > **Save As** and select the **MyWidget** folder.
- Save the widget in this folder as '**Confidence Intervals**'.



To open a saved widget:

- Press [doc▾] > **Insert** > **Widget**. Select the widget you wish to use.
- Alternatively, open a **New** document or press [ctrl] [doc▾] in an existing document and select **Add Widget**. Select the widget you wish to use.



Understanding confidence intervals for sample proportions

Question

Assume that the true population proportion of Australian adults with hypertension is $p = 0.3$.

When different random samples are drawn from this population and a confidence interval is calculated for each sample's proportion, those intervals will vary due to sampling variability.

Use simulation to investigate how confidence intervals for p vary when calculated from random samples of size $n = 200$ taken from this population.

(a) Compare intervals based on the same point estimate, $\hat{p}$, but with confidence levels of 99%, 95%, 90% and 50%.

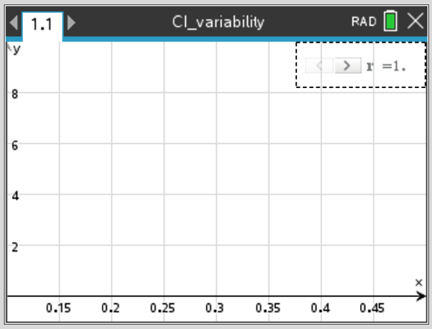
(b) Observe whether a calculated confidence interval contains the true population proportion, p, and interpret the meaning of a C% confidence interval.

Solution

To set up a slider that will be used to trigger the taking of a new sample, on a **Graphs** page in a new document:

- Press [menu] > **Window/Zoom > Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = **0.1** Xmax = **0.5** XScale = **0.05**
  YMin = **−2** YMax = **10** YScale = **2**
- Press [menu] > **Actions > Insert Slider**. Enter the settings:

  variable: **r**, min: **1,** max: **100,** step:**1,** minimise: ☑**.**

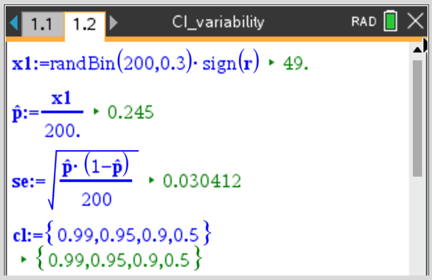


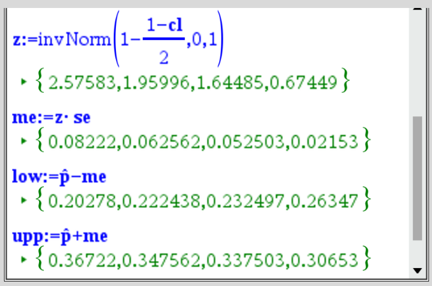
To set up the simulation, add a **Notes** page, then:

- Press [ctrl] [M] to add a **Maths Box**
- In the Maths Box, enter ***x*1:= randBin(200,0.3)×sign(*r*)**, pressing [📖] [1] [R] or [S] to select **randBin** or **sign**.

*Note: **sign(r)**=+1 when **r** >0 and has no effect on the result. However, changing the value of **r** triggers a new sample.*

- Enter the following, with each entry in a new **Maths Box**.
- $\hat{p} := x1/200$ (select $\hat{p}$ from the [∞β°] options).
- $se := \sqrt{\dfrac{\hat{p}\times(1-\hat{p})}{200.}}$ (estimates the standard deviation)
- $cl := \{0.99, 0.95, 0.90, 0.5\}$ (sets the confidence level)
- $z :=$ invNorm$\left(1-\dfrac{1-cl}{2}, 0, 1\right)$ (corresponding z-scores)
- $me := z \times se$ (calculates the margins of error for each CI).
- $low := \hat{p} - me$ (Lower fence for each CI).
- $upp := \hat{p} + me$ (Upper fence for each CI).





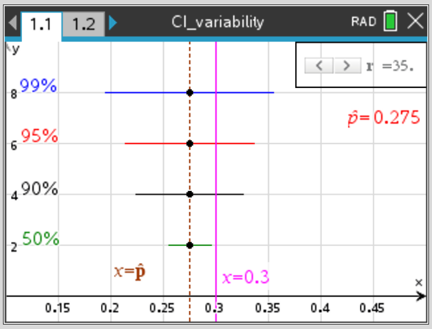
... continued

Solution (continued)

To display the confidence intervals, on the **Graphs** page 1.1:

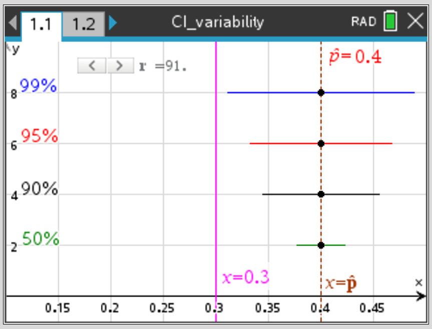
- Press [ctrl] [G] and enter:
- $f1(x)=8\,|\,\boldsymbol{low}[1]\le x\le \boldsymbol{upp}[1]$
- $f2(x)=6\,|\,\boldsymbol{low}[2]\le x\le \boldsymbol{upp}[2]$
- $f3(x)=4\,|\,\boldsymbol{low}[3]\le x\le \boldsymbol{upp}[3]$
- $f4(x)=2\,|\,\boldsymbol{low}[4]\le x\le \boldsymbol{upp}[4]$

Note: ***low**[1] and **upp**[1] denotes the first values in the lists **low** and **upp**, which correspond to 99% CI.*



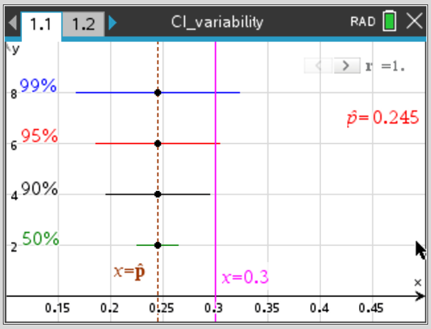
- Press [menu] > **Graph Entry/Edit** > **Relation**.
- Enter $x=\hat{p}$ then $x=0.3$.
- Press [menu] > **Geometry** > **Points & Lines** > **Intersection Points**. Click line $x=\hat{p}$ then each horizontal interval.
- Press [menu] > **Show/Hide**. Click to hide unwanted labels, then [esc].

Use the slider to take samples and observe which intervals contain the p (i.e. intersect the line $x=0.3$).

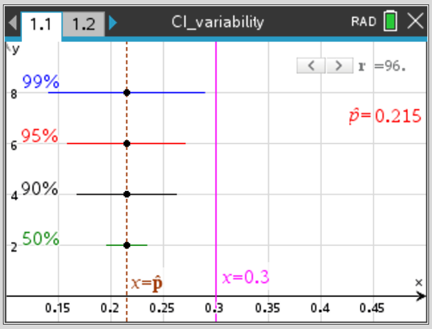


(a) Comparing intervals based on the same $\hat{p}$, but with confidence levels of 99%, 95%, 90% and 50%.

Answer: The simulation demonstrates that to have a greater level of confidence that the interval will contain the true population proportion, *p*, requires a greater margin of error. Conversely, a smaller margin of error results in less confidence that the interval will contain *p*. In the example shown, the two confidence intervals containing *p* have a greater radius (margin of error) than the 50% CI.



(b) Interpret the meaning of a *C*% confidence interval.



Answer: The simulation demonstrates that if many random samples of the same size are taken from the population and a confidence interval is constructed from each one, then about *C*% of those intervals will contain the true population proportion *p*, and about $(100-C)\%$ of intervals will not contain *p*. In the example shown, none of the confidence intervals contain *p*, regardless of the confidence level.

What a *C*% confidence interval for *p* does **not** mean:

A *C*% CI does **not** mean that there is a *C*% chance that the realised CI contains *p*. Once a sample has been taken and the CI calculated, the realised CI will either contain *p* or it will not contain *p*; there is no longer randomness.

The confidence level (99%, 95% etc.) associated with CIs for *p* relates to the reliability of the estimation **procedure** used to produce an interval that will contain the true value of *p*. It does not relate to a specific calculated interval.

Appendix: TI-Nspire shortcuts and tips

(With thanks to Neale Woods. Note that a tick (✓) indicates where the shortcut is applicable. MacOS users – substitute CMD for ctrl.)

Shortcut	Handheld	Computer	Glob	Calc	Gra	Geo	L&S	D&S	Note	Data	Prog	Pyth	Result
ctrl A	✓	✓		✓					✓	✓	✓	✓	Select all text
ctrl B	✓	✓									✓	✓	Check Syntax & Store/Save
ctrl C	✓	✓	✓										Copy text
ctrl E	✓	✓							✓				Insert a Chem Box
ctrl F	✓	✓									✓	✓	Find
ctrl G	✓	✓			✓								Toggle Graph entry line
ctrl G	✓	✓					✓				✓	✓	Go to line/cell
ctrl H	✓	✓									✓	✓	Find and replace
ctrl I	✓	✓	✓										Insert page
ctrl J		✓	✓										Take screen capture
ctrl K	✓	✓	✓										Select page (in split screen)
ctrl L	✓	✓		✓	✓	✓			✓		✓		Display variable list
ctrl L	✓	✓					✓						Store or link to variable
ctrl M	✓	✓							✓				Insert Math Box
ctrl N	✓	✓	✓										New document
ctrl O	✓	✓	✓										Open document
ctrl P		✓	✓										Print document
ctrl R	✓	✓					✓						Recalculate
ctrl R	✓	✓									✓	✓	Checks syntax/run program
ctrl S	✓	✓	✓										Save document
ctrl T	✓	✓			✓		✓						Toggle Table/No Table
ctrl T	✓	✓									✓	✓	Comment/Uncomment
ctrl V	✓	✓	✓										Paste text
ctrl W	✓	✓	✓										Close current document
ctrl X	✓	✓	✓										Cut text
ctrl Y	✓	✓	✓										Redo
ctrl Z	✓	✓	✓										Undo

Appendix: TI-Nspire shortcuts and tips (continued)

(Note that a tick (✓) indicates where the shortcut is applicable. MacOS users – substitute CMD for ctrl *.)*

Shortcut	Handheld	Computer	Global	Calc	Graph	Geom	L&S	D&S	Notes	DataQ	Prog	Python	Result
ctrl 1	✓	✓		✓			✓		✓	✓	✓	✓	Jump to last line
ctrl 2	✓	✓		✓			✓		✓	✓	✓	✓	Jump to end of line/last cell
ctrl 3	✓	✓		✓			✓		✓	✓	✓	✓	Page down
ctrl 4	✓	✓	✓										Merge two pages into split screen.
ctrl 6	✓	✓	✓										Convert split screen into two pages
ctrl 7	✓	✓		✓			✓		✓	✓	✓	✓	Jump to first line
ctrl 8	✓	✓		✓			✓		✓	✓	✓	✓	Jump to start of line/first cell
ctrl 9	✓	✓		✓			✓		✓			✓	Page up
ctrl ␣	✓	✓	✓										Underscore
ctrl tab	✓		✓										Toggle b/w split screen windows
ctrl tab		✓	✓										Toggle b/w open documents
tab	✓	✓		✓	✓		✓	✓	✓	✓	✓		Move through fields or zones
tab	✓	✓			✓								Display graph entry line
tab	✓	✓										✓	Indent
⇧shift ◀▶▲▼	✓	✓	✓										Highlight selected text
⇧shift tab	✓	✓			✓		✓	✓	✓	✓			Move back through fields or zones
⇧shift tab	✓	✓										✓	Remove indent
⇧shift +	✓			✓	✓		✓	✓	✓		✓		Derivative
⇧shift −	✓			✓	✓		✓	✓	✓		✓		Integral
⇧shift ↵	✓			✓	✓			✓	✓		✓		Add a column to a matrix
⇧shift enter		✓		✓	✓			✓	✓		✓		Add a column to a matrix
↵	✓			✓	✓			✓	✓		✓		Add a row to a matrix
Alt enter		✓		✓	✓			✓	✓		✓		Add a row to a matrix
P	✓	✓			✓	✓							Add a point

Appendix: TI-Nspire Shortcuts and Tips (continued)

From the Handheld or Computer Keyboard

To enter this:	Type this shortcut:
π	**pi**
θ	**theta**
∞	**infinity**
≤	**<=**
≥	**>=**
≠	**/=**
⇒ (logical implication)	**=>**
⇔ (logical double implication, XNOR)	**<=>**
→ (store operator)	**=:**
| | (absolute value)	**abs(...)**
√()	**sqrt(...)**
Σ() (Sum template)	**sumSeq(...)**
Π() (Product template)	**prodSeq(...)**
$\sin^{-1}()$, $\cos^{-1}()$, ...	**arcsin(...), arccos(...), ...**
ΔList()	**deltaList(...)**
ΔtmpCnv()	**deltaTmpCnv(…)**

From the Computer Keyboard

To enter this:	Type this shortcut:
e (natural log base e)	**@e**
E (scientific notation)	**@E**
^{T} (transpose)	**@t**
^{r} (radians)	**@r**
° (degrees)	**@d**
^{g} (gradians)	**@g**
∠ (angle)	**@<**
▶ (conversion)	**@>**
▶ Decimal, ▶ approxFraction(), and so on.	**@>Decimal, @>approxFraction(), and so on.**
*c*1, *c*2, … (constants)	**@c1, @c2, …**
*n*1, *n*2, … (integer constants)	**@n1, @n2, …**
i (imaginary constant)	**@i**

Useful functions/commands available in the Catalog not available in the menus.

Function/Command name	Function/Command purpose
and	Boolean 'and', useful for specifying restrictions.
domain(expr,var)	Displays the domain of a function.
euler(…)	Generates a table of values using Euler's method.
isPrime(…)	Displays 'true' if prime and 'false' if composite.
true	Displays 'true' if two expressions are equivalent.

Texas Instruments Australia

Email: teacher-support@list.ti.com

Website: **education.ti.com/australia**

stationary points

composite functions

discrete random variables

properties of integrals

function modelling

statistical inference

sample proportion

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + c, n \neq -1$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$