VCE Specialist Mathematics Teacher Resource Book *for*

TI-Nspire™ CX II CAS graphing calculator

Teachers Teaching with Technology™
Professional Development from Texas Instruments

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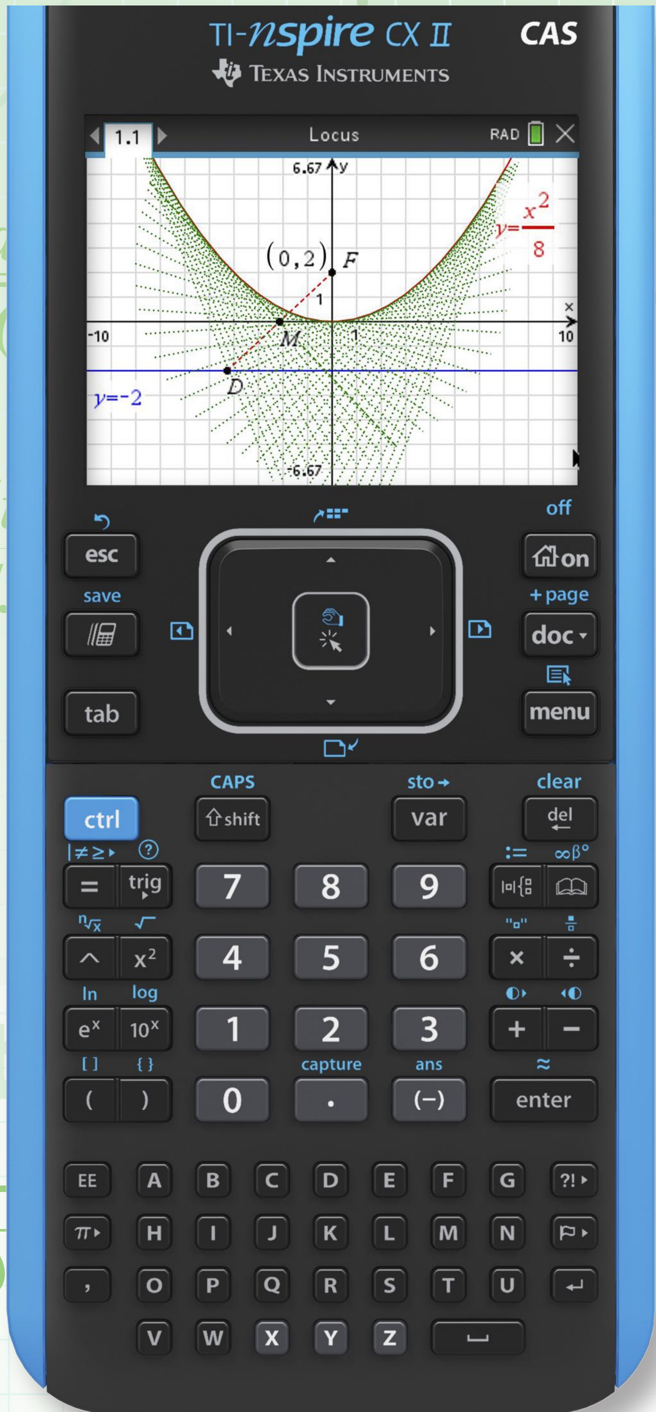

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Teachers Teaching with Technology™

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The Teachers Teaching with Technology™ (T³™) Australia professional learning organization is comprised of some of the most creative and innovative mathematics and STEM teachers in the world. They are dynamic and passionate educators who share their knowledge and expertise with secondary teachers and students through professional development events and resource creation.

$$tan(x-y) = \frac{tan(x) - tan(y)}{1 + tan(x)\ tan(y)}$$

circular functions

algebra

$$z^n = r^n cis(n\theta)$$

number

$$z = x + iy = r(\cos(\theta) + i\sin(\theta)) = rcis(\theta)$$

structure

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Introduction

This publication, *VCE Specialist Mathematics Teacher Resource Book for the TI-Nspire™ CX II CAS*, is intended to support senior secondary school mathematics teachers in Victoria as they seek to teach the *VCAA Mathematics Study Design 2023*.

Specifically, the publication highlights ways in which *TI-Nspire CAS* technology might be used to assist in the teaching, learning and assessment of *VCE Specialist Mathematics Units 1 to 4*.

It is not a complete manual for using this technology, rather it tries to look at each syllabus dot point and make suggestions for possible classroom use.

It has been developed by experienced educators and reviewed by senior mathematics teachers from Victorian schools. We hope you find this to be a useful and supportive publication.

[**Note:** A digital version of this publication can be found at *https://education.ti.com/aus/VIC*].

Notes for teachers

To maximise the usefulness of *VCE Specialist Mathematics Teacher Resource Book for the TI-Nspire™ CX II CAS* to teachers, the authors have provided the following explanatory notes.

- It is assumed that the user of this teacher resource book has a basic familiarity with navigating calculator documents and pages. Readers requiring an introduction to this are referred to tutorials at https://education.ti.com/aus and https://www.youtube.com/@TIAustralia.
- Throughout this publication, unless otherwise specified, the default calculator document settings have been used. The calculator user interface language has been set to *English (U.K.).*
- For each example task, it is desirable to start a new calculator document ([ctrl] [N] or [on] [1]). Alternatively, insert a new problem ([doc] > **Insert** > **Problem**).
- When working with functions, use of the **assign** command (via '**:=**') has been privileged over the **define** command. While both commands essentially perform the same role, the **assign** command is a more natural command to use in the **Notes** and **Calculator** applications.
- Implied multiplication has been assumed when working with products such as $6x$. However, where it is necessary to use the multiplication key when entering the product *bx*, for example, the symbol '·' or '×' is used to denote multiplication.
- In some instances in this publication, space has been added to the syntax of commands to improve readability, even though in general spaces should **not** be used in calculator commands without a clear reason. For example, when entering a function, the authors may have expressed this as ***f*(*x*) := *a* + *bx*,** but on the calculator, it will appear as *f*(*x*):=*a*+*b*·*x*.
- There will be some variation in the formatting of commands and text to be entered, but the authors have attempted to use bold formatting when referring to commands to be entered or accessed via the calculator.
- For screenshots from the **Graphs** application, grid and label settings will vary. Use [menu] commands (or [ctrl] [menu]) to modify these settings for an open document. The default settings (for all documents) for grids and labels can be edited by pressing [menu] and then select **Settings**.
- When catalogue commands are mentioned, pressing [catalog] and then [1] will display catalog commands in alphabetical order. Pressing the first letter of the desired command will locate it more quickly.

To make this publication as practical and concise as possible, mathematical problems considered have been restricted to those that can be attempted by teachers and students without using pre-prepared files. For more interactive digital resources aligned to *VCE Specialist Mathematics Units 1 to 4*, go to https://education.ti.com/aus.

VCE Specialist Mathematics Unit 1

1.1 Algebra, number and structure

1.1.1 Proof and number

Rationalising the denominator of a surd

Question

Rationalise the denominator of $\dfrac{5+\sqrt{6}}{5-\sqrt{6}}$.

Solution

To enter $\dfrac{5+\sqrt{6}}{5-\sqrt{6}}$ on a **Calculator** page:

- Press [ctrl] [÷] to access the **Fraction** template.
- Press [ctrl] [x²] to access $\sqrt{\square}$.
- Enter as shown.

Answer: $\dfrac{10\sqrt{6}+31}{19}$

$$\frac{5+\sqrt{6}}{5-\sqrt{6}}\times\frac{5+\sqrt{6}}{5+\sqrt{6}}=\frac{\left(5+\sqrt{6}\right)^2}{25-6}=\frac{25+2\left(5\sqrt{6}\right)+6}{19}=\frac{10\sqrt{6}+31}{19}$$

Listing factors

A positive integer d is a factor of a positive integer n if there exists a positive integer k such that $n = dk$.

***Note**: In this section, a library of commands and functions related to number theory are used. For example, **divisors(n)** and **listprimediv(n)** are examples of such commands. All such commands are accessible from the Catalog via* [📖] [6].

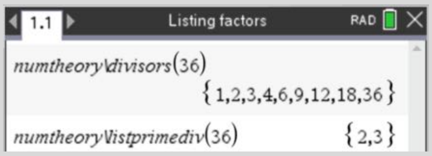
Question

List all the factors of 36.

Solution

On a **Calculator** page:

- Press [📖] [6] > **numtheory** > **divisors**.
- Enter as shown.



Answer: 1, 2, 3, 4, 6, 9, 12, 18, 36

***Note:** The command **listprimediv(n)** gives the list of prime factors of a positive integer **n**.*

Listing factors using the Programme Editor

As mentioned previously, a positive integer d is a factor of a positive integer n if there exists a positive integer k such that $n = dk$.

Alternatively, a positive integer d is a factor of a positive integer n if $\frac{n}{d}$ has a remainder of zero.

***Note**: The TI-Nspire CX II CAS has a command which returns the remainder when a positive integer **n** is divided by a positive integer **d**. Its syntax is **remain(n,d)**.*

The following describes an algorithm (in pseudocode) and matching example code using the TI-Nspire **Programme Editor** to produce a factor list for any positive integer n:

Pseudocode	***Programme Editor code***
$i \leftarrow 0$	*i*:=0
list ← empty	*list*:={}
for *d* **from** 1 **to** *n*	**For** *d*,1,*n*,1
if remainder (*n*/*d*) = 0	**If** remain(*n*,*d*)=0 **Then**
$i \leftarrow i + 1$	*i*:=*i*+1
list(*i*) ← *d*	*list*[*i*]:=*d*
end if	**EndIf**
end for	**EndFor**
print *list*	**Disp** *list*

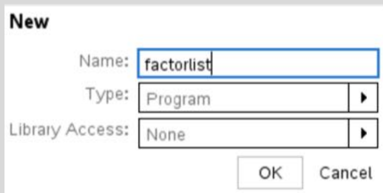
Question

Enter the code from the **Programme Editor** column (above right), then run the program **factorlist(*n*)** to test its accuracy for n = 12 and 84.

Solution

To start coding, in a new **Problem** or a new **Document**:

- Select **Add Programme Editor > New**.
- In the dialog box that follows, enter as shown.



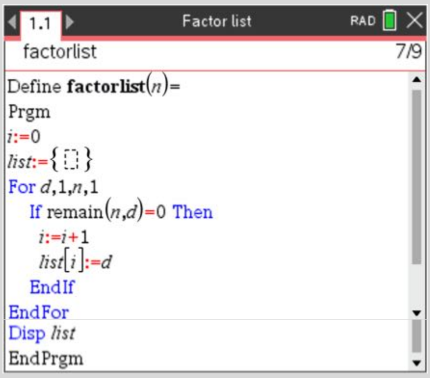
The **Program Editor** will load, ready to accept the code.

To name the inputs ***n***, at the top of the page:

- Enter ***n*** in the brackets for **factorlist(*n*)**=

To initialise the values of ***i*** and ***list***, after the "Prgm' line:

- Enter ***i*:=0** to set the initial value of *i*.
- Enter ***list*:={ }** to set *list* to be initially empty.

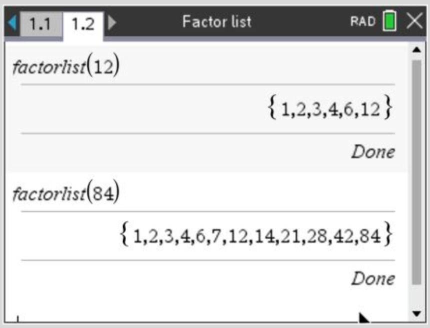


To create the **For** loop block of code:

- Press menu **> Control > For…EndFor**.
- Enter **For *d*,1,*n*,1** as shown.

To create the **If** block of code:

- Press menu **> Control > If …Then…EndIf**.
- In the **If** line, enter **remain(*n*,*d*)=0** (as shown).
- In the lines between the **If** and **EndIf** lines, enter ***i*:=*i*+1** to add one to the value of *i*, then enter ***list*[*i*]:=*d*** to add a factor to the list of factors (as shown).



To display the list of factors, after the **EndFor** line:

- Press menu **> I/O > Disp** and enter **Disp *list***.

To check syntax, save and run the program on a new page:

- Press ctrl B followed by ctrl R (or menu **> Check Syntax & Store > Run**).

Answer: The factors of 12 and 84 are displayed at right.

Listing prime numbers

A positive integer p is a prime number if $p > 1$, and the only factors of p are 1 and p itself.
If a positive integer n, where $n > 1$, is not prime, it is said to be composite. The integer 1 is neither prime nor composite.

Question

Determine all prime numbers between 1 and 100.

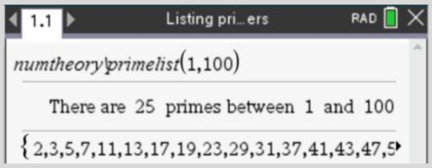
Solution

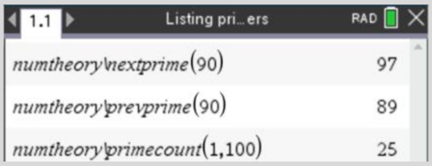
On a **Calculator** page:

- Press 6 > **numtheory** > **primelist**.
- Enter as shown.

Answer: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

***Note:** The command **nextprime(n)** gives the first prime after n. The command **prevprime(n)** gives the last prime before n. The command **primecount(a, b)** gives the number of primes between a and b inclusive. There are 25 primes between 1 and 100.*





Finding prime factorisations

Every positive integer greater than 1 is either prime or can be written as a product of primes. Prime factorisations are unique, apart from the order in which the prime factors are written.

Question

Find the prime factorisation of 78.

Solution

On a **Calculator** page:

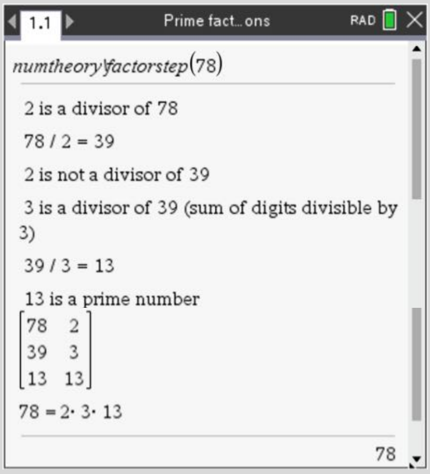
- Press menu > **Number** > **Factor**.
- Enter as shown.

Answer: $2\times3\times13$.

Alternatively on a **Calculator** page:

- Press 6 > **numtheory** > **factorstep**.
- Enter as shown.

***Note:** The command **factorstep(Int)** shows the steps taken to determine a prime factorisation. Can you see how it works?*



Finding Fermat primes

Fermat primes, F_n, are of the form $2^{2^n}+1$ where n is a non-negative integer.

Question

(a) Find the first five Fermat primes.

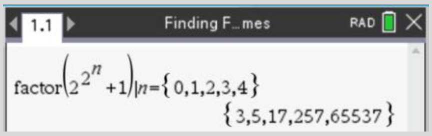
Fermat thought that all numbers of the form $2^{2^n}+1$ were prime.

(b) By examining the case $n=5$, show that Fermat was incorrect.

Solution

To complete parts **(a)** and **(b)**, on a **Calculator** page:

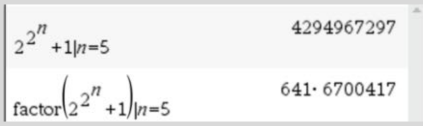
- Press [menu] > **Number** > **Factor**.
- Press [ctrl] [=] to access | the 'with' or 'given' symbol.
- Press [ctrl] [)] to access { }.
- Enter as shown.



Answer: (a) 3, 5, 17, 257, 65 537

To examine the case $n=5$:

- Press [menu] > **Number** > **Factor**.
- Press [ctrl] [=] to access the 'with' or 'given' symbol |.
- Enter as shown.



Answer: (b) $4\,294\,967\,297=641\times6\,700\,417$

Hence, Fermat was incorrect.

Alternatively, to confirm the result for $n=5$:

- Press [catalog] [1] [I], scroll down and select **isPrime(**.
- Enter as shown.

Proof: RTP that $641\,|\,2^{32}+1$.

$641=2^4+5^4$ and $641=2^7\times5+1\Rightarrow2^7\times5=641-1$

$2^{28}\times5^4=(641-1)^4=641n+1$ where n is an integer

$641=2^4+5^4\Rightarrow5^4=641-2^4$

$2^{28}\left(641-2^4\right)=641n+1$

$641\times2^{28}-2^{32}=641n+1$

$2^{32}+1=641\left(2^{28}-n\right)$

Hence $641\,|\,2^{32}+1$.

Note: *Encourage students to investigate properties of Mersenne primes. For example, to investigate if an even number is perfect if and only if it has the form* $2^{p-1}\left(2^p-1\right)$ *where* 2^p-1 *is prime.*

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Finding HCF (GCD) and LCM using prime factorisation

The *highest common factor* (HCF), or *greatest common divisor* (GCD) of two non-negative integers a and b is the largest non-negative integer that is a factor of both a and b. The least (lowest) common multiple (LCM) of positive integers a and b is the smallest positive integer that is a multiple of both a and b.

***Note**: To match common calculator syntax and case, hereafter we will use **gcd**(a,b) and **lcm**(a,b). For example,* $\mathbf{gcd}(24,36)=12$ *and* $\mathbf{lcm}(24,36)=72$.

Integers a and b are relatively prime (or coprime) if $\gcd(a,b)=1$.

Prime factorisation can be used to find $\mathbf{gcd}(a,b)$ and $\mathbf{lcm}(a,b)$.

- The $\mathbf{gcd}(a,b)$ is found by taking the lower power of each prime factor.
- The $\mathbf{lcm}(a,b)$ is found by taking the higher power of each prime factor.

Question

Use prime factorisation to find the following

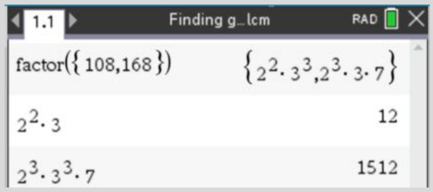
(a) $\gcd(108,168)$. **(b)** $\text{lcm}(108,168)$.

(c) Confirm your results to parts **(a)** and **(b)** by finding $\gcd(108,168)$ and $\text{lcm}(108,168)$ directly.

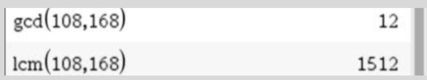
Solution

Parts **(a)** and **(b)** on a **Calculator** page:

- Press [menu] > **Number** > **Factor**.
- Press [ctrl] [)] to access { }.
- Enter as shown.



Answers: (a) Taking the lower power of each prime factor, $\gcd(108,168)=2^2\times 3=12$. **(b)** Taking the higher power of each prime factor, $\text{lcm}(108,168)=2^3\times 3^3\times 7=1512$.

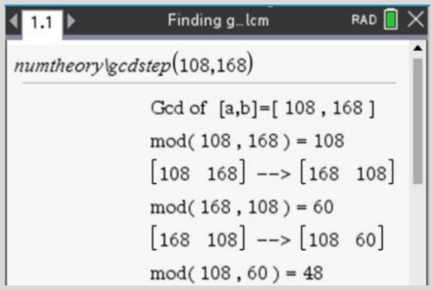


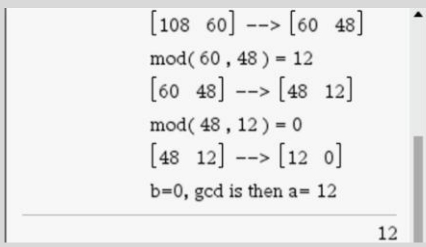
Part **(c)** on a **Calculator** page:

- Press [menu] > **Number** > **Greatest Common Divisor**.
- Press [menu] > **Number** > **Least Common Multiple**.
- Enter as shown.

Answer: (c) $\gcd(108,168)=12$ and $\text{lcm}(108,168)=1512$.

***Note:** Press* [📖] [6] *> **numtheory** > **gcdstep**. The command **gcdstep(n₁, n₂)** shows the steps taken to determine the gcd. It uses Euclid's algorithm (see page 34). Can you see how it works?*





Finding the number of factors

Fundamental theorem of arithmetic: Every integer greater than 1 can be written in the form

$$p_1^{n_1} p_2^{n_2} \dots p_k^{n_k} \text{ where } n_i \geq 0 \text{ (} n_i \text{ are non-negative integers) and } p_i \text{ are distinct primes.}$$

The factorisation is unique apart from rearrangement of the order of the factors.

Question

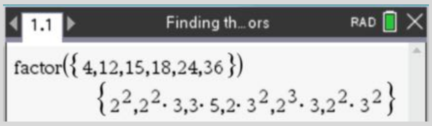
Use prime factorisation to determine a formula that identifies the number of factors a number *N* has.

Extension: Find the smallest number, *N*, that has 30 factors.

Solution

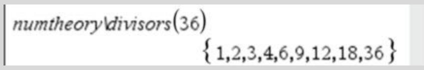
To determine a prime factorisation on a **Calculator** page:

- Press menu > **Number** > **Factor**.
- If desired, press ctrl) to access { }.



To determine the list of factors a number *N* has:

- Press 📖 6 > **numtheory** > **divisors**.
- Enter as shown.



To count the number of factors a number *N* has:

- Press 📖 1 C, scroll down and select **count(**.
- Enter as shown.

Answer: Consider the example: $4312 = 2^3 \times 7^2 \times 11$.
Each factor is of the form $2^\alpha \times 7^\beta \times 11^\gamma$ where $\alpha = 0,1,2,3$, $\beta = 0,1,2$ and $\gamma = 0,1$. So 4312 has $4 \times 3 \times 2 = 24$ factors.

In general, to find the total number of factors:

- Express the number as the product of its prime factors in index form
- Add 1 to each exponent: $(n_1+1),(n_2+1),\dots,(n_k+1)$.
- Multiply these together.

The total number of factors is $(n_1+1)(n_2+1)\dots(n_k+1)$.

This is because the prime p_1 can appear as any of its (n_1+1) different powers, and so on with each prime p_i.

Extension: Consider the number of factors 30 has. Any number with 30 factors must take one of the following forms:

$$p_1^{29},\ p_1^{14} \times p_2,\ p_1^{9} \times p_2^{2},\ p_1^{5} \times p_2^{4},\ p_1^{4} \times p_2^{2} \times p_3$$

The smallest number with 30 factors is $2^4 \times 3^2 \times 5 = 720$.

Linking gcd and lcm

Question

Find the following

(a) $\gcd(70,120)$. **(b)** $\text{lcm}(70,120)$. **(c)** $\gcd(70,120)\times\text{lcm}(70,120)$. **(d)** 70×120.

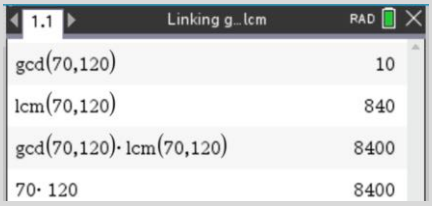
(e) Determine $\gcd(a,b)$, $\text{lcm}(a,b)$ and ab for various positive integers a and b.

(f) Conjecture a result which connects $\gcd(a,b)$, $\text{lcm}(a,b)$ and ab.

Solution

Parts **(a)**, **(b)**, **(c)** and **(d)** on a **Calculator** page:

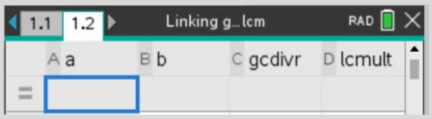
- Press [menu] > **Number > Greatest Common Divisor**.
- Press [menu] > **Number** > **Least Common Multiple**.
- Enter as shown.



Answers: (a) $\gcd(70,120)=10$, **(b)** $\text{lcm}(70,120)=840$, **(c)** $\gcd(70,120)\times\text{lcm}(70,120)=8400$, **(d)** $70\times120=8400$.

Part **(e)** on a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the variable ***a***.
- In the column B heading cell, enter the variable ***b***.
- In the column C heading cell, enter the variable ***gcdivr***.
- In the column D heading cell, enter the variable ***lcmult***.
- In the column E heading cell, enter the variable ***prod1***.
- In the column F heading cell, enter the variable ***prod2***.

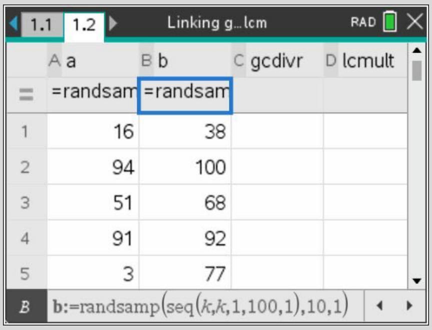


Generate the required sequences of values as follows:

Firstly, generate ten different random integers from 1 to 100 in column A and in column B.

To enter $\boldsymbol{a := \text{randSamp}(\text{seq}(k,k,1,100,1),10,1)}$ in the column A formula cell and $\boldsymbol{b := \text{randSamp}(\text{seq}(k,k,1,100,1),10,1)}$ in the column B formula cell:

- Press [menu] > **Data > Random > Sample**.
- Press [📖][1][S], scroll down and select **seq(**.
- Enter as shown.



Note: The ***randSamp*** *command syntax is* ***randSamp(List, #Trials [, noRepl])****. Enter 1 to generate a random sample without replacement.*

Note: The syntax for expressing a sequence as a list is ***seq(Expression, Variable, Low, High [, Step])****.*
The default value for ***Step*** *is 1.*

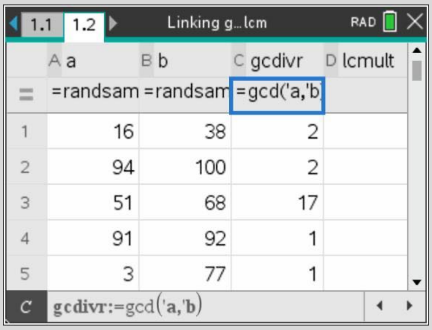
... continued

Solution (continued)

Note: *Shortcuts for navigating around the* ***Lists & Spreadsheet*** *cells. Press* [ctrl] [1] *to go to the last entry in a column. Press* [ctrl] [7] *to go to the first entry in a column. Press* [ctrl] [3] *to go down a page and* [ctrl] [9] *to go up a page. To go to a specific cell, press* [ctrl] [G] *and enter the cell reference.*

To enter $\boldsymbol{gcdivr} := \mathbf{gcd}(\boldsymbol{'a}, \boldsymbol{'b})$ in the column C formula cell:

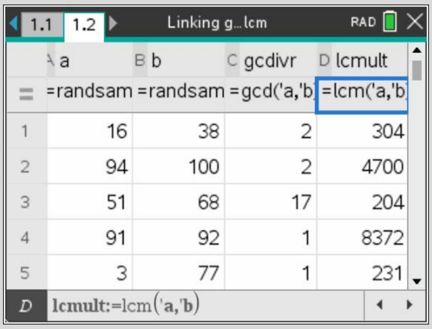
- Press [📖] [1] [G] and select **gcd(**.
- Press [var] to select ***a*** and ***b***.
- Enter as shown.



Note: *The symbol* **'** *in* ***'a*** *specifies* ***a*** *as a variable reference. Otherwise, TI-Nspire CX II CAS will consider* ***a*** *as a column reference. If the* **'** *symbol is omitted, a prompt may appear asking whether the variable being used refers to a variable or to a spreadsheet column.*

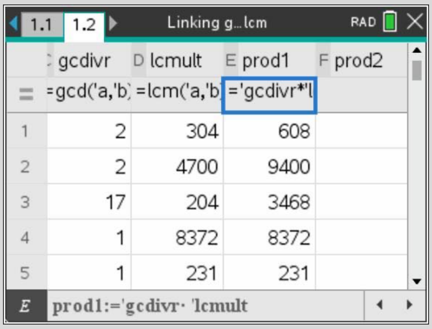
To enter $\boldsymbol{lcmult} := \mathbf{lcm}(\boldsymbol{'a}, \boldsymbol{'b})$ in the column D formula cell:

- Press [📖] [1] [L], scroll down and select **lcm(**.
- Press [var] to select ***a*** and ***b***.
- Enter as shown.



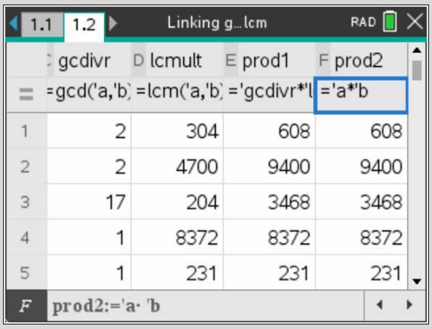
To enter $\boldsymbol{prod1} := \boldsymbol{'gcdivr} \times \boldsymbol{'lcmult}$ in the column E formula cell:

- Press [var] to select ***gcdivr*** and ***lcmult***.
- Enter as shown.



To enter $\boldsymbol{prod2} := \boldsymbol{'a} \times \boldsymbol{'b}$ in the column F formula cell:

- Press [var] to select ***a*** and ***b***.
- Enter as shown.



Answers: (e) Answers will vary.

(f) $\gcd(a,b) \times \operatorname{lcm}(a,b) = ab$

The proof of this result is beyond the scope of the course.

Divisibility and modular arithmetic: a brief background

When a positive integer a is divided by another positive integer m then $a = km + r$ where $0 \le r < m$, k is the quotient and r is the remainder. For example, $39 = 5 \times 7 + 4$.

In terms of modular arithmetic (the study of the properties of remainders), a is said to be congruent to r modulo m if $m \mid (a - r)$. For example, $7 \mid (39 - 4)$.

For any integer a there exists a congruence $a \equiv r \pmod{m}$. For example, $39 \equiv 4 \pmod{7}$.

Congruence modulo m generalizes the notion of divisibility since $a \equiv 0 \pmod{m} \Leftrightarrow m \mid a$.

In summary, if $a = km + r$ then $a \equiv r \pmod{m}$, since $m \mid (a - r)$.

Expressing rational numbers as recurring decimals

Every rational number can be written as a terminating or recurring decimal.

A real number has a terminating decimal representation if and only if it can be written as $\dfrac{m}{2^{\alpha}5^{\beta}}$ for some $m \in \mathbb{Z}$ and some non-negative integers α, β.

When a rational number, $\dfrac{m}{n}$, where m, n are relatively prime (i.e. m, n have no common factor other than 1), is expanded, the periodic decimal expansion begins after s digits and has length t digits, where s and t are the smallest numbers that satisfy $10^{s} \equiv 10^{s+t} \pmod{n}$.

Question

Consider the decimal expansions of fractions with denominator 84.

(a) Find the period, s.

(b) Find the length, t.

(c) Repeat parts **(a)** and **(b)** for decimal expansions of fractions with denominator 7.

Solution

To find the smallest values of s and t that satisfy $10^{s} \equiv 10^{s+t} \pmod{84}$, find the remainders of the powers of $10 \pmod{84}$.

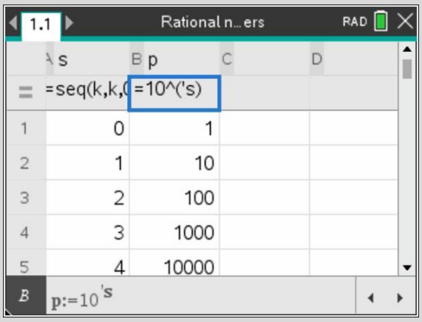
Parts **(a)** and **(b)** on a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the variable ***s***.
- In the column B heading cell, enter the variable ***p***.
- In the column C heading cell, enter the variable ***r***.

Generate the required sequences of values as follows:

To enter $\boldsymbol{s := \mathrm{seq}(k, k, 0, 10)}$ in the column A formula cell:

- Press 📖 1 S, scroll down and select **seq(**.
- Enter as shown.



Note: *The syntax for expressing a sequence as a list is* ***seq(Expression, Variable, Low, High[, Step])****. The default value for* ***Step*** *is 1.*

... continued

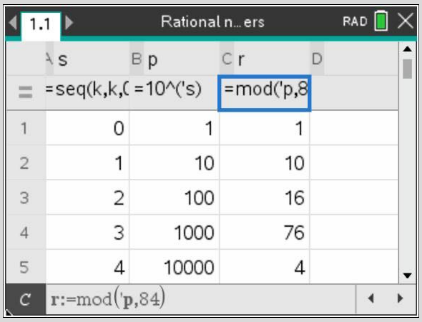
Solution (continued)

To enter $\boldsymbol{p := 10^{'s}}$ in the column B formula cell:

- Press [var] to select ***s***.
- Enter as shown.

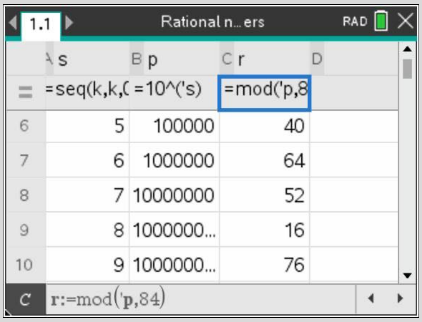
Note: *The symbol* **'** *in* **'s** *specifies* **s** *as a variable reference. Otherwise, TI-Nspire CX II CAS will consider* **s** *as a column reference. If the* **'** *symbol is omitted, a prompt may appear asking whether the variable being used refers to a variable or to a spreadsheet column.*

To enter $\boldsymbol{r := \text{mod}('p, 84)}$ in the column C formula cell:



- Press [📖] [1] [M], scroll down and select **mod(**.
- Press [var] to select ***p***.
- Enter as shown.

Notes: *The* ***remain(*** *command, accessed from the* ***Catalog****, can be used instead of* ***mod(****. To access* ***mod(*** *on a* ***Calculator*** *page, press* [menu] ***> Number > Number Tools****. To access* ***remain(*** *on a* ***Calculator*** *page, press* [menu] ***> Number > Remainder****.*

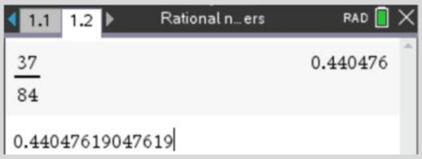


Column C gives the remainders of the powers of $10 \pmod{84}$ for $s = 0,1,2,3,4,5,6,7,8,9,10$.
The remainders are $1, 10, \mathbf{16}, 76, 4, 40, 64, 52, \mathbf{16}, 76, 4$.

Answers: From these remainders:
(a) $s = 2$ i.e. the period begins after two digits.

(b) $t = 6$ i.e. the length is 6.

On a **Calculator** page:

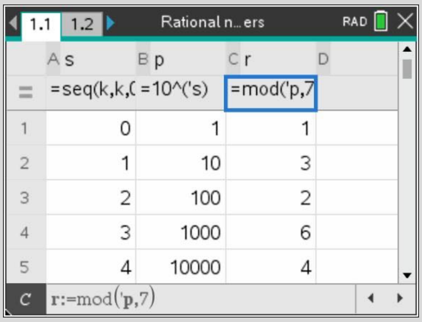


- Enter $\frac{37}{84}$ (press [ctrl] [÷] to access the **Fraction** template) and press [ctrl] [enter] to obtain a decimal number.
- Press ▲ [enter].

The decimal output is 0.44047619047619.
This decimal expansion confirms that $s = 2$, $t = 6$ and
$\frac{37}{84} = 0.44\overline{047619}$.

Part **(c)** on a **Lists & Spreadsheet** page:

Repeat the above instructions with $n = 7$:



- In the Column C formula cell, replace 84 with 7.

Column C now gives the remainders of the powers of $10 \pmod{7}$ for $s = 0,1,2,3,4,5,6,7,8,9,10$.

The remainders are $1, 3, 2, 6, 4, 5, 1, 3, 2, 6, 4$.

... continued

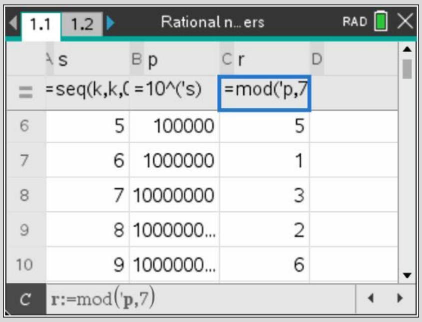
Solution (continued)

Answer: From these remainders:

$s = 0$ i.e. the period begins immediately.

$t = 6$ i.e. the length is 6.

For example, $\frac{2}{7} = 0.\overline{285714}$.



When the denominator of $\frac{m}{n}$ has the form $n = n_0 \cdot 2^{\alpha} \cdot 5^{\beta}$ and $(n_0, 10) = 1$, the period begins after μ digits, where μ is the larger of α and β. The length of the period is the exponent to which 10 belongs $(\bmod\, n_0)$.

So when $n = 84 = 2^2 \times 21$, the period starts after the second digit and has length 6 since 10 belongs to the exponent $6(\bmod\, 21)$.

Now consider when the denominator of the fraction, n, to be expanded does not have factors 2 or 5.

Since n is relatively prime to 10, then $10^t \equiv 1(\bmod\, n)$.

So $s = 0$ and the period starts with the first digit.

This is a purely periodic expansion.

Proving by contradiction

Question

Consider integers x and y such that $x^2 + y^2$ is divisible by 4.

Prove by contradiction that x and y cannot both be odd.

Solution

Start with an exploratory verification that both x and y cannot be odd when $x^2 + y^2$ is exactly divisible by 4.

Explore cases where x and y are both odd.

On a **Lists & Spreadsheet** page:

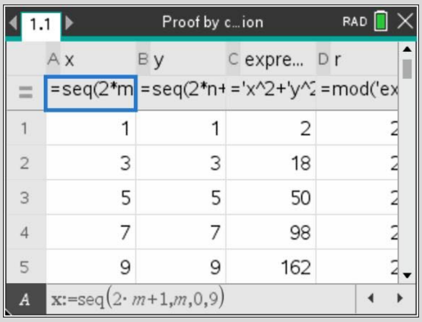
- In the column A heading cell, enter the variable ***x***.
- In the column B heading cell, enter the variable ***y***.
- In the column C heading cell, enter the variable ***expression***.
- In the column D heading cell, enter the variable ***r***.

... continued

Solution (continued)

Generate the required sequences of values as follows:

To enter $\boldsymbol{x} := \mathbf{seq}(\mathbf{2}\boldsymbol{m}+\mathbf{1},\boldsymbol{m},\mathbf{0},\mathbf{9})$ in the column A formula cell:



- Press [catalog] [1] [S], scroll down and select **seq(**.
- Enter as shown.

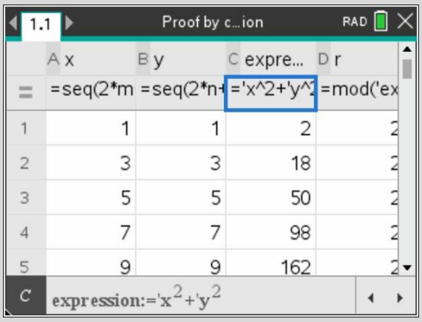
Note: *The syntax for expressing a sequence as a list is* ***seq(Expression, Variable, Low, High[, Step])****.*
The default value for ***Step*** *is 1.*

To enter $\boldsymbol{y} := \mathbf{seq}(\mathbf{2}\boldsymbol{n}+\mathbf{1},\boldsymbol{n},\mathbf{0},\mathbf{9})$ in the column B formula cell:

- Press [catalog] [1] [S], scroll down and select **seq(**.
- Enter as shown.

Note: *Press* [ctrl] [1] *to go to the last entry in a column.*
Press [ctrl] [7] *to go to the first entry in a column.*
Press [ctrl] [3] *to go down a page and* [ctrl] [9] *to go up a page.*
To go to a specific cell, press [ctrl] [G] *and enter the cell reference.*

To enter $\boldsymbol{expression} := \boldsymbol{'x}^2 + \boldsymbol{'y}^2$ in the column C formula cell:

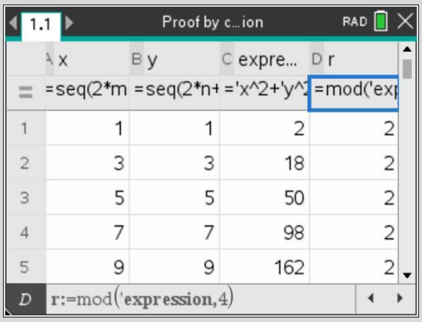


- Press [var] to select ***x*** and ***y***.
- Enter as shown.

Note: *The symbol* **'** *in* **'x** *specifies x as a variable reference. Otherwise, TI-Nspire CX II CAS will consider x as a column reference. If the* **'** *symbol is omitted, a prompt may appear asking whether the variable being used refers to a variable or to a spreadsheet column.*

To enter $\boldsymbol{r} := \mathbf{mod}(\boldsymbol{'expression},\mathbf{4})$ in the column D formula cell:

- Press [catalog] [1] [M], scroll down and select **mod(**.
- Press [var] to select ***expression***.
- Enter as shown.



Note: *The* ***remain(*** *command, accessed from the* ***Catalog****, can be used instead of* ***mod(****.*

Column D of the spreadsheet indicates that when x and y are both odd, x^2+y^2 is not divisible by 4. In each case, the remainder is 2. This result forms an important part of the proof.

Note: *To access* ***mod(*** *on a* ***Calculator*** *page, press* [menu] > ***Number*** > ***Number Tools****.*

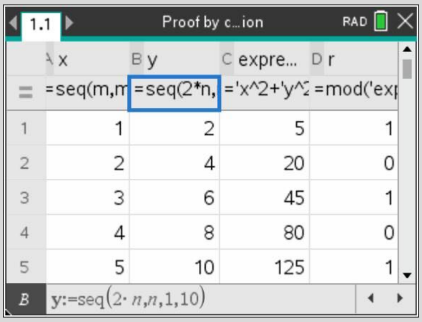
... continued

Solution (continued)

To explore the case where y is even:

- Enter $\boldsymbol{x := \mathbf{seq}(m,m,1,10)}$ in the column A formula cell.
- Enter $\boldsymbol{y := \mathbf{seq}(2\cdot n,n,1,10)}$ in the column B formula cell.

Note: *Alternatively in the column B formula cell, enter* $\boldsymbol{y := \mathbf{seqn}(2\cdot n,10)}$ *where* ***seqn(****, found in the* ***Catalog****, generates a list beginning with n=1.*

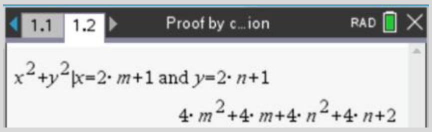


Column D of the spreadsheet now indicates that when x and y are both even, x^2+y^2 is divisible by 4. In each case, the remainder is 0. When x is odd and y is even (or vice versa), x^2+y^2 is not divisible by 4. In each case, the remainder is 1.

Proof: Assume that x and y are both odd.

Then $x=2m+1$ and $y=2n+1$ where $m,n\in\mathbb{Z}$.

$$\begin{aligned}x^2+y^2&=(2m+1)^2+(2n+1)^2\\&=4m^2+4m+1+4n^2+4n+1\\&=4(m^2+m+n^2+n)+2\end{aligned}$$



$4(m^2+m+n^2+n)$ is divisible by 4.

However, 2 is not divisible by 4.

So x^2+y^2 is not divisible by 4, which is a contradiction.

Hence x and y cannot both be odd.

Note*: A number is divisible by 4 if the number formed by its last two digits is divisible by 4.*

Using examples and counterexamples

Question

Consider the set of numbers S of the form $n^2 - n + 41$ where $n \in \mathbb{Z}^+$.

(a) Prove that all elements of S are odd.

The first five elements of S are $\{41, 43, 47, 53, 61\}$. These are all prime numbers.

(b) Prove by use of a counterexample that not all elements of S are prime.

Solution

(a) Proof: $(n^2 - n) + 41 = n(n-1) + 41$

Either n is even or $n-1$ is even so $n^2 - n = n(n-1)$ is even (the product of an even number and an odd number is even).

Adding 41 to an even number gives an odd number.

So $n^2 - n + 41$ is odd.

(b) Legendre (1798) discovered the prime-generating polynomial $n^2 - n + 41$. Note that $n^2 - n + 41$ is also prime for $n = 0$.

An exploratory verification can help in finding a counterexample that proves that not all elements of S are prime.

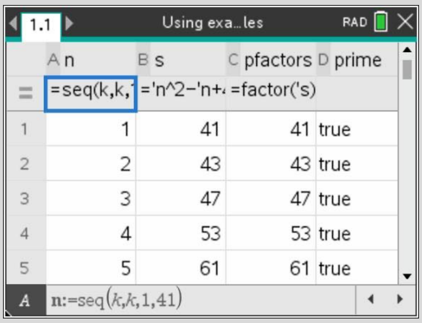
On a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the variable ***n***.
- In the column B heading cell, enter the variable ***s***.
- In the column C heading cell, enter the variable ***pfactors***.
- In the column D heading cell, enter the variable ***prime***.

Generate the required sequences of values as follows:

To enter $\boldsymbol{n := \text{seq}(k, k, 1, 41)}$ in the column A formula cell:

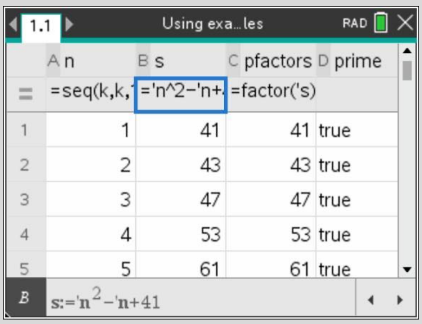
- Press 📖 1 S, scroll down and select **seq(**.
- Enter as shown.



***Note:** The syntax for expressing a sequence as a list is **seq(Expression, Variable, Low, High[, Step])**. The default value for **Step** is 1.*

To enter $\boldsymbol{s := 'n^2 - 'n + 41}$ in the column B formula cell:

- Press var to select ***n***.
- Enter as shown.



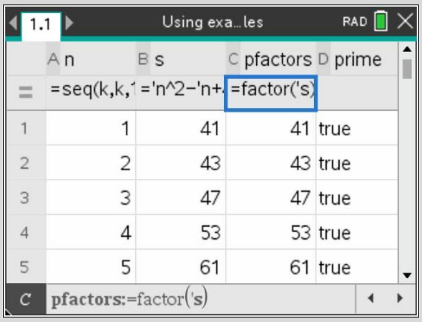
***Note:** The symbol ' in **'n** specifies **n** as a variable reference. Otherwise, TI-Nspire CX II CAS will consider **n** as a column reference. If the ' symbol is omitted, a prompt may appear asking whether the variable being used refers to a variable or to a spreadsheet column.*

... continued

Solution (continued)

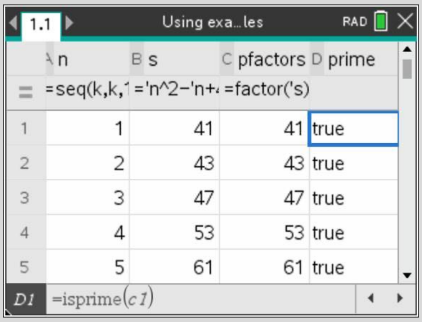
To enter $\boldsymbol{pfactors} := \textbf{factor}(\textbf{'}s)$ in the column C formula cell:

- Press [book] [1] [F] and select **factor(**.
- Enter as shown.



In cell D1:

- Press [=] [book] [1] [I], scroll down and select **isPrime(**.
- Enter $= \textbf{isprime}(\textbf{c1})$ where **c1** denotes the cell reference.



To fill down to cell D41:

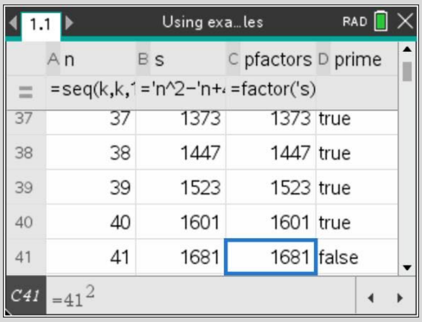
- Press [menu] > **Data** > **Fill**.
- Press ▼ to extend a rectangular box down to and including cell D41.
- Press [enter].

Note:** Alternatively, press* [ctrl] [menu] *>* ***Fill.

The cells D1 through to D41 will be filled with either the output **'true'** or the output **'false'**.

Cell C41 shows that when $n = 41$, $n^2 - n + 41 = 1681 = 41^2$.

The output **'false'** in cell D41 confirms that $n^2 - n + 41$ is not prime when $n = 41$.

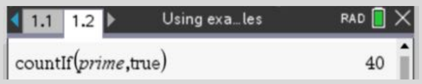


Proof: Consider the constant term, 41.

Substituting $n = 41$ into $n^2 - n + 41$, for example, gives $41^2 - 41 + 41 = 41^2$ which has 41 as a factor and hence is not prime.

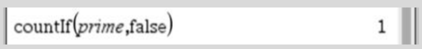
Alternatively on a **Calculator** page:

- Press [book] [1] [C], scroll down and select **countIf(**.
- Press [var] and select **prime**.
- Press [,] [book] [1] [T], scroll down and select **true**.



The output confirms that the first 40 integers are prime.

- Press [book] [1] [C], scroll down and select **countIf(**.
- Press [var] and select **prime**.
- Press [,] [book] [1] [F], scroll down and select **false**.



The output confirms that one integer (the 41st integer) is prime.

- Press [menu] > **Number** > **Factor**.
- Press [var] to select ***s***.

This generates the list of integers in factored form where possible.

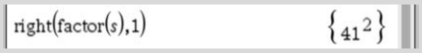
The last integer in the list, 1681, can be expressed as 41^2.

... continued

Solution (continued)

To see this integer in the list without scrolling:

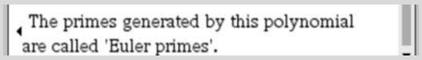
- Press [catalog] [1] [R], scroll down and select **right(**.
- Press [menu] > **Number** > **Factor**.
- Press [var] to select ***s***.
- Enter as shown.



This confirms that $1681 = 41^2$.

Note: *The syntax for returning the rightmost elements in a list is* ***right(List[,Num])****.*

To add a comment to a **Calculator** page:



- Press [menu] > **Actions** > **Insert Comment**.

The primes generated by this polynomial are called ‘Euler primes’.

Proving results involving integers using the contrapositive

Question

By proving the contrapositive, prove that if $n^2 - 6n + 5$ is even then n is odd, $\forall n \in \mathbb{Z}$.

Solution

Start with an exploratory verification that $n^2 - 6n + 5$ is even when n is odd, $\forall n \in \mathbb{Z}$.

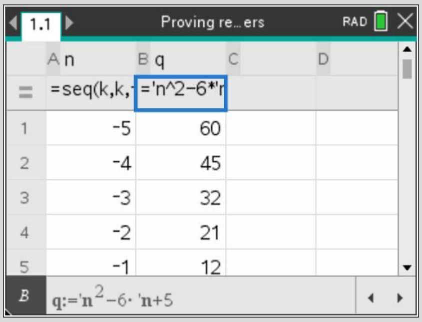
On a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the variable ***n***.
- In the column B heading cell, enter the variable ***q***.

Generate the required sequences of values as follows:

To enter $\boldsymbol{n := \mathbf{seq}(k, k, -5, 5)}$ in the column A formula cell:

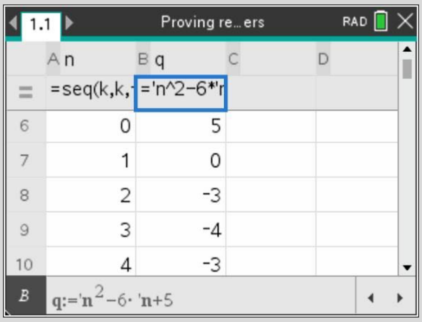
- Press [📖] [1] [S], scroll down and select **seq(**.
- Enter as shown.



Note: *The syntax for expressing a sequence as a list is **seq(Expression, Variable, Low, High[, Step])**. The default value for **Step** is 1.*

To enter $\boldsymbol{q := 'n^2 - 6 \cdot 'n + 5}$ in the column B formula cell:

- Press [var] to select ***n***.
- Enter as shown.



Note: *The symbol* **'** *in* **'n** *specifies n as a variable reference. Otherwise, TI-Nspire CX II CAS will consider n as a column reference. If the* **'** *symbol is omitted, a prompt may appear asking whether the variable being used refers to a variable or to a spreadsheet column.*

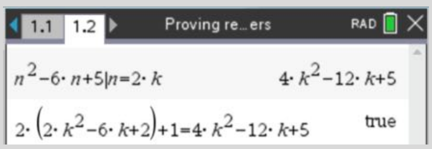
This numerical exploration suggests that when n is odd, $n^2 - 6n + 5$ is even and when n is even, $n^2 - 6n + 5$ is odd.

Proof: The contrapositive statement is:

If n is even, then $n^2 - 6n + 5$ is odd, $\forall n \in \mathbb{Z}$.

Let $n = 2k$ for $k \in \mathbb{Z}$.

$$\begin{aligned} n^2 - 6n + 5 &= (2k)^2 - 6(2k) + 5 \\ &= 4k^2 - 12k + 5 \\ &= 4k^2 - 12k + 4 + 1 \\ &= 2(2k^2 - 6k + 2) + 1 \end{aligned}$$



So $n^2 - 6n + 5 = 2a + 1$ where a is the integer $2k^2 - 6k + 2$.

Thus $n^2 - 6n + 5$ is odd.

Hence, by the contrapositive, if $n^2 - 6n + 5$ is even then n is odd, $\forall n \in \mathbb{Z}$.

Proving by mathematical induction

Inductive proof involves an initial statement, assumption statement, inductive step and conclusion.

In practice, there are three main parts to an induction proof:

- Verify the statement for any initial terms.
- Prove the implication that if the statement is true for some integer k then it is true for the next integer $(k+1)$.
- Provide a concluding statement of truth that appeals to the principle of mathematical induction.

Mathematical induction is used to prove divisibility results for any positive integer n.

Question

Use mathematical induction to prove that 5^n+9^n+2 is divisible by 4, for $n \in \mathbb{Z}^+$.

Solution

Start with an exploratory verification establishing that 5^n+9^n+2 is divisible by 4 for positive integers between 1 and 10.

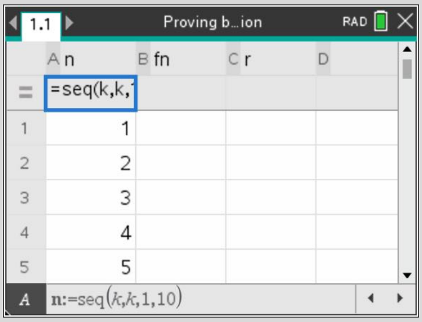
On a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the variable ***n***.
- In the column B heading cell, enter the variable ***fn***.
- In the column C heading cell, enter the variable ***r***.

Generate the required sequences of values as follows:

To enter $\boldsymbol{n := \text{seq}(k,k,1,10)}$ in the column A formula cell:

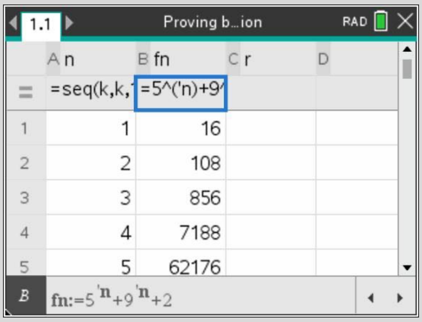
- Press 📖 1 S, scroll down and select **seq(**.
- Enter as shown.



__Note:__ The syntax for expressing a sequence as a list is __seq(Expression, Variable, Low, High[, Step])__. The default value for __Step__ is 1.

To enter $\boldsymbol{fn := 5^{'n}+9^{'n}+2}$ in the column B formula cell:

- Press var to access ***n***.
- Enter as shown.



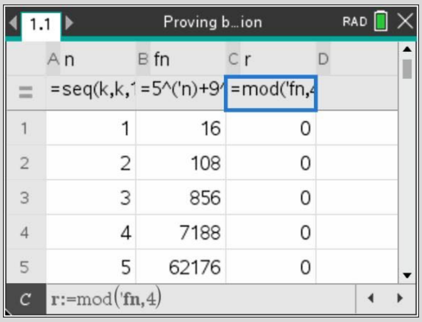
__Note:__ The symbol __'__ in __'n__ specifies n as a variable reference. Otherwise, TI-Nspire CX II CAS will consider n as a column reference. If the __'__ symbol is omitted, a prompt may appear asking whether the variable being used refers to a variable or to a spreadsheet column.

... continued

Solution (continued)

To enter $\boldsymbol{r := \mathbf{mod}('fn,4)}$ in the column C formula cell:

- Press [catalog] [1] [M], scroll down and select **mod(**.
- Press [var] to select ***fn***.
- Enter as shown.



Note: *The* ***remain(*** *command, accessed from the* ***Catalog****, can be used instead of* ***mod(****.*

Note*: Press [ctrl] [1] to go to the last entry in a column. Press [ctrl] [7] to go to the first entry in a column. Press [ctrl] [3] to go down a page and [ctrl] [9] to go up a page. To go to a specific cell, press [ctrl] [G] and enter the cell reference.*

Column C of the spreadsheet shows that $5^n + 9^n + 2$ is divisible by 4 for positive integers between 1 and 10.

In each case, the remainder is 0.

Note: *A number is exactly divisible by 4 if the number formed by its last two digits is divisible by 4.*

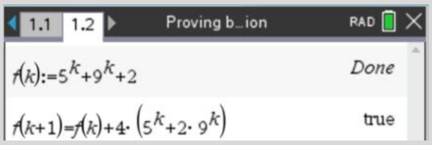
Proof: Let $f(n) = 5^n + 9^n + 2$ and let P_n be the proposition that $f(n)$ is divisible by 4.

$f(1) = 16$ and so P_1 is true.

Assume P_k is true for $n = k$, i.e. $f(k)$ is divisible by 4.

Consider $f(k+1)$.

$$\begin{aligned} f(k+1) &= 5^{k+1} + 9^{k+1} + 2 \\ &= 5^k(4+1) + 9^k(8+1) + 2 \\ &= 4 \times 5^k + 5^k + 8 \times 9^k + 9^k + 2 \\ &= f(k) + 4(5^k + 2 \times 9^k) \end{aligned}$$



Both terms are divisible by 4, so $f(k+1)$ is divisible by 4.

Since P_1 is true and P_k true $\Rightarrow P_{k+1}$ true, P_n is proved true by mathematical induction for $n \in \mathbb{Z}^+$.

1.1.2 Graph theory

Using adjacency matrices to represent an undirected graph

An adjacency matrix, A, is an $n \times n$ matrix that shows the number of connections between the vertices of a graph.

A loop is a single edge connecting a vertex to itself.

Loops are counted as one edge.

In a graph, a walk is a sequence of edges that connect successive vertices.

A walk starts at one vertex and follows any route to finish at another vertex.

Walks are specified by listing the vertices in the order they are visited.

Question

The adjacency matrix, A, of a graph G, with vertices P, Q, R, S and T is given by

$$A = \begin{array}{c} \\ P \\ Q \\ R \\ S \\ T \end{array} \begin{array}{c} \begin{array}{ccccc} P & Q & R & S & T \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix} \end{array}.$$

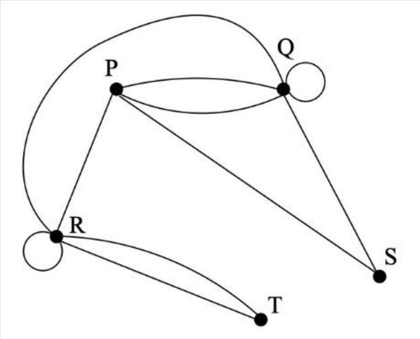
(a) Draw the graph G.

(b) Find the number of walks of length 5 from P to Q.

(c) Determine the pairs of distinct vertices that have more than 15 walks of length 3 between them.

Solution

(a) Graph G is shown at right:



From A, for example, there are two edges connecting P and Q, one edge connecting P and R and one edge connecting P and S.

The leading diagonal of A indicates there is a loop (single edge) connecting Q to itself and R to itself.

(b) To find the number of walks of length 5 from P to Q, calculate A^5 where:

$$A = \begin{array}{c} \\ P \\ Q \\ R \\ S \\ T \end{array} \begin{array}{c} \begin{array}{ccccc} P & Q & R & S & T \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix} \end{array}$$

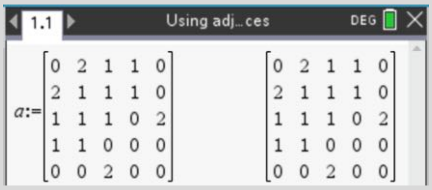
... continued

Solution (continued)

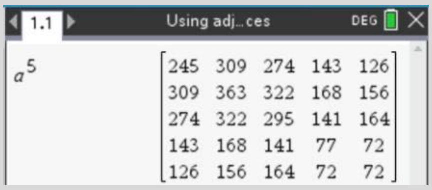
Parts **(b)** and **(c)** on a **Calculator** page.

To assign A:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [book] [1] [5] and select the **m-by-n Matrix** template.
- Set the number of rows to be 5 and the number of columns to be 5.
- Enter A^5 as shown.



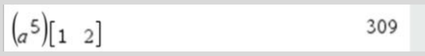
$$A^5 = \begin{bmatrix} 245 & 309 & 274 & 143 & 126 \\ 309 & 363 & 322 & 168 & 156 \\ 274 & 322 & 295 & 141 & 164 \\ 143 & 168 & 141 & 77 & 72 \\ 126 & 156 & 164 & 72 & 72 \end{bmatrix}$$



The number of walks of length 5 from P to Q is given by the element in row 1 and column 2 of A^5.

To access this element:

- Enter $\left(A^5\right)[1,2]$ as shown.

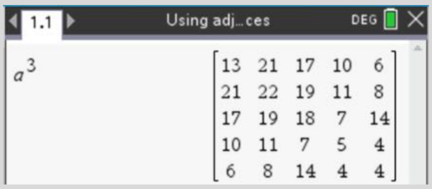


Answer: (b) There are 309 walks of length 5 from P to Q.

Note: *The first element in* $\begin{bmatrix}1 & 2\end{bmatrix}$ *indicates the row number and the second element indicates the column number.*

(c) To determine the pairs of distinct vertices that have more than 15 walks of length 3 between them, we calculate A^3.

$$A^3 = \begin{bmatrix} 13 & 21 & 17 & 10 & 6 \\ 21 & 22 & 19 & 11 & 8 \\ 17 & 19 & 18 & 7 & 14 \\ 10 & 11 & 7 & 5 & 4 \\ 6 & 8 & 14 & 4 & 4 \end{bmatrix}$$



To identify these pairs of distinct vertices, we look for elements in A^3 that are >15 and not on the leading (main) diagonal. We need only look above or below the leading diagonal (note symmetry).

Answer: (c) The pairs of distinct vertices are PQ (21), PR (17) and QR (19).

Demonstrating planarity with the Geometry application

A graph G is planar if it can be drawn in the plane with edges only crossing at vertices.

Such a representation of G is called a plane representation.

If a connected graph G is planar with v vertices, e edges and f faces, then it satisfies Euler's formula $v-e+f=2$.

The complete graph K_n is a simple graph with n vertices in which every vertex is adjacent to every other vertex.

The complete graph K_n has $\dfrac{n(n-1)}{2}$ edges.

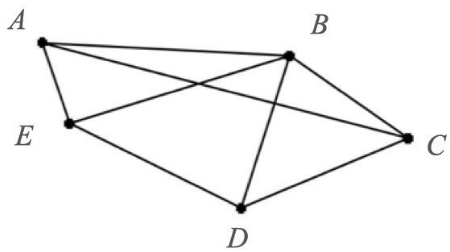
If a simple connected graph with $v \ge 3$ vertices is planar, then $e \le 3v-6$.

Question

The **Geometry** application can be used to construct graphs.

Vertices can be moved to help visualise whether a graph is planar.

Consider the graph G shown below.



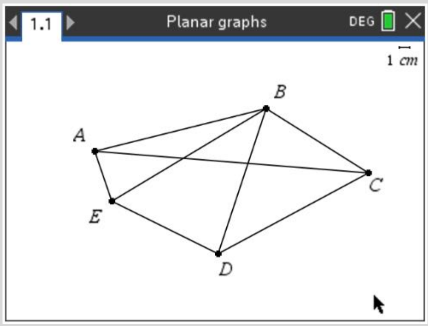
Use the **Geometry** application to construct G and determine whether it is planar.

Extension: Determine which complete graphs K_n are planar.

Solution

To construct G, on a **Geometry** page:

- Press menu > **Points & Lines** > **Point**.
- Click on the **Geometry** page in five locations to represent the vertices in the diagram above.
- Press esc to exit the **Point** tool.
- Press menu > **Points & Lines** > **Segment**.
- Click on each pair of vertices joined by an edge (as per the above diagram).
- Press esc to exit the **Segment** tool.



To label the vertices A to E:

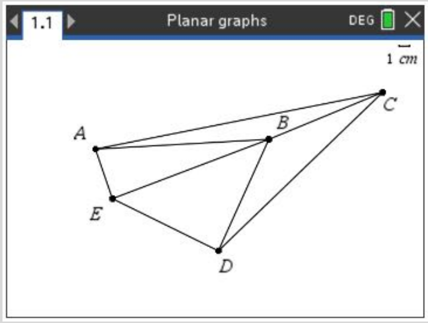
- Click on a vertex, press ctrl menu and select **Label**.
- Enter the label text 'A'.
- Repeat the above for the remaining vertices.

... continued

Solution (continued)

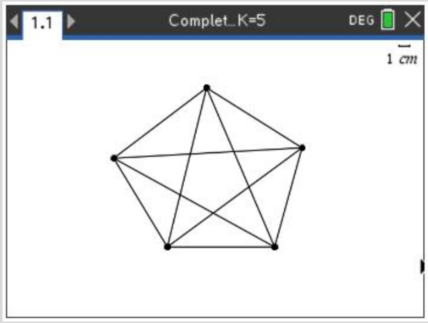
To verify that G is planar:

- Click and drag point C upwards and to the right so that the line segment AC no longer crosses the line segments BD and BE.



Answer: By moving point C, we can see more clearly that there are $v = 5$ vertices, $e = 8$ edges and $f = 5$ faces.
$v - e + f = 5 - 8 + 5 = 2$ and hence G is planar.

***Note**: It is also useful to use such constructions to highlight that a graph might not be planar. For example, complete graphs K_n, where $n \in Z^+, n > 4$, are not planar. The complete graph K_5 is shown at right.*



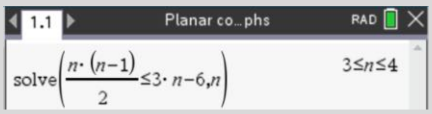
Extension: To determine which complete graphs K_n are planar on a **Calculator** page:

Suppose the simple, connected, complete graph K_n is planar.

Then $e \le 3v - 6$.

However, $e = \dfrac{n(n-1)}{2}$ and $v = n$.

- Press [menu] > **Algebra** > **Solve**.
- Press [ctrl] [=] to access $\le$.
- Enter as shown.



Solving $\dfrac{n(n-1)}{2} \le 3n - 6$ for n gives $3 \le n \le 4$ i.e. $n = 3$ or 4.

K_n is planar $\Leftrightarrow$ $n = 3$ or 4. K_1 and K_2 are also planar.

Answer: The only complete graphs that are planar are K_1, K_2, K_3 and K_4.

Determining a property of a graph G and G′

If G is a simple graph, then its complement is the simple graph G' where:

- G and G' have the same set of vertices.
- two vertices are adjacent in G' if and only if they are not adjacent in G.

Question

Consider a simple, connected graph G with at least 11 vertices.

Prove that G and G' cannot both be planar.

Solution

G has v vertices where $v \geq 11$ and e edges.

G' has v vertices and $\frac{v(v-1)}{2} - e$ edges.

Suppose both G and G' are planar.

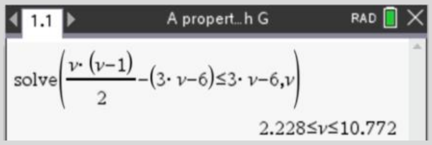
Then $e \leq 3v-6$ (1) and $\frac{v(v-1)}{2} - e \leq 3v-6$ (2).

From (2): $e \geq \frac{v(v-1)}{2} - (3v-6)$.

Hence $\frac{v(v-1)}{2} - (3v-6) \leq 3v-6$.

On a **Calculator** page:

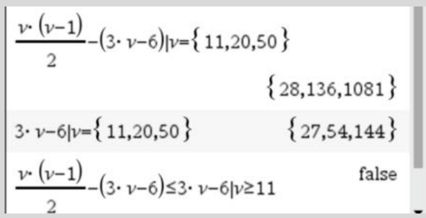
- Press menu > **Algebra** > **Solve**.
- Press ctrl = to access $\leq$.
- Enter as shown.



Solving $\frac{v(v-1)}{2} - (3v-6) \leq 3v-6$ for v gives $2.228 \leq v \leq 10.772$.

To test the inequality for some $v \geq 11$:

- Press ctrl) to access $\{\ \}$.
- Press ctrl = to access |, the 'with' or 'given' symbol.
- Enter as shown.



Answer: For $v \geq 11$, $\frac{v(v-1)}{2} - (3v-6) > 3v-6$ which contradicts the above inequality and so G and G' cannot both be planar.

Finding the number of spanning trees for $K_{n,n}$

A tree is a connected graph that contains no cycles. A tree with n vertices has $n-1$ edges.

A spanning tree of a graph G is a connected subgraph of G with no cycles, and which contains all the vertices of G.

A bipartite graph is a graph whose vertices can be divided into two disjoint subsets, A and B, such that each edge joins a vertex in A and a vertex in B.

Question

The graph $K_{n,n}$ is the complete bipartite graph where A and B both have n vertices.

It is known that $K_{4,4}$ has 4096 spanning trees.

Conjecture a formula for the number of spanning trees for $K_{n,n}$.

Solution

For $K_{1,1}$ there is 1 spanning tree.

For $K_{2,2}$, there are 2 choices for the vertex of degree 2 from one subset and 2 from the other.

So there are 4 spanning trees.

For $K_{3,3}$ there are 4 cases to consider:

(1) There are 3 ways to choose a vertex of degree 3 on the top and 3 ways to choose a vertex on the bottom.

(2) There are 3 ways to choose a vertex of degree 3 on the top and 3 ways to choose a vertex of degree 1 on the bottom and 2 ways to choose how the vertices of degree 2 on the bottom connect to the vertices of degree 1 on the top.

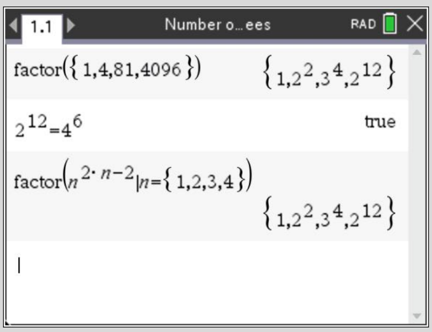
(3) is a symmetrical case of (2).

(4) There are 3 ways to choose a vertex of degree 1 on the top and 3 ways to choose a vertex of degree 1 on the bottom and 2 ways to choose which vertex of degree 1 on the top connects to and 2 ways to choose which vertex of degree 1 on the bottom connects to.

Hence $(3\times3)+2(3\times3\times2)+(3\times3\times2\times2)=81$.

On a **Calculator** page:

- Press [menu] > **Algebra** > **Factor**.
- Press [ctrl] [)] to access $\{\ \}$.
- Enter as shown.



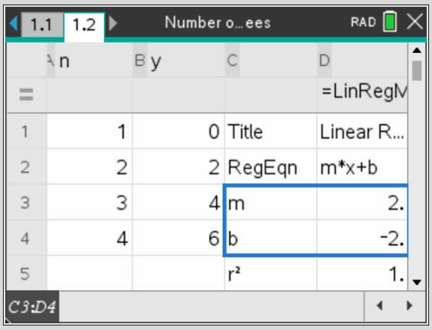
Answer: $K_{1,1}$ has 1^0 spanning trees, $K_{2,2}$ has 2^2 spanning trees, $K_{3,3}$ has 3^4 spanning trees and $K_{4,4}$ has 4^6 spanning trees. Conjecture that $K_{n,n}$ has n^{2n-2} spanning trees.

... continued

Solution (continued)

Alternatively, to determine/confirm the form of the expression's power on a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the variable ***n***.
- In the column B heading cell, enter the variable ***y***.
- Enter as shown.
- Press [menu] > **Statistics** > **Stat Calculations** > **Linear Regression (mx+b)**.
- In the dialog box that follows:
    - For **X List**, select ***n***.
    - For **Y List**, select ***y***.
- Press [enter].



The value of m and the value of b are displayed in column D.
As $m = 2$ and $b = -2$, the expression's power is $2n - 2$.

1.1.3 Logic and algorithms

Simplifying Boolean expressions

To represent switches in electrical circuits, three Boolean operations, $\vee$, $\wedge$ and $'$, acting on the set $\{0,1\}$ can be used as follows:

- 0 for an open (off) switch.
- 1 for a closed (on) switch.
- $x \vee y$, read as 'x or y', for two switches x and y connected in parallel.
- $x \wedge y$, read as 'x and y', for two switches x and y connected in series.
- x' for the complement of x.

Question

Evaluate $(1 \vee 0) \wedge 1'$.

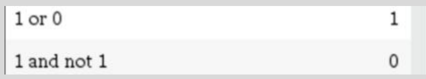
Solution

Use 'or' for $\vee$, 'and' for $\wedge$ and 'not' for $'$.

Enter as shown on a **Calculator** page:

***Note:** Press ␣ to insert a space. It is easiest to type the words 'and', 'or' and 'not'.*

Answer: $(1 \vee 0) \wedge 1' = 0$



$$\begin{aligned}(1 \vee 0) \wedge 1' &= 1 \wedge 1' \\ &= 1 \wedge 0 \\ &= 0\end{aligned}$$

***Notes**: The simplification steps can be shown with TI-Nspire CX II CAS.*

The method of constructing a truth table (shown on page 35) can be used to represent switches in parallel, switches in series and complementary switches. These tables can be extended to show the operation of an electrical circuit.

Verifying axioms of Boolean algebra

A Boolean algebra is a set B with operations $\vee$, $\wedge$ and $'$ and elements 0, 1 that follow a set of axioms for $\forall x, y, z \in B$. One such axiom is the distributivity axiom:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

Question

Verify that

(a) $1 \vee (0 \wedge 1) = (1 \vee 0) \wedge (1 \vee 1)$.

(b) $1 \wedge (0 \vee 1) = (1 \wedge 0) \vee (1 \wedge 1)$.

Solution

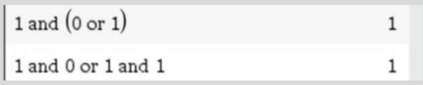
Use 'or' for $\vee$, 'and' for $\wedge$ and 'not' for $'$.

Part **(a)** enter as shown on a **Calculator** page:

Note: *Press ⎵ to insert a space. It is easiest to type the words 'and' and 'or'. Use brackets as required – note that some typed brackets are removed by the calculator after entry.*

Answer: $1 \vee (0 \wedge 1) = (1 \vee 0) \wedge (1 \vee 1) = 1$

Part **(b)** enter as shown on a **Calculator** page:



Answer: $1 \wedge (0 \vee 1) = (1 \wedge 0) \vee (1 \wedge 1) = 1$

Notes*: Part **(a)** is an example of $\vee$ distributing over $\wedge$ and part **(b)** is $\wedge$ distributing over $\vee$.*

It is worthwhile verifying the distributive axiom using different combinations of the elements 0 and 1.

Constructing truth tables for compound statements

Every statement is either true or false.

In logic, a statement is built from simple statements using the logical connectives, $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\Rightarrow$ (implies) and $\Leftrightarrow$ (is equivalent to).

The truth or falsity of a statement built with logical connectives depends on the truth or falsity of its components.

A truth table is a table of rows and columns showing the truth value (T or F) of every possible combination of the given statements as operated by logical connectives.

It shows how the truth or falsity of a compound statement depends on the truth or falsity of the simple statements from which it is constructed.

Question

Construct the truth table for $\neg(p \wedge q)$.

Solution

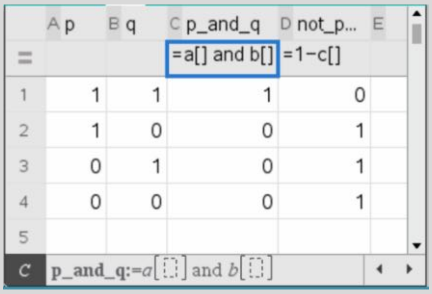
Use 1 for T (true) and 0 for F (false)

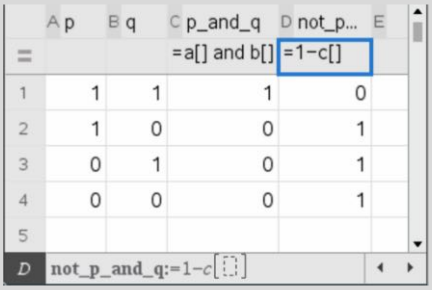
Use 'and' for conjunction, $\wedge$, and '$1-$' for negation, $\neg$.

***Note**: If required, use 'or' for inclusive disjunction, $\vee$.*

On a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the statement ***p***.
- In the column B heading cell, enter the statement ***q***.
- In column A and in column B, enter the truth values as shown.
- In the column C heading cell, enter the statement ***p_and_q***.
- In the column D heading cell, enter the statement ***not_p_and_q***.
- Enter the highlighted formulas as shown into the column C and D formula cells.





***Notes**: Press* [ctrl] [(] *to access the square brackets.*

Press [␣] *to insert a space.*

It is easiest to type the words 'and' and 'not'.

Answer:

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

***Notes**: The above approach can be used, for example, to show the logical equivalence of two statements, show whether a statement is a tautology (true under all circumstances) or show whether a statement is a contradiction (false under all circumstances).*

The 'truth table' approach can be used to represent circuits (switches or logic gates), Boolean expressions and functions. It can also be used to simplify circuits and Boolean expressions and functions. They can also be used to verify the equivalence of two Boolean expressions or two Boolean functions.

Converting binary and decimal numbers

Integers written in binary (base 2) are very important in computer science.

Integers are written in binary using powers of 2 and digits from the set $\mathbb{Z}_2 = \{0,1\}$ as their coefficients:

$$a_n 2^n + a_{n-1} 2^{n-1} + \ldots + a_2 2^2 + a_1 2^1 + a_0 \text{ where each } a_i \in \mathbb{Z}_2 \text{ and } \mathbb{Z}_2 = \{0,1\}$$

Question

Convert

(a) $101\ 101_2$ to a base 10 integer.

(b) the base 10 integer 13 into binary.

Solution

Part **(a)** on a **Calculator** page:

- Enter as shown using the prefix **0b** (indicates base 2).

Answer: $101101_2 = 45_{10}$

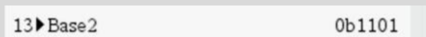
$101101_2 = 1\times 2^5 + 0\times 2^4 + 1\times 2^3 + 1\times 2^2 + 0\times 2^1 + 1\times 2^0$

Alternatively on a **Calculator** page:

- Enter as shown using the prefix **0b**.
- Press 📖 1 B, scroll down and select ►**Base10**.

Part **(b)** on a **Calculator** page:

- Press 📖 1 B, scroll down and select ►**Base2**.



Answer: $13_{10} = 1101_2$

Start with 13. Continuously divide by 2 keeping track of the quotient and remainder.

$13 \div 2 = 6R1$

$6 \div 2 = 3R0$

$3 \div 2 = 1R1$

$1 \div 2 = 0R1$

The binary representation is formed by writing down the remainders from first to last, 1, 0, 1, 1 in reverse order.
The remainders in reverse order are 1, 1, 0, 1 and so 1101_2.

Introducing Euclid's division algorithm

Question

Euclid's algorithm finds the *highest common factor* (hcf) (also known as the *greatest common divisor* (gcd)) of two integers without listing their factors.

The steps to find the hcf or gcd of $a = 720$ and $b = 168$ are illustrated below.

Step 1. Divide a by b to get quotient q and remainder r.

$\frac{720}{168} = 4\,r\,48$	$720 = 168 \times 4 + 48$	gcd(720, 168) = gcd(168, 48)

Step 2. Replace a with b and replace b with the remainder r.

$\frac{168}{48} = 3\,r\,24$	$168 = 48 \times 3 + 24$	gcd(720, 168) = gcd(168, 48) = gcd(48, 24)

Step 3. Repeat until the remainder, $r = 0$. The last non-zero divisor is the *hcf* or *gcd*.

$\frac{48}{24} = 2\,r\,0$	$48 = \mathbf{24} \times 2 + 0$	gcd(720, 168) = gcd(168, 48) = gcd(48, **24**) = 24

(a) Write pseudocode for this algorithm using a 'while' loop. Assume $a = 720$ and $b = 95$.

(b) Explain the pseudocode in terms of the constructs: sequencing, decision, and repetition.

(c) Implement the pseudocode for the algorithm in the Lists & Spreadsheet application

Solution

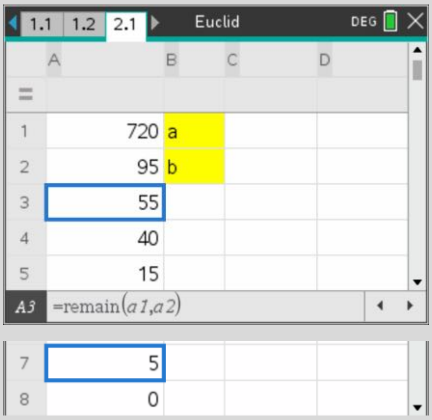
Answer:

(a) Possible pseudocode $a \leftarrow 720$ $b \leftarrow 95$ **while** $b \neq 0$ $r \leftarrow$ remain(a,b) $a \leftarrow b$ $b \leftarrow r$ **end while** **print** a	**(b) Decision**. In the loop boundary. Equivalent to **if** $b \neq 0$ continue loop, **else** end loop and print a. **Repetition**. The **while** loop keeps repeating until the remainder is zero. **Sequencing**. Updating the variables: replace a with b and b with the remainder, r.

Note:** The steps in the* ***Calculator *application.*

(c) To implement the algorithm in **Lists & Spreadsheet**:

- Enter the values of a and b in cells A1 and A2, as shown.
- In cell A3, enter the formula **=remain(*a*1,*a*2)** by pressing [catalog] [1] [R] to select **remain**.
- Navigate to cell A3 and press [menu] > **Data** > **Fill**. Press the ▼ key to, say, A10 then press [enter]. The relative cell references A1 and A2 will renew when filled down.



Answer: The last non-zero divisor, 5, is the hcf or gcd.

Note: *Edit cells A1, A2 to repeat for other values of a and b.*

Implementing Euclid's division algorithm in the Python application

Question

A version of pseudocode for Euclid's division algorithm was developed in the previous problem.

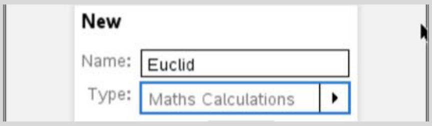
(a) Implement Euclid's algorithm in the Python application. Assume $a = 67860$ and $b = 1428$.

(b) Test the code with various values of a and b. Check the results using the gcd() command.

Solution

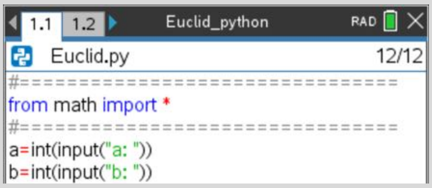
(a) To start coding, in a new **Document** (or a new **Problem**):

- Select **Add Python** > **New**.
- In the dialog box that follows, enter as shown.



To request user inputs for integers ***a*** and ***b*** (where $a > b$):

- Enter ***a*** **= int(input("*a*: "))** and ***b*** **= int(input("*a*: "))** by pressing [menu] > **Built-ins** > **Type** for **int()** and **…** > **I/O** for **input**. To select the text quotation marks, **"**, press [?!▸].

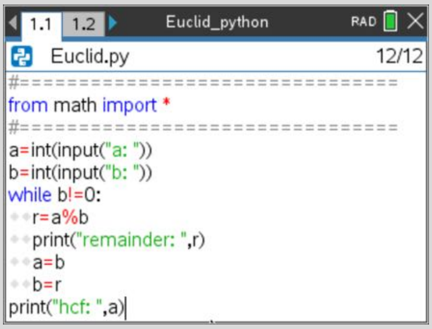


To initiate the **while** loop with boundary condition $b \neq 0$:

- Press [menu] > **Built-ins** > **Control** to select **while**. Enter **while b!= 0:** by pressing [ctrl] [=] (≠≥▸) to select **!=** for ≠.

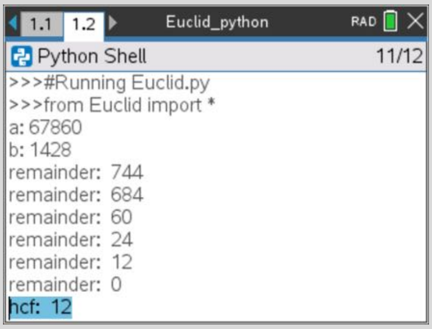
To calculate and display the remainder and update the variables, ensuring that the indentation is as shown:

- Enter ***r*** **=** ***a*%*b*** for remainder(a,b) by pressing [?!▸] for **%**.
- Enter **print("remainder: ",*r*)** by pressing [menu] > **Built-ins** > **I/O** to select **print**.
- Enter ***a*** **=** ***b*** and ***b*** **=** ***r***
- Enter **print("hcf: ",*a*)**.



To check syntax, save and run the program:

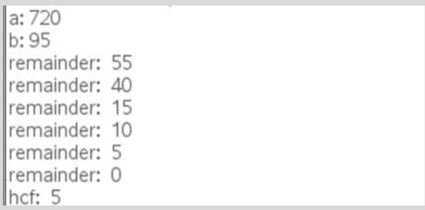
- Press [ctrl] [B] followed by [ctrl] [R] (or [menu] > **Run** …).
- In the **Python Shell** page that follows, use the prompts to enter ***a*: 67860** then ***b*: 1428**.



Answer: The highest common factor 67860 and 1428 is 12.

(b) To test the code for $a = 720$, $b = 95$:

- Press [ctrl] [R] and enter ***a*: 720** then ***b*: 95**.



Answer: The highest common factor is 5, confirming the result obtained in the previous problem.

To check the results against the inbuilt **gcd()** command, on a **Calculator** page:

- Press [📖] [1] [G] and select **gcd()**.
- Enter **gcd(67860,1428)**.
- Similarly, enter **gcd(720,95)**.

Answer: The previous hcf results of 12 and 5 are confirmed.

Using the Programme Editor to implement Euclid's division algorithm

Question

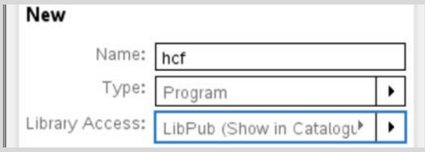
Euclid's division algorithm, including pseudocode, was developed in previous problems.

Implement Euclid's algorithm in the Programme Manager, with the inputs being entered through a user-defined *hcf*(*a*,*b*). Test the code by comparing with previous results.

Solution

To start coding, in a new **Problem** or a new **Document**:

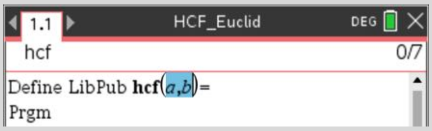
- Select **Add Programme Editor > New**.
- In the dialog box that follows, enter as shown.



The **Program Editor** will follow, ready to accept the code.

To name the inputs ***a*** and ***b***, in line 0:

- Enter ***a*,*b*** in the brackets for **Define LibPub hcf(*a*,*b*)**=.

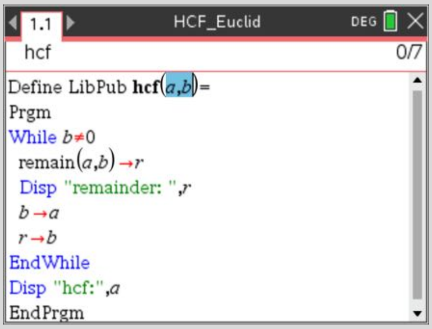


To initiate the **while** loop with boundary condition $b \neq 0$:

- In line 1, press menu > **Control > While … End While**.
- Enter **While *b* ≠ 0** by pressing ctrl = (≠≥▸) to select ≠.

To calculate and display the remainder and update variables:

- Enter **remain(*a*,*b*) → *r*** by pressing ctrl 1 R to select **remain** and ctrl var (sto→) for the **store** symbol, →.
- Enter **Disp "remainder: ",*r*** by pressing menu > **I/O** to select **Disp** and ?!▸ to select text quotation marks, ".
- Enter ***b* → *a*** and ***r* → *b***
- After **EndWhile**, enter **print("hcf: ",*a*)** to display the result.

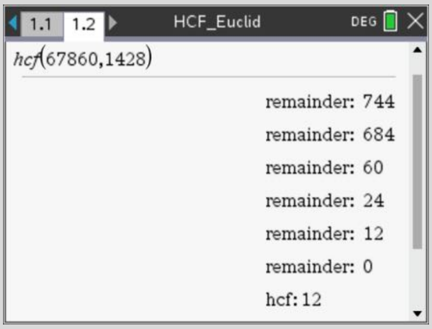


To check syntax, save and run the program:

- Press ctrl B followed by ctrl R (or menu > **Run** …).

To find the highest common factor of $a = 67860$, $b = 1428$:

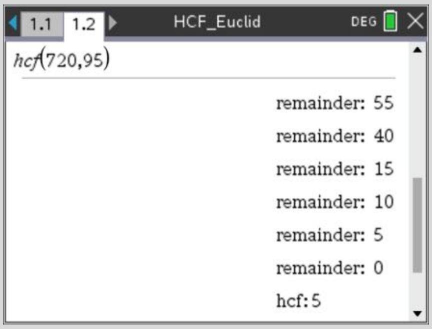
- In the **Calculator** page that follows, enter the values of *a* and *b* in the brackets, ***hcf*(67860, 1428)**, as shown.



Answer: The highest common factor is 12, consistent with results obtained in previous problems.

To find the highest common factor of $a = 720$, $b = 95$:

- On any **Calculator** page within the problem, enter ***hcf*(720, 95)** either by keying it in, or by pressing 📖 6 > **Current Problem** and selecting ***hcf***.



Answer: The highest common factor is 5, consistent with results obtained in previous problems.

1.2 Discrete mathematics

1.2.1 Sequences and series

Working with arithmetic sequences and series

The *n*th term of an arithmetic sequence is given by $t_n = a + (n-1)d$ where *a* is the first term and *d* is the common difference between any two consecutive terms.

That is, $d = t_n - t_{n-1}$, where $n \in \mathbb{Z}^+, n > 1$.

The sum of the first *n* terms of an arithmetic sequence is given by $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$ or

$S_n = \frac{n}{2}(a + l)$ where $l = t_n = a + (n-1)d$.

Question

The sum of the first six terms of an arithmetic sequence is 81.

The sum of its first eleven terms is 231.

(a) Find the first term and the common difference.

(b) Find t_{20}.

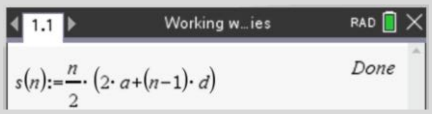
(c) Find S_{20}.

Solution

Parts **(a), (b)** and **(c)** on a **Calculator** page.

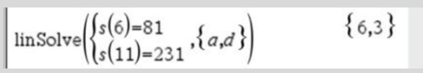
(a) Assign S_n (represented by $s(n)$) as follows:

- Press [ctrl] [⊞] to access the **Assign** [:=] command.
- Enter as shown.



To solve $S_6 = 81$ and $S_{11} = 231$ for *a* and *d*:

- Press [menu] > **Algebra** > **Solve System of Equations** >**Solve System of Linear Equations**.
- In the dialog box that follows:
  - For **Number of equations**, enter **2**.
  - For **Variables**, enter ***a*,*d***.
- Enter as shown.



Answer: (a) $a = 6$ and $d = 3$

$S_6 = 81 \Rightarrow 81 = \frac{6}{2}(2a + 5d)$ and so $2a + 5d = 27$ (1)

$S_{11} = 231 \Rightarrow 231 = \frac{11}{2}(2a + 10d)$ and so $a + 5d = 21$ (2)

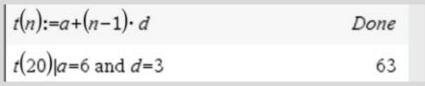
Solving (1) and (2) simultaneously for *a* and *d* gives $a = 6$ and $d = 3$.

... continued

Solution (continued)

(b) Assign t_n (represented by $t(n)$) as follows:

- Press ctrl [templates] to access the **Assign** := command.
- Press ctrl = to access the 'with' or 'given' symbol |.
- Enter as shown.

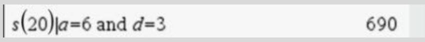


Answer: (b) $t_{20} = 63$

***Note**: Here it is not necessary to assign $t(n)$. However, this is a good strategy when attempting a multi-part question.*

(c) Find S_{20}:

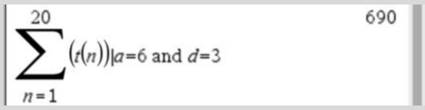
- Press ctrl = to access the 'with' or 'given' symbol |.
- Enter as shown.



Answer: (c) $S_{20} = 690$

Alternatively, S_{20} can be found using the **Sum** command as follows:

- Press menu > **Calculus** > **Sum**.
- Press ctrl = to access the 'with' or 'given' symbol |.
- Enter as shown.



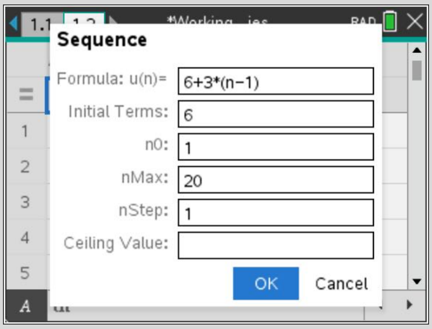
TI-Nspire CX II CAS can represent sequences and partial sums of sequences in tabular form.

Alternatively, parts **(b)** and **(c)** on a **Lists & Spreadsheet** page:

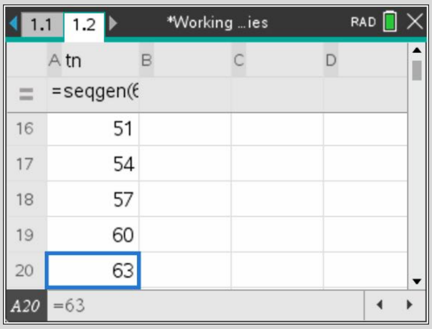
- In the column A heading cell enter the variable ***tn***.

In the column A formula cell generate the sequence as follows:

- Press menu > **Data** > **Generate Sequence**.
- Complete the **Sequence** dialog box as shown.



This confirms that $t_{20} = 63$.



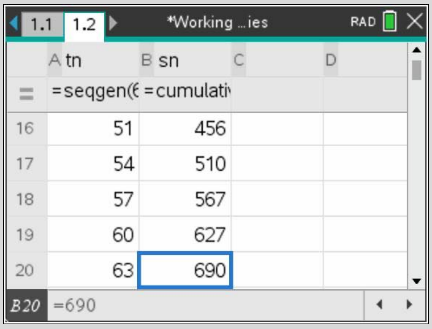
***Note:** Press ctrl 1 to go to the last entry in a column. Press ctrl 7 to go to the first entry in a column. Press ctrl 3 to go down a page and ctrl 9 to go up a page. To go to a specific cell, press ctrl G and enter the cell reference.*

... continued

Solution (continued)

To determine S_{20}:

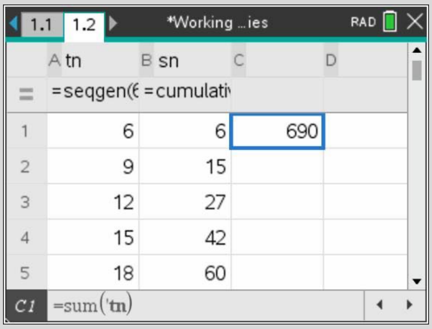
- In the column B heading cell enter the variable ***sn***.
- In the column B formula cell, press menu > **Data** > **List Operations** > **Cumulative Sum List**.
- Press var and select ***tn***.



Can you see how this command works? It confirms that $S_{20} = 690$.

Alternatively:

- In cell C1, press = menu > **Data** > **List Maths** > **Sum of Elements**.
- Press var and select ***tn***.

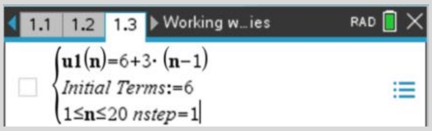


Again, confirming that $S_{20} = 690$.

TI-Nspire CX II CAS can represent arithmetic sequences in graphical form.

Part **(b)** on a **Graphs** page:

- Press menu > **Graph Entry/Edit** > **Sequence** > **Sequence**.
- Enter as shown.

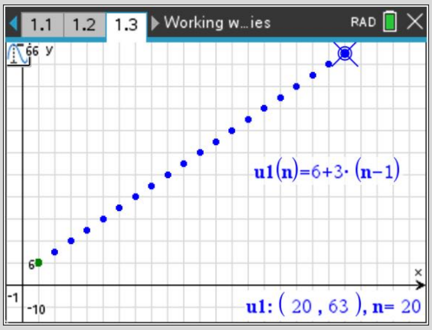


Note: *For sequences defined explicitly, it is optional to enter a value for* ***Initial Terms****.*

- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –1 XMax = 25 XScale = 1
  YMin = –10 YMax = 66 YScale = 6

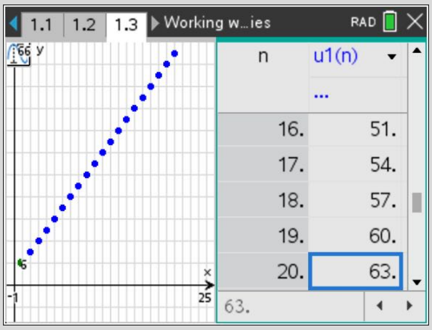
To add a grid:

- Press menu > **View** > **Grid** > **Lined Grid**.



To confirm that $t_{20} = 63$ using a graph and a table:

- Press menu > **Trace** > **Graph Trace**.
- Trace as shown using ▶ and ◀.
- Press ctrl T to display the sequence in tabular form (ctrl T toggles the table on/off).



Notes: *To display the table on a separate page, press* doc▾ > ***Page Layout*** > ***Ungroup****.*

If desired, press menu > ***Graph Entry/Edit*** > ***Function*** *and graph* $f1(x) = 6 + 3(x-1) | 1 \le x \le 20$. *However, remind students that number sequences are discrete.*

Sequences can also be graphed on a **Data & Statistics** page. To do this, ensure the **Lists & Spreadsheet** page has a column of accompanying n values.

Working with geometric sequences and series

The nth term of a geometric sequence is given by $t_n = ar^{n-1}$ where a is the first term and r is the common ratio between any two consecutive terms.

That is, $r = \dfrac{t_n}{t_{n-1}}$, where $n \in \mathbb{Z}^+, n > 1$.

The sum of the first n terms of a geometric sequence, $S_n = a + ar + ar^2 + ... + ar^{n-1}$, is given by

$S_n = \dfrac{a(r^n - 1)}{r-1}$, $r \neq 1$ (most convenient to use if $r > 1$) and

$S_n = \dfrac{a(1 - r^n)}{1-r}$, $r \neq 1$ (most convenient to use if $r < 1$).

Question

The amount of salt v_n tonnes produced by Vera's Salt Refinery in the nth month can be modelled by

$$v_n = 4000 + 400(n-1).$$

The amount of salt w_n tonnes produced by Wally's Salt Refinery in the nth month can be modelled by

$$w_n = 1000(1.1)^{n-1}.$$

The total amount of salt produced by Vera's Salt Refinery in the first n months is R_n and the total amount of salt produced by Wally's Salt Refinery in the first n months is S_n.

(a) Determine the total amount of salt produced by each refinery in the first 12 months of operation. For Wally's Salt Refinery, give your answer correct to the nearest tonne.

(b) Given that the two refineries started producing salt at the same time, find in which month the monthly production of Wally's Salt Refinery first exceeds that of Vera's Salt Refinery.

Solution

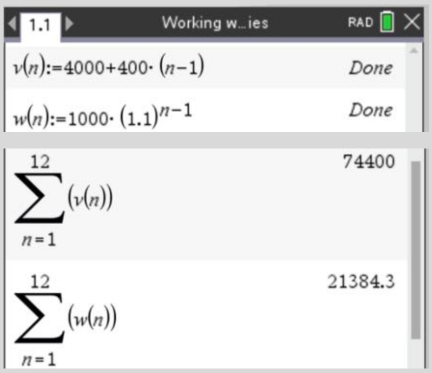
Parts **(a)** and **(b)** on a **Calculator** page:

(a) Assign v_n and w_n (represented by $v(n)$ and $w(n)$ respectively) as follows:

- Press ctrl [templates] to access the **Assign** [:=] command.
- Enter as shown.

R_{12} and S_{12} can be found using the **Sum** command as follows.

- Press menu > **Calculus** > **Sum**.
- Enter as shown.



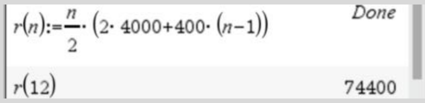
Answer: (a) $R_{12} = 74400$ (tonnes) and $S_{12} = 21384$ (tonnes) (correct to the nearest tonne).

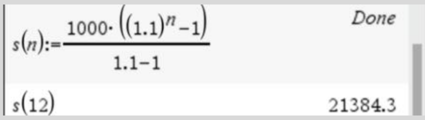
... continued

Solution (continued)

Substitute $n = 12$, $a = 4000$ and $d = 400$ into $R_n = \frac{n}{2}\left[2a + (n-1)d\right]$ and substitute $n = 12$, $a = 4000$ and $r = 1.1$ into $S_n = \frac{a(r^n - 1)}{r - 1}$.

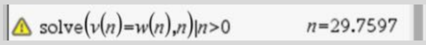
***Note:** R_n and S_n could be assigned/defined and then R_{12} and S_{12} calculated in the same fashion that S_{20} was calculated in the previous example. This approach is shown at right.*





(b) Find the least value of n such that $w_n > v_n$ as follows:

- Press [menu] > **Algebra** > **Solve**.
- Press [ctrl] [=] to access the 'with' or 'given' symbol | and the > symbol.
- Enter as shown.



Answer: (b) Solving $v_n = w_n$ for n with $n > 0$ gives $n = 29.75...$. In the 30th month, the monthly production of Wally's Salt Refinery first exceeds that of Vera's Salt Refinery.

TI-Nspire CX II CAS can represent sequences and partial sums in tabular form.

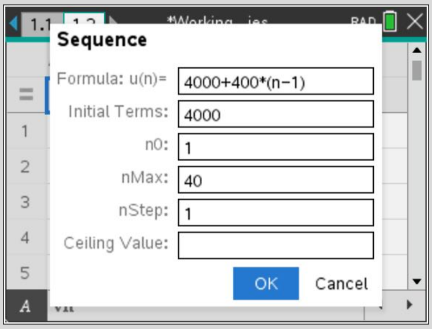
Alternatively, parts **(a)** and **(b)** on a **Lists & Spreadsheet** page:

(a) With v_n and w_n (represented by $v(n)$ and $w(n)$ respectively) assigned:

- In the column A heading cell enter the variable ***vn***.
- In the column C heading cell enter the variable ***wn***.

In the column A formula cell generate the sequence as follows:

- Press [menu] > **Data** > **Generate Sequence**.
- Complete the **Sequence** dialog box as shown.



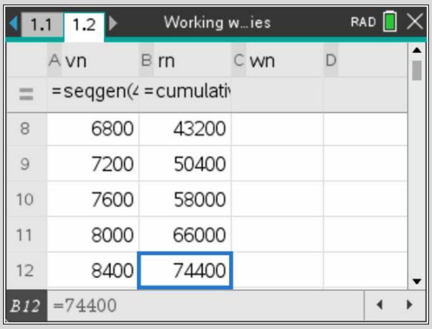
***Note:** With v_n assigned, $\boldsymbol{v(n)}$ can be entered in the first field. For sequences defined explicitly, it is optional to enter a value for **Initial Terms**.*

... continued

Solution (continued)

To determine R_{12}:

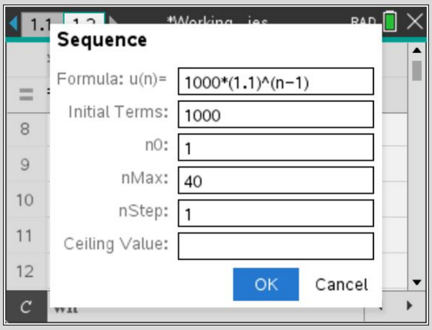
- In the column B heading cell enter the variable ***rn***.
- In the column B formula cell, press menu > **Data** > **List Operations** > **Cumulative Sum List**.
- Press var and select ***vn***.



Can you see how this commands works? It confirms that $R_{12} = 74400$.

In the column C formula cell generate the sequence as follows:

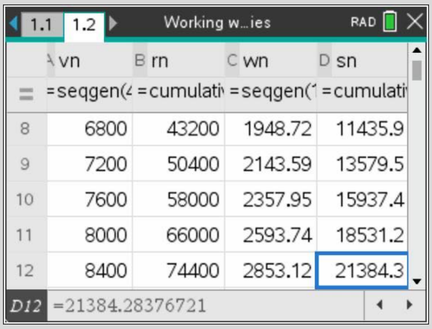
- Press menu > **Data** > **Generate Sequence**.
- Complete the **Sequence** dialog box as shown.



Note: *With* w_n *assigned,* $\mathbf{w}(\mathbf{n})$ *can be entered in the first field. For sequences defined explicitly, it is optional to enter a value for* ***Initial Terms****.*

To determine S_{12}:

- In the column D heading cell enter the variable ***sn***.
- In the column D formula cell, press menu > **Data** > **List Operations** > **Cumulative Sum List**.
- Press var and select ***wn***.



It confirms that $S_{12} = 21384$, correct to the nearest tonne.

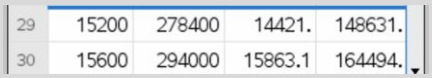
Note*: Alternatively, in a cell in an empty column, press* = menu ***> Data > List Maths > Sum of Elements****. Enter as shown to find* R_{12}.

Note: *Press* ctrl 1 *to go to the last entry in a column. Press* ctrl 7 *to go to the first entry in a column. Press* ctrl 3 *to go down a page and* ctrl 9 *to go up a page. To go to a specific cell, press* ctrl G *and enter the cell reference.*

(b) Find the least value of n such that $w_n > v_n$ as follows:

- Scroll down the spreadsheet until it seen where $v_n > w_n$ becomes $v_n < w_n$.

In the 29th month, $v_n > w_n$ and in the 30th month, $v_n < w_n$.

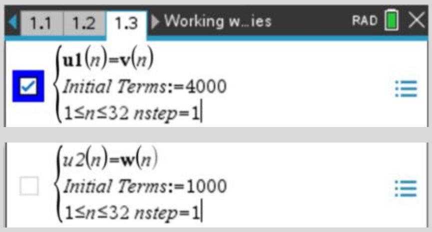


In the 30th month, the monthly production of Wally's Salt Refinery first exceeds that of Vera's Salt Refinery.

TI-Nspire CX II CAS can represent sequences in graphical form.

... continued

Solution (continued)



Part **(b)** on a **Graphs** page with v_n and w_n (represented by $v(n)$ and $w(n)$ respectively) assigned:

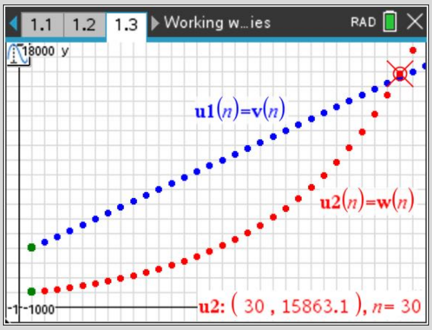
- Press [menu] > **Graph Entry/Edit** > **Sequence** > **Sequence**.
- Enter as shown.

Note: *For sequences defined explicitly, it is optional to enter a value for **Initial Terms**.*

- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –1 XMax = 32 XScale = 1
  YMin = –1000 YMax = 16000 YScale = 500

To add a grid:

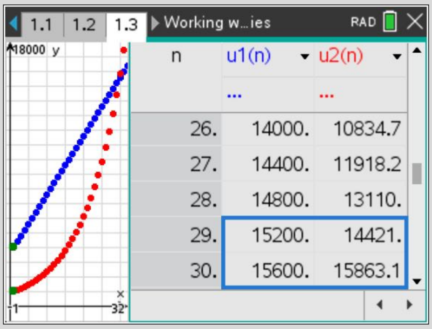
- Press [menu] > **View** > **Grid** > **Lined Grid**.



To confirm that $v_n < w_n$ for the first time in the 30th month using a graph and a table:

- Press [menu] > **Trace** > **Graph Trace**.
- Trace as shown using ▶, ◀, ▲ and ▼.
- Press [ctrl] [T] to display the sequence in tabular form ([ctrl] [T] toggles the table on/off).

If desired, graphing in function mode will overlay a straight line and a curve over the corresponding sequence. However, remind students that number sequences are discrete.



Notes *To display the table on a separate page, press* [doc▾] > ***Page Layout** > **Ungroup**.*

Note: *Sequences can also be graphed on a **Data & Statistics** page. To do this, ensure the **Lists & Spreadsheet** page has a column of accompanying n values.*

Working with infinite geometric sequences

If $-1<r<1$, then the sum of the infinite geometric sequence, $a, ar, ar^2, \ldots$, is convergent and the sum to infinity is given by $S_\infty = \dfrac{a}{1-r}$.

The limiting behaviour as $n \to \infty$ of t_n in a geometric sequence is dependent on the value of r.

Question

A rubber ball is dropped from a height of 12 metres.

Each time it hits the ground, it rebounds half the distance of the previous fall.

Let D_n represent the total distance travelled by the ball in a downwards direction.

(a) Show graphically how D_n changes with each rebound.

(b) As the number of rebounds increases, what happens to D_n?

(c) If the ball could rebound in this way indefinitely, find the total distance it would travel.

Solution

Part **(a)** on a **Graphs** page:

The sequence of downwards distances travelled is $12, 6, 3, \ldots$.

This is a geometric sequence with $a=12$ and $r=\dfrac{1}{2}$.

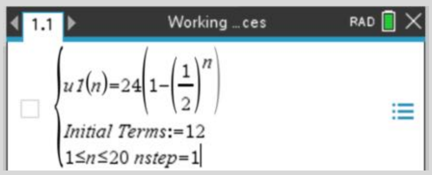
The partial sum of the sequence is denoted by D_n where

$$D_n = \frac{12\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} = 24\left(1-\left(\frac{1}{2}\right)^n\right).$$

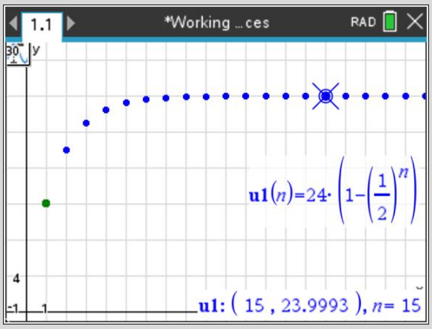
To show how D_n changes with each rebound graphically:

- Press menu > **Graph Entry/Edit** > **Sequence** > **Sequence**.
- Enter as shown.

__Note:__ For sequences defined explicitly, it is optional to enter a value for __Initial Terms__.



- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = −1 XMax = 20 XScale = 1
  YMin = −1 YMax = 30 YScale = 4



To add a grid:

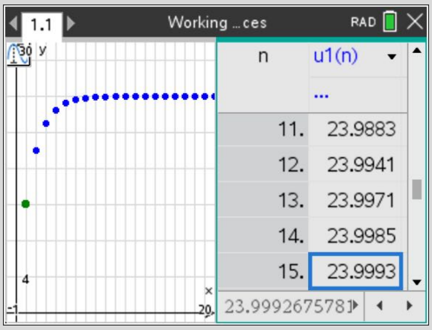
- Press menu > **View** > **Grid** > **Lined Grid**.

... continued

Solution (continued)

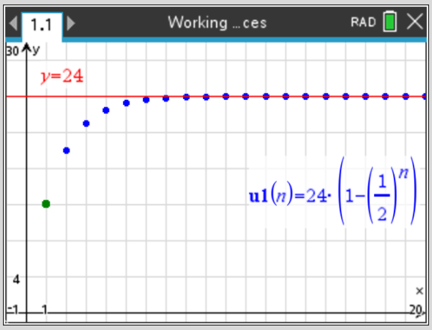
To trace the behaviour of D_n using a graph and a table:

- Press menu > **Trace** > **Graph Trace**.
- Trace as shown using ▶ and ◀.
- Press ctrl T to also display the sequence in tabular form (ctrl T toggles the table on/off).



Note: *To display the table on a separate page, press* doc▾ *>* ***Page Layout*** *>* ***Ungroup****.*

- Press menu > **Graph Entry/Edit** > **Relation** and graph $y = 24$.



Note*: Sequences can also be graphed on a* ***Data & Statistics*** *page. To do this, ensure the* ***Lists & Spreadsheet*** *page has a column of accompanying n values.*

(b) As the number of rebounds increases ($n \to \infty$), $D_n \to 24$.

Part **(c)** on a **Calculator** page:

To find the total distance the ball would travel, we need to consider the downward motion and upward motion separately.

Downward motion: $12 + 6 + 3 + ...$

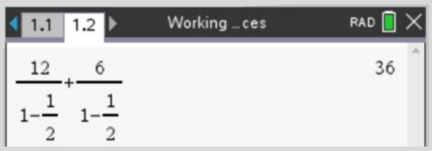
Use $S_\infty = \dfrac{a}{1-r}$ with $a = 12$ and $r = 0.5$:

$$S_\infty = \frac{12}{1-0.5} = 24$$

Upward motion: $6 + 3 + 1.5 + ...$

Use $S_\infty = \dfrac{a}{1-r}$ with $a = 6$ and $r = 0.5$:

$$S_\infty = \frac{6}{1-0.5} = 12$$



The total distance the ball would travel is $24 + 12 = 36$ (m).

Working with sequences generated by recursion

Arithmetic sequences are defined recursively by $t_n = t_{n-1} + d$.

Geometric sequences are defined recursively by $t_n = rt_{n-1}$.

For an arithmetic-geometric sequence, $t_n = rt_{n-1} + d$, where r, d are constants.

The case where $r = 1$ corresponds to an arithmetic sequence.

The case where $d = 0$ corresponds to a geometric sequence.

For a sequence defined recursively by $t_n = rt_{n-1} + d$ where $r \neq 1$, the nth term is given by

$$t_n = r^{n-1}t_1 + \frac{d\left(r^{n-1} - 1\right)}{r - 1}, \text{ where } t_1 \text{ is the first term.}$$

Question

The volume of water, V, in a tank on the morning of the nth day can be modelled by

$$V_{n+1} = 0.9V_n + 2000 \text{ where } V_1 = 45000 \text{ litres.}$$

(a) Determine how many litres of water will be in the tank on the morning of the fourth day.

(b) On the morning of which day will the volume of water in the tank first be below 30 000 litres?

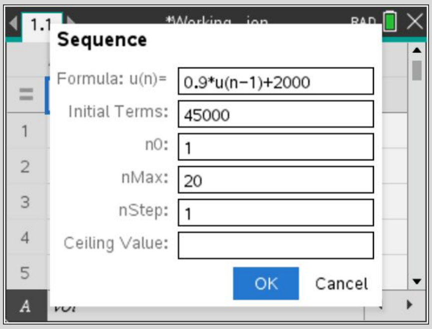
Solution

Parts **(a)** and **(b)** on a **Lists & Spreadsheet** page.

- In the column A heading cell enter the variable ***vol***.

In the column A formula cell, generate the sequence as follows:

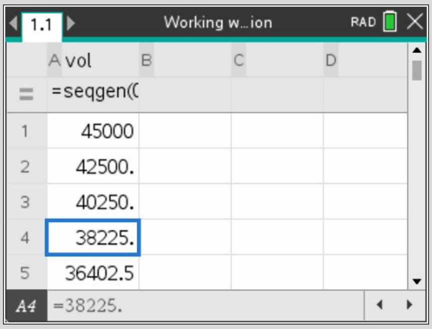
- Press [menu] > **Data** > **Generate Sequence**.
- Complete the **Sequence** dialog box as shown.



(a) To find the volume of water in the tank on the fourth morning, we require the fourth term in the sequence, i.e. V_4.

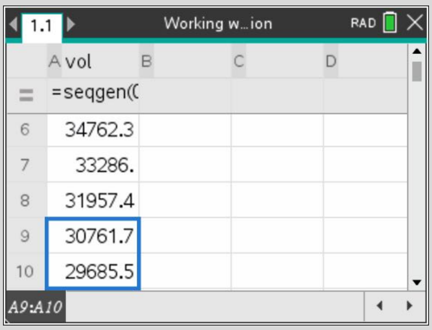
Scrolling down column A, there will be 38 225 litres of water in the tank on the fourth morning.

Note: *Press [ctrl][1] to go to the last entry in a column. Press [ctrl][7] to go to the first entry in a column. Press [ctrl][3] to go down a page and [ctrl][9] to go up a page. To go to a specific cell, press [ctrl][G] and enter the cell reference.*



(b) From the table of values generated there will be, correct to the nearest litre, 30 762 litres in the tank on the ninth morning and 29 686 litres in the tank on the tenth morning.

Hence the volume of water in the tank will first fall below 30 000 litres on the tenth morning.



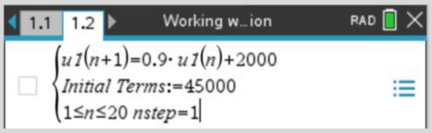
... continued

Solution (continued)

TI-Nspire CX II CAS can represent sequences defined recursively in graphical form.

Parts **(a)** and **(b)** on a **Graphs** page.

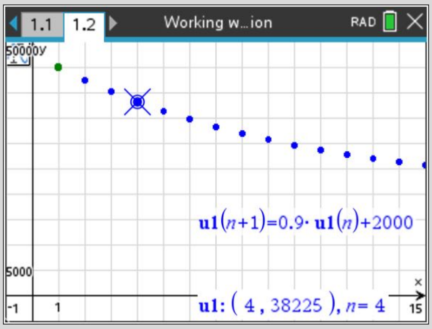
- Press [menu] > **Graph Entry/Edit** > **Sequence** > **Sequence**.
- Enter as shown.



- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –1 XMax = 15 XScale = 1
  YMin = –5000 YMax = 50000 YScale = 2000

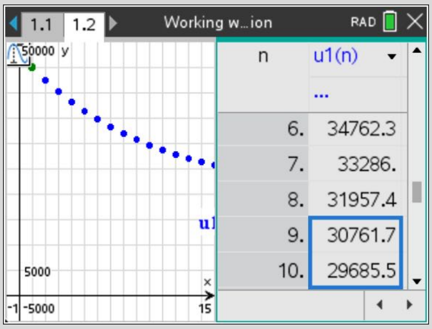
To add a grid:

- Press [menu] > **View** > **Grid** > **Lined Grid**.



To confirm there will be 38 225 litres of water in the tank on the fourth morning and the volume of water in the tank will first fall below 30 000 litres on the tenth morning using a graph and a table:

- Press [menu] > **Trace** > **Graph Trace**.
- Trace as shown using ▶, ◀, ▲ and ▼.
- Press [ctrl] [T] to display the sequence in tabular form ([ctrl] [T] toggles the table on/off).



***Note:** To display the table on a separate page, press [doc▾] > **Page Layout** > **Ungroup**.*

***Note**: Sequences can also be graphed on a **Data & Statistics** page. To do this, ensure the **Lists & Spreadsheet** page has a column of accompanying n values.*

1.2.2 Combinatorics

Using the pigeon-hole principle

If $n+1$ or more objects are placed into n holes, then some hole contains at least two objects.

Question

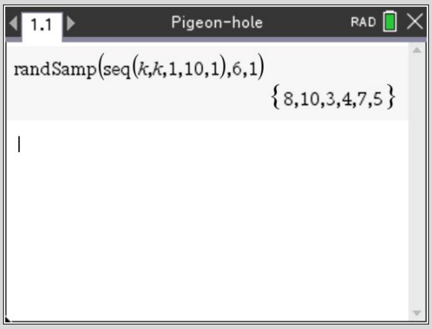
Consider six different random integers from 1 to 10.

Prove that at least one pair of these integers must sum to 11.

***Note**: To 'seed' or initialise the pseudo-random number generator, on a **Calculator** page, press [menu] **> Probability > Random > Seed** then enter, say, the last few digits of your mobile phone number, e.g. **RandSeed** 74839.*

Solution

Generate six different random integers from 1 to 10.



On a **Calculator** page:

- Press [menu] **> Probability > Random > Sample**.
- Press [menu] **> Statistics > List Operations > Sequence**.
- Enter as shown.

***Notes:** The **randSamp** command syntax is **randSamp(List, #Trials[, noRepl])**. Enter 1 to generate a random sample without replacement.*

*The syntax for expressing a sequence as a list is **seq(Expression, Variable, Low, High[, Step])**. The default value for **Step** is 1.*

Assign the variable **list** to the six integers and order them from smallest to largest:

- Enter **list** and press [ctrl] [⊞] to access the **Assign** [:=] command.
- Press [menu] **> Statistics > List Operations > Sort Ascending**, then press [var], select **list**, and press [enter] to complete the sort.
- Press [var], select **list**, press [enter] to view the sorted list.

***Note:** To sort the elements of a list, the list must have a name.*

Answer: The six integers generated in this random sample are $\{3,4,5,7,8,10\}$. Note that $3+8=11$ and $4+7=11$.

Proof: Pair up the integers from 1 to 10 as follows:

(1,10), (2,9), (3,8), (4,7) and (5,6)

There are five possible pairs that sum to 11 i.e. $n=5$.

Hence at least two of the six integers must be in the same pair.

***Note:** The objects ('pigeons') are the six random integers and the holes are the five ordered pairs. It is worthwhile generating a number of random samples and also discussing the lists {1,2,3,4,5,6} and {1,2,3,8,9,10}.*

Using the generalised pigeon-hole principle

If at least $mn+1$ objects are placed into n holes, then some hole contains at least $m+1$ objects.

Question

Consider a set of 13 different random positive integers.

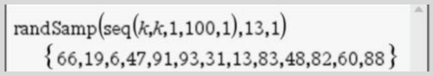
Prove that at least four of these positive integers have the same remainder when divided by 4.

Solution

Generate 13 different random integers from 1 to 100.

On a **Calculator** page:

- Press [menu] > **Probability** > **Random** > **Sample**.
- Press [menu] > **Statistics** > **List Operations** > **Sequence**.
- Enter as shown.



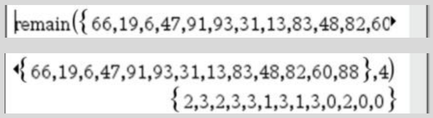
Notes: *The* ***randSamp*** *command syntax is* ***randSamp(List, #Trials[, noRepl])****. Enter 1 to generate a random sample without replacement.*

The syntax for expressing a sequence as a list is ***seq(Expression, Variable, Low, High[, Step])****.*
The default value for ***Step*** *is 1.*

A list of random integers can also be generated on a ***Lists & Spreadsheet*** *page.*

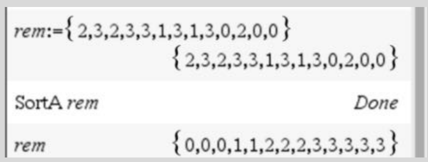
To determine the remainder when each integer is divided by 4:

- Press [menu] > **Number** > **Remainder**.
- Enter as shown.



Assign the variable **rem** to these 13 remainders and order them from smallest to largest:

- Enter **rem** and press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [menu] > **Statistics** > **List Operations** > **Sort Ascending**, then press [var], select **rem**, and press [enter] to complete the sort.
- Press [var], select **rem**, press [enter] to view the sorted list.



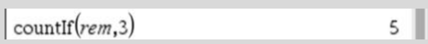
Note: *To sort the elements of a list, the list must have a name.*

Answer: The remainders generated from this random sample of 13 integers are {0,0,0,1,1,2,2,2,3,3,3,3,3}.

There are three remainders of 0, two remainders of 1, three remainders of 2 and five (at least four) remainders of 3.

To confirm there are five remainders of 3, for example:

- Press [catalog] [1] [C], scroll down and select **countIf(**.
- Press [var], select **rem** and enter as shown.



... continued

Solution (continued)

Note: *The* ***countIf(*** *command syntax is* ***countIf(list, criteria)****.*

Proof: There are four possible remainders (four holes) when dividing a positive integer by 4.

These remainders are 0, 1, 2, 3.

There are 13 positive integers to be placed into four holes.

Since $13 = 3\times 4+1$, there is a hole with at least four $(3+1)$ positive integers that have the same remainder when divided by 4.

Note: *The objects ('pigeons') are the 13 positive integers and the holes are the four possible remainders.*

Using the inclusion-exclusion principle

The inclusion-exclusion principles for two sets and three sets are:

$$|A\cup B| = |A|+|B|-|A\cap B|$$

$$|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$$

A common application of the inclusion-exclusion principle is finding the number of integers in a range that are not divisible by a set of primes.

Question

How many integers from 1 to 32 inclusive are not divisible by 2, 3 or 5?

Solution

The **floor** function (greatest integer function), $\lfloor x \rfloor$, gives the greatest integer less than or equal to a given real number, *x*.

To find the number of integers from 1 to 32 inclusive that are not divisible by 2, 3 or 5:

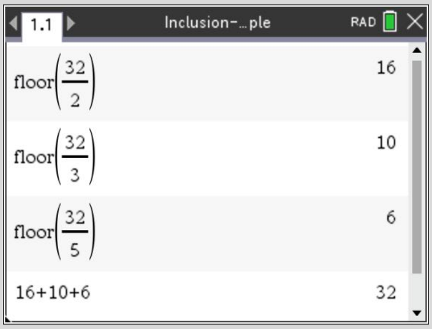
(1) Count the individual sets.

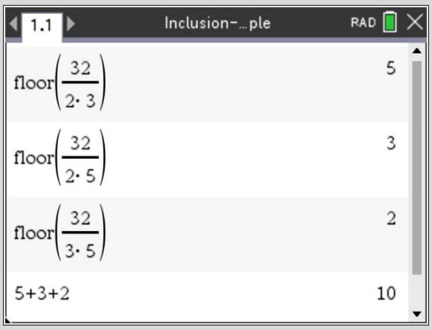
(2) Subtract the pairwise intersections.

(3) Add back the three-way intersection.

On a **Calculator** page:

- Press menu > **Number** > **Number Tools** > **Floor**.
- Enter as shown.



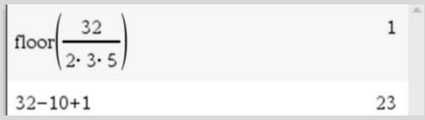


Note: *The* ***int(*** *command, found in the* ***Catalog****, is identical to the* ***Floor*** *command. The* ***Integer Part*** *command (press* menu *>* ***Number*** *>* ***Number Tools*** *>* ***Integer Part*** *gives the same output as the floor function for positive real numbers.*

... continued

Solution (continued)

Answer: There are 23 integers not divisible by 2, 3 or 5.



Let A, B and C be the sets of all integers from 1 to 32 that are divisible by 2, 3 and 5 respectively.

$|A| = 32 \div 2 = 16$, $|B| = \lfloor 32 \div 3 \rfloor = 10$ and $|C| = \lfloor 32 \div 5 \rfloor = 6$

$|A \cap B| = \lfloor 32 \div 6 \rfloor = 5$, $|A \cap C| = \lfloor 32 \div 10 \rfloor = 3$,
$|B \cap C| = \lfloor 32 \div 15 \rfloor = 2$

$|A \cap B \cap C| = \lfloor 32 \div 30 \rfloor = 1$

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap \\ &= 16 + 10 + 6 - 5 - 3 - 2 + 1 \\ &= 23 \end{aligned}$$

Hence the number of integers not divisible by 2, 3 or 5 is $32 - 9 = 23$.

Using factorial notation

Question

Find the value of each of the following

(a) $10!$ **(b)** $\dfrac{100!}{98!}$ **(c)** $6 \times 4! - 7 \times 3!$

Solution

Text in the form of comments can be added to a **Calculator** page.

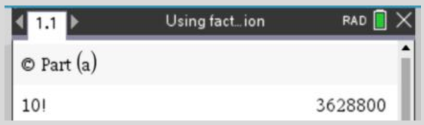
To add a comment such as '© Part (a)':

- Press [menu] > **Actions** > **Insert Comment**.

To access the factorial symbol:

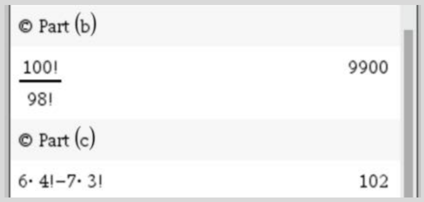
- Press [menu] > **Probability** > **Factorial (!)**.

To evaluate parts **(a)**, **(b)** and **(c)**, enter as shown.



Answer: (a) $10! = 3\,628\,800$ where $10! = 10 \times 9 \times 8 \times \ldots \times 3 \times 2 \times 1$.

- Press [ctrl] [÷] to access the **Fraction** template.



Answer: (b) $\dfrac{100!}{98!} = 9900$ where $\dfrac{100!}{98!} = \dfrac{100 \times 99 \times 98!}{98!} = 100 \times 99$.

Answer: (c) $6 \times 4! - 7 \times 3! = 102$

Defining and using permutations

Question

Adele has seven different books but there is only room for three of these books on her bookshelf. Find the number of ways Adele can randomly select the books and arrange them on her bookshelf using

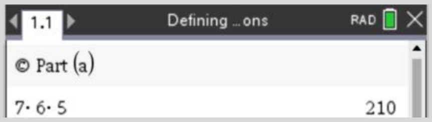
(a) the multiplication principle. **(b)** $\dfrac{n!}{(n-r)!}$. **(c)** ${}^{n}P_{r}$.

Solution

To add a comment to a **Calculator** page:

- Press [menu] > **Actions** > **Insert Comment**.

Answer: (a) 7 books can be placed in the first position, 6 remain for the second and 5 for the third. So $7\times6\times5=210$.



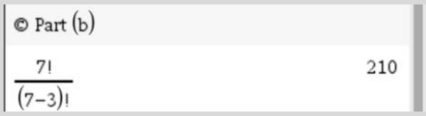
To access the factorial symbol:

- Press [menu] > **Probability** > **Factorial (!)**.

To access the **Fraction** template:

- Press [ctrl] [÷].

Answer: (b) $\dfrac{7!}{(7-3)!}=210$ where $\dfrac{7!}{4!}=\dfrac{7\times6\times5\times4!}{4!}=210$.



$\dfrac{n!}{(n-r)!}$ can be interpreted as

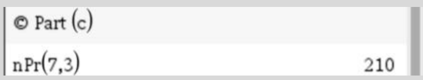
$$\frac{(\text{total number of objects})!}{(\text{total number of objects} - \text{number of objects to be arranged})!}$$

The number of ways to arrange r objects from a total of n objects is ${}^{n}P_{r}$ where ${}^{n}P_{r}=\dfrac{n!}{(n-r)!}$.

To access the **Permutations** command:

- Press [menu] > **Probability** > **Permutations**.

Answer: (c) ${}^{7}P_{3}=210$



Note: *${}^{n}P_{r}$ represents the number of ways of arranging r objects from n distinct objects where order is important.*

Solving problems using permutations

Question

In how many ways can 4 cats and 3 dogs be arranged in a row if

(a) they are placed randomly?

(b) the 4 cats are kept together, and the 3 dogs are kept together?

(c) no cat is next to another cat?

Solution

To add a comment to a **Calculator** page:

- Press [menu] > **Actions** > **Insert Comment**.

To access the factorial symbol:

- Press [menu] > **Probability** > **Factorial (!)**.

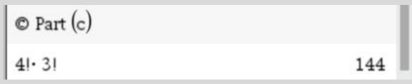
Answer: (a) There are $7! = 5040$ arrangements.

Answer: (b) There are 4! ways of keeping the 4 cats together and for each of these, 3! ways of keeping the dogs together. Also, there are 2 ways of arranging the group of cats and the group of dogs. The number of ways is $2 \times 4! \times 3! = 288$.

Answer: (c) If no cat is next to another cat, the arrangement must be CDCDCDC.

There are 4! ways of arranging the 4 cats and for each of these, 3! ways of arranging the 3 dogs.

The number of ways is $4! \times 3! = 144$.



Evaluating ^nC_r

Question

Evaluate ^6C_r for $r = 0, 1, 2, 3, 4, 5, 6$.

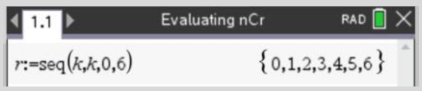
Extension: Verify that the sum of the entries in row n of Pascal's triangle is 2^n i.e. $^nC_0 + {}^nC_1 + \ldots + {}^nC_{n-1} + {}^nC_n = 2^n$.

Solution

One way to evaluate ^6C_r for $r = 0, 1, 2, 3, 4, 5, 6$ is to use the **Sequence** command.

On a **Calculator** page, assign the values of r as a sequence.

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [menu] > **Statistics** > **List Operations** > **Sequence**.
- Enter as shown.



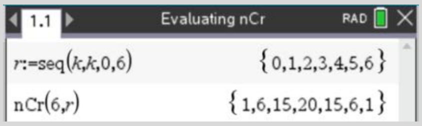
... continued

Solution (continued)

***Note:** The syntax for expressing a sequence as a list is **seq(Expression, Variable, Low, High[, Step])**. The default value for **Step** is 1.*

To access the **Combinations** command:

- Press [menu] > **Probability** > **Combinations**.
- Enter as shown.



Answer:

$^6C_0=1, {}^6C_1=6, {}^6C_2=15, {}^6C_3=20, {}^6C_4=15, {}^6C_5=6, {}^6C_6=1$

***Notes:** This is the $n=6$ row of Pascal's triangle. The $n=0$ row of Pascal's triangle is the first row.*

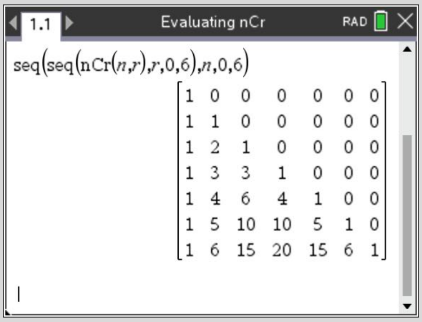
Alternatively, enter on a **Calculator** page as shown.
To access {}, press [ctrl] [)].

Extension:

A way to generate rows of Pascal's triangle is to write a command for "nested" sequences as shown on the **Calculator** page at right.

The first 7 rows of Pascal's triangle are displayed.

Can you see how it works?

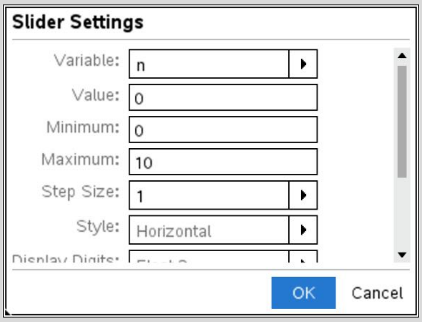


Alternatively on a **Notes** page:

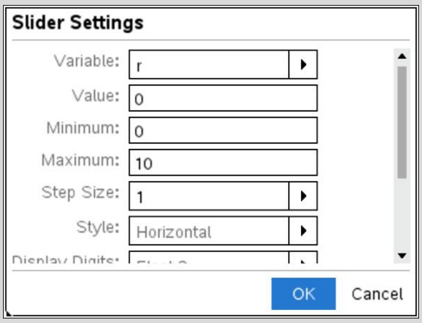
***Note:** Press [⌂ on] > **Settings** > **Document Settings** and set **Calculation Mode** to **Exact**.*

Insert a **Slider** to control the value of n as follows:

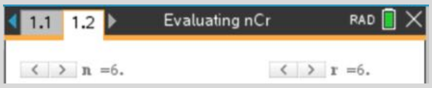
- Press [menu] > **Insert** > **Slider**.
- Set the **Slider Settings** as shown.
- Check the **Minimised** box.



Repeat the above instructions to insert a slider for r.



Position the sliders as shown.



Insert a **Maths Box** as follows:

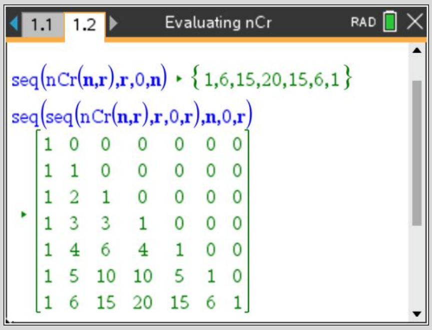
- Press [menu] > **Insert** > **Maths Box**.

***Note:** Alternatively, to insert a **Maths Box**, press [ctrl] [M].*

... continued

Solution (continued)

- Enter $\mathbf{seq(nCr(}\boldsymbol{n,r}\mathbf{),}\boldsymbol{r}\mathbf{,0,}\boldsymbol{n}\mathbf{)}$ into the **Maths Box** as shown.



Insert another **Maths Box**:

- Enter $\mathbf{seq(seq(nCr(}\boldsymbol{n,r}\mathbf{),}\boldsymbol{r}\mathbf{,0,}\boldsymbol{r}\mathbf{),}\boldsymbol{n}\mathbf{,0,}\boldsymbol{r}\mathbf{)}$ as shown.

Click on the sliders to change the value of *n* and *r*.

The screenshot displays the $n = 6$ row of Pascal's triangle and the first 7 rows of Pascal's triangle.

Extension: The sum of the entries in row *n* of Pascal's triangle is 2^n i.e. ${}^nC_0 + {}^nC_1 + ... + {}^nC_{n-1} + {}^nC_n = 2^n$.

To verify this result for particular values of *n*:

- Enter $\mathbf{factor}\left(\mathbf{sum}\left(\mathbf{seq(nCr(}\boldsymbol{n,r}\mathbf{),}\boldsymbol{r}\mathbf{,0,}\boldsymbol{n}\mathbf{)}\right)\right)$ into a **Maths Box** as shown.
- Click on the slider to vary the value of *n*.

To access the **Factor** command:

- Press [menu] > **Calculations** > **Algebra** > **Factor**.

To access the **Sum of Elements** command:

- Press [menu] > **Calculations** > **Statistics** > **List Maths** > **Sum of Elements**.

To change the display of a **Maths Box**, for example, to display an equals sign:

- Click on the **Maths Box**.
- Press [menu] > **Maths Box Options** > **Maths Box Attributes**.
- Press [tab] to highlight the **Insert Symbol** field.
- Press ▶ and select =.

***Note: Maths Box Attributes** can also be accessed within a **Maths Box** by pressing* [ctrl] [menu].

Solving equations involving $^{n}C_{r}$

Question

Solve $3\times {}^{n}C_{6}=11\times {}^{n}C_{4}$ for n where n is a positive integer.

Solution

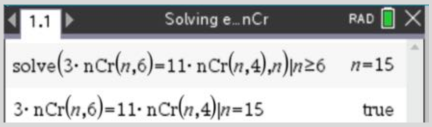
Note that ${}^{n}C_{6}\geq 1$ and ${}^{n}C_{4}>1$ for $n\in\mathbb{Z}^{+}, n\geq 6$.

On a **Calculator** page:

- Press [menu] > **Algebra** > **Solve**.
- Press [menu] > **Probability** > **Combinations**.
- Enter as shown.

To add the constraint $n\geq 6$:

- Press [ctrl] [=] to access the 'with' or 'given' symbol | and the ≥ symbol.



Answer: Solving $3\times {}^{n}C_{6}=11\times {}^{n}C_{4}$ for n with $n\geq 6$ gives $n=15$.

***Note:** Entering the equation with $n=15$ gives the output **'true'**.*

Solving problems involving combinations

Question

A committee of three must be chosen from a cricket team of 11 players.

How many different committees are possible if:

(a) there are no restrictions?

(b) the captain of the team must be on the committee?

Solution

To add a comment to a **Calculator** page:

- Press [menu] > **Actions** > **Insert Comment**.

To access the **Combinations** command:

- Press [menu] > **Probability** > **Combinations**.

Answer: (a) There are ${}^{11}C_{3}=165$ possible committees.

Answer: (b) As the captain of the team must be on the committee, we simply need to select two of the remaining players.

There are ${}^{10}C_{2}=45$ possible committees.

Deriving and applying simple combinatorial identities

Each entry in Pascal's triangle is the sum of the two entries immediately above.

This property is described by Pascal's rule which states that ${}^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}$, where $1 \le r < n$ and ${}^{n}C_{r}$ is the coefficient of the x^{r} term in the expansion $(1+x)^{n}$.

Question

Prove that ${}^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}$, where $1 \le r < n$.

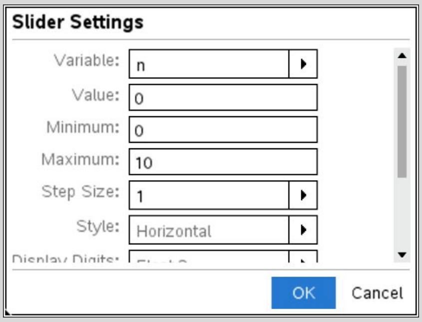
Solution

On a **Notes** page:

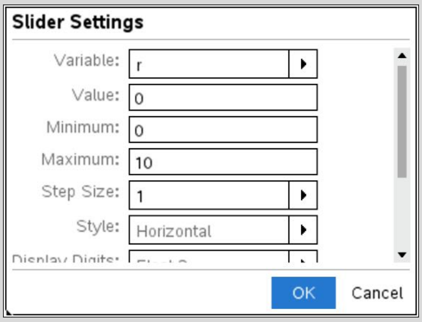
***Note:** For verifying Pascal's rule on a **Notes** page, press* [⌂ on] ***> Settings > Document Settings** and set **Calculation Mode** to **Exact**.*

Insert a **Slider** to control the value of *n* as follows:

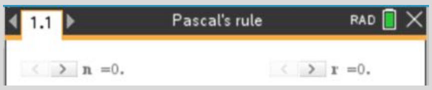
- Press [menu] **> Insert** > **Slider**.
- Set the **Slider Settings** as shown.
- Check the **Minimised** box.



Repeat the above instructions to insert a slider for *r*.



Position the sliders as shown.



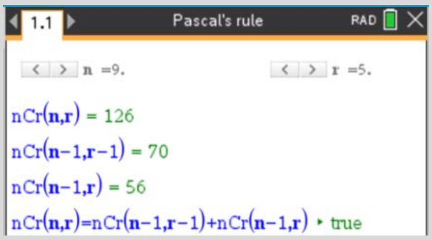
Insert four **Maths Boxes** as follows:

- Press [menu] **> Insert** > **Maths Box**.

***Note:** Alternatively, to insert a **Maths Box**, press* [ctrl] [M].

To access the **Combinations** command:

- Press [menu] **> Calculations > Probability > Combinations**.
- Enter as shown.
- Click on each slider to vary the value of *n* and of *r*.



To change the display of a **Maths Box**, for example, to display an equals sign:

- Click on the **Maths Box**.
- Press [menu] **> Maths Box Options > Maths Box Attributes**.
- Press [tab] to highlight the **Insert Symbol** field.
- Press ▶ and select =.

... continued

Solution (continued)

***Note: Maths Box Attributes** can also be accessed within a **Maths Box** by pressing* ctrl menu.

The output is always 'true'.

Proof: The number of subsets of $\{1, 2, ..., n\}$ containing exactly r elements is $^{n}C_{r}$. Each of these subsets can be put into either (1) a group containing n or (2) a group not containing n.

(1) Each of the remaining $r-1$ elements must be chosen from $\{1, 2, ..., n-1\}$. Hence this group contains $^{n-1}C_{r-1}$ subsets.

(2) Each of the r elements must be chosen from $\{1, 2, ..., n-1\}$. Hence this group contains $^{n-1}C_{r}$ subsets.

The two groups together contain all $^{n}C_{r}$ subsets and so $^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}$, where $1 \le r < n$.

***Note:** Pascal's rule can also be proven algebraically by using the formula for* $^{n}C_{r}$.

1.2.3 Matrices

Understanding the matrix definition and notation

A matrix is a rectangular array of elements.

The dimension of a matrix is $m \times n$, where m is the number of rows and n is the number of columns.

Defining, adding and subtracting matrices

Two matrices can be added or subtracted if they have the same dimension.

Addition and subtraction are performed by adding or subtracting the corresponding elements in each matrix.

In general for 2×2 matrices, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then $A \pm B = \begin{bmatrix} a \pm e & b \pm f \\ c \pm g & d \pm h \end{bmatrix}$.

Question

A fruit and vegetable cooperative has three shops, S_1, S_2 and S_3.

On a particular Monday:

- S_1 sold 45 avocados, 18 lettuces and 11 watermelons.
- S_2 sold 35 avocados, 18 lettuces and 9 watermelons.
- S_3 sold 47 avocados, 29 lettuces and 10 watermelons.

On a particular Tuesday:

- S_1 sold 28 avocados, 13 lettuces and 16 watermelons.
- S_2 sold 31 avocados, 17 lettuces and 13 watermelons.
- S_3 sold 29 avocados, 28 lettuces and 19 watermelons.

Let the sales for the Monday be denoted by matrix M and the sales for the Tuesday be denoted by matrix T.

(a) Find the sales for the three shops over the two days.

(b) Find the number of lettuces sold by S_2 over the two days.

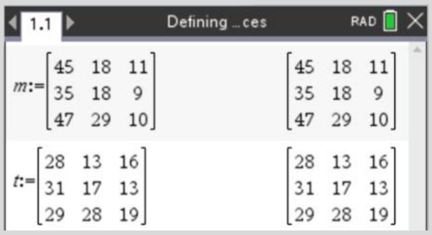
Solution

$$M = \begin{bmatrix} 45 & 18 & 11 \\ 35 & 18 & 9 \\ 47 & 29 & 10 \end{bmatrix} \text{ and } T = \begin{bmatrix} 28 & 13 & 16 \\ 31 & 17 & 13 \\ 29 & 28 & 19 \end{bmatrix}$$

Parts **(a)** and **(b)** on a **Calculator** page:

Assign M and T as follows:

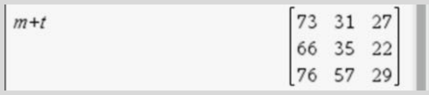
- Press ctrl [templates key] to access the **Assign** := command.
- Press menu > **Matrix & Vector** > **Create** > **Matrix**.
- Set the number of rows to be 3 and the number of columns to be 3.
- Enter as shown and calculate $M + T$ as shown.



... continued

Solution (continued)

Answer: (a) $M+T=\begin{bmatrix}73 & 31 & 27\\ 66 & 35 & 22\\ 76 & 57 & 29\end{bmatrix}$



Note: *Alternatively, to create a 3 x 3 matrix, press [book] 5 (or [template]), select the* ***m-by-n Matrix*** *template and complete as above.*

The number of lettuces sold by S_2 over the two days is given by the element $(2,2)$.

- To access element $(2,2)$, enter **(*M*+*T*)[2,2]** as shown.

Note: *The first element in* $\begin{bmatrix}2 & 2\end{bmatrix}$ *indicates the row number and the second element indicates the column number. When entering this, separate values with a comma as shown. Note that the comma is removed when [enter] is pressed.*

Answer: (b) Element $(2,2)$ indicates that S_2 sold 35 lettuces on the Monday and the Tuesday.

Note*: The* ***Define*** *command ([menu] >* ***Actions*** *>* ***Define****) and the* ***Store*** *command (press [ctrl] [var] to access [sto→]) can also be used.*

Verifying matrix addition properties

The following matrix properties are important:

- $A+B=B+A$ (commutative law for addition)
- $A+O=A$ (additive identity)
- $A+(-A)=O$ (additive inverse)

Question

Let $A=\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ and $B=\begin{bmatrix} c & d \\ e & f \end{bmatrix}$.

Verify that

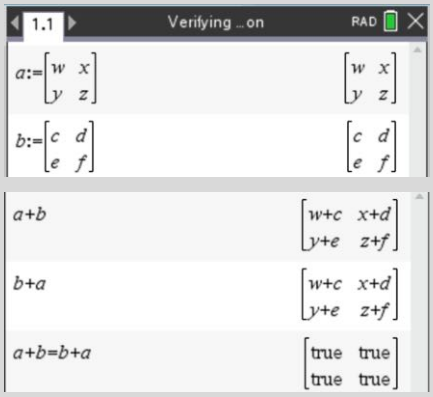
(a) $A+B=B+A$ **(b)** $A+O=A$ **(c)** $A+(-A)=O$

Solution

Parts **(a)**, **(b)** and **(c)** on a **Calculator** page:

Assign A and B as follows:

- Press [ctrl] [:=] to access the **Assign** [:=] command.
- Press [📖] [5], select the **2-by-2 Matrix** template and enter as shown.

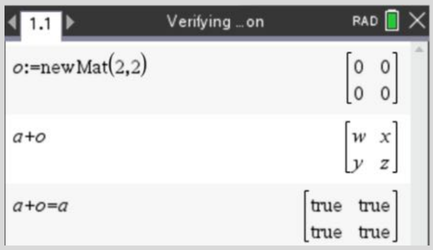


Answer: (a) $A+B=\begin{bmatrix} w+c & x+d \\ y+e & z+f \end{bmatrix}$, $B+A=\begin{bmatrix} w+c & x+d \\ y+e & z+f \end{bmatrix}$.

Note: *Entering* $A+B=B+A$ *gives the output* $\begin{bmatrix} \text{true} & \text{true} \\ \text{true} & \text{true} \end{bmatrix}$.

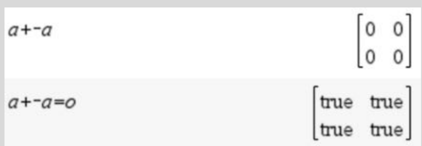
To create O:

- Press [ctrl] [:=] to access the **Assign** [:=] command.
- Press [menu] **> Matrix & Vector > Create > Zero Matrix**.
- Enter as shown.



Answer: (b) $A+O=\begin{bmatrix} w & x \\ y & z \end{bmatrix}=A$.

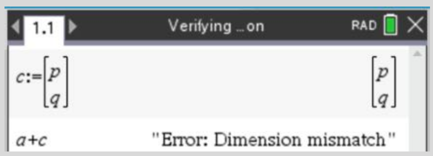
Note: *Entering* $A+O=A$ *gives the output* $\begin{bmatrix} \text{true} & \text{true} \\ \text{true} & \text{true} \end{bmatrix}$.



Answer: (c) $A+(-A)=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}=O$.

Notes: *Entering* $A+(-A)=O$ *gives the output* $\begin{bmatrix} \text{true} & \text{true} \\ \text{true} & \text{true} \end{bmatrix}$.

When attempting to add two matrices of different dimension, a 'dimension mismatch' error message is displayed. An example of this is shown at right. To enter C, press [📖] [5], *select the* ***2-by-1 Matrix*** *template and enter as shown.*



Defining scalar and matrix multiplication

Matrices can be multiplied by scalar (real number) quantities.

In general for 2×2 matrices, if $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $kA=\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$ where k is a scalar.

Two matrices can be multiplied together if the number of columns in the first matrix equals the number of rows in the second matrix.

In general, if an $m\times n$ matrix is multiplied by an $n\times p$ matrix, the resulting matrix will be of dimension $m\times p$, that is, $(m\times \boldsymbol{n})\times(\boldsymbol{n}\times p)=m\times p$.

Note that, in general, for two matrices, A and B, $AB\neq BA$.

In other words, matrix multiplication in general is not commutative.

The following matrix properties are important:

- $A(B+C)=AB+AC$ (left distributive law)
- $(B+C)A=BA+CA$ (right distributive law)

Question

Let $A=\begin{bmatrix} 9 & 4 \\ -2 & 3 \end{bmatrix}$, $B=\begin{bmatrix} 10 & 7 \\ 2 & -4 \end{bmatrix}$ and $C=\begin{bmatrix} 4 & 0 \\ -3 & 2 \end{bmatrix}$.

Verify that

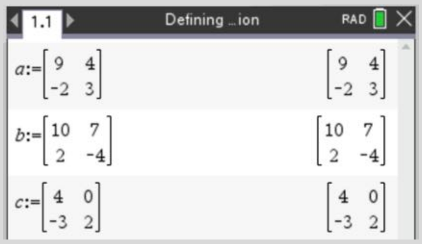
(a) $AB\neq BA$ **(b)** $A(B+C)=AB+AC$ **(c)** $(B+C)A=BA+CA$

Solution

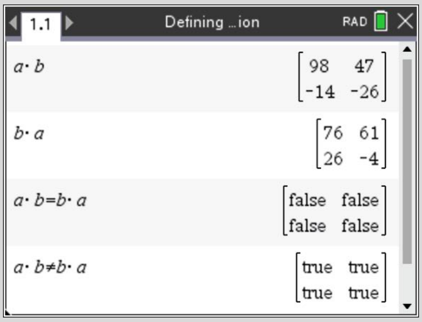
Parts **(a)**, **(b)** and **(c)** on a **Calculator** page:

Assign A, B and C as follows:

- Press [ctrl] [template key] to access the **Assign** [:=] command.
- Press [catalog key] [5], select the **2-by-2 Matrix** template and enter as shown.



Answer: (a) $AB=\begin{bmatrix} 98 & 47 \\ -14 & -26 \end{bmatrix}$ and $BA=\begin{bmatrix} 76 & 61 \\ 26 & -4 \end{bmatrix}$.

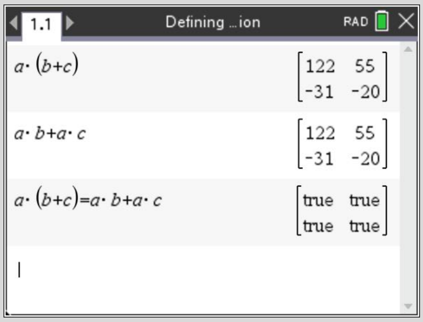


***Notes:** Entering $AB=BA$ gives the output $\begin{bmatrix} \text{false} & \text{false} \\ \text{false} & \text{false} \end{bmatrix}$ and entering $AB\neq BA$ gives the output $\begin{bmatrix} \text{true} & \text{true} \\ \text{true} & \text{true} \end{bmatrix}$. When performing matrix multiplication, always use the multiplication key, [×].*

... continued

Solution (continued)

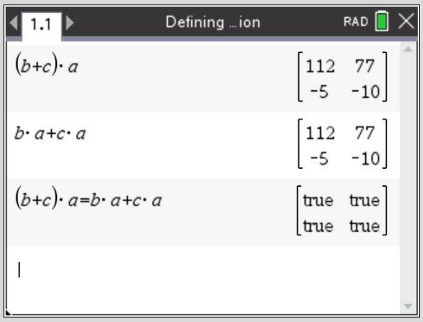
Enter $A(B+C)$ and $AB+AC$ as shown.



Answer: (b) $A(B+C)=\begin{bmatrix}122 & 55\\ -31 & -20\end{bmatrix}$ and

$AB+AC=\begin{bmatrix}122 & 55\\ -31 & -20\end{bmatrix}$.

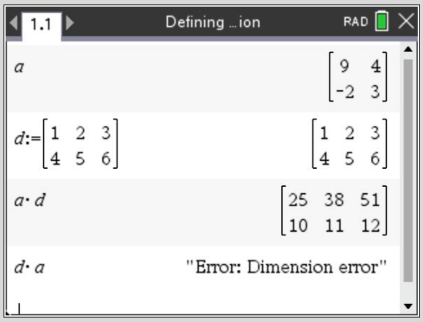
Note: *Entering* $A(B+C)=AB+AC$ *gives the output* $\begin{bmatrix}\text{true} & \text{true}\\ \text{true} & \text{true}\end{bmatrix}$.



Answer: (c) $(B+C)A=\begin{bmatrix}112 & 77\\ -5 & -10\end{bmatrix}$ and

$BA+CA=\begin{bmatrix}112 & 77\\ -5 & -10\end{bmatrix}$.

Note: *Entering* $(B+C)A=BA+CA$ *gives the output* $\begin{bmatrix}\text{true} & \text{true}\\ \text{true} & \text{true}\end{bmatrix}$.



On a **Calculator** page, assign D as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **m-by-n Matrix** template, fix the dimensions as 2-by-3 and enter as shown.

AD can be found because A is a 2×2 matrix and D is a 2×3 matrix ($2=2$). However, DA cannot be found because D is a 2×3 matrix and A is a 2×2 matrix ($3\neq 2$).

Note: *When attempting to multiply two matrices of different dimension, a 'dimension error' message is displayed when the number of columns in the first matrix does not equal the number of rows in the second matrix.*

Raising a matrix to a power

Question

Consider $M = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$.

(a) Find M^2, M^3 and M^4.

(b) Hence, infer a general result for M^n where $n \in \mathbb{Z}^+$.

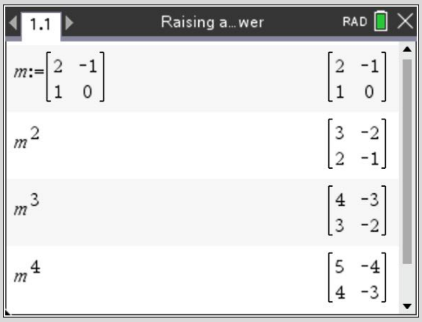
(c) Use your result to determine M^{2025} and check your answer with *TI-Nspire CX II CAS*.

Solution

Part **(a)** on a **Calculator** page.

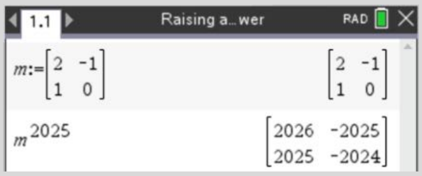
Assign M as follows:

- Press ctrl [template key] to access the **Assign** [:=] command.
- Press [book] 5, select the **2-by-2 Matrix** template and enter as shown.



Answer: (a) $M^2 = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$, $M^3 = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$ and $M^4 = \begin{bmatrix} 5 & -4 \\ 4 & -3 \end{bmatrix}$.

Answer: (b) $M^n = \begin{bmatrix} n+1 & -n \\ n & 1-n \end{bmatrix}$.

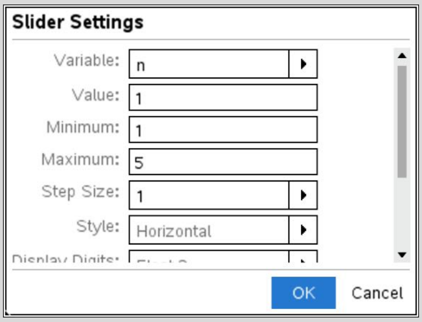


Answer (c) $M^{2025} = \begin{bmatrix} 2025+1 & -2025 \\ 2025 & 1-2025 \end{bmatrix} = \begin{bmatrix} 2026 & -2025 \\ 2025 & -2024 \end{bmatrix}$.

Alternatively parts **(a)** and **(c)**, on a **Notes** page:

Insert a **Slider** to control the value of n as follows:

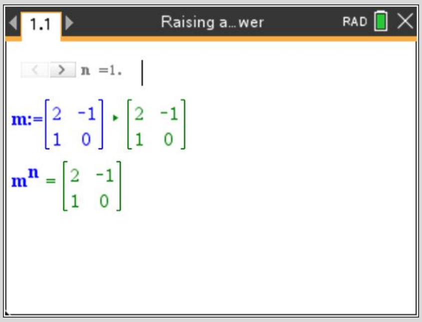
- Press menu > **Insert** > **Slider**.
- Set the **Slider Settings** as shown.
- Check the **Minimised** box.



Insert a **Maths Box** as follows:

- Press menu > **Insert** > **Maths Box**.

***Note:** Alternatively, to insert a **Maths Box**, press* ctrl M.



... continued

Solution (continued)

Assign M as follows:

- Press ctrl to access the **Assign** := command.
- Press 5, select the **2-by-2 Matrix** template and enter as shown.

Now:

- Insert another **Maths Box** and enter m^n.

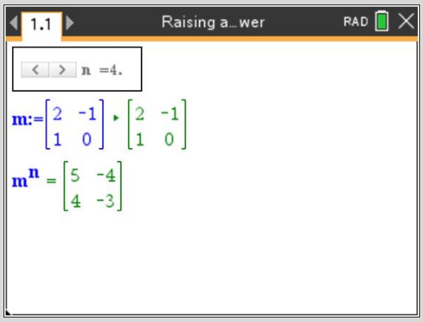
To change the display of a **Maths Box**, for example, to display an equals sign:

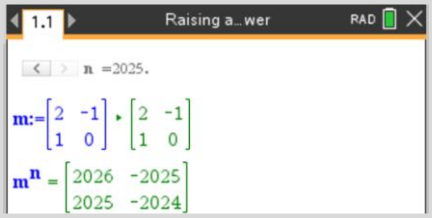
- Click on the **Maths Box**.
- Press menu > **Maths Box Options** > **Maths Box Attributes**.
- Press tab to highlight the **Insert Symbol** field.
- Press ▶ and select =.

Note: Maths Box Attributes *can also be accessed within a* ***Maths Box*** *by pressing* ctrl menu.

Click on the slider to change the value of n.

To calculate M^n for $n = 2025$, manually change the value of n to 2025 in the slider box.





Introducing the multiplicative identity and multiplicative inverse

An identity matrix, I, is a square matrix that has the number 1 for all elements on the main diagonal and 0 for all other elements.

The following matrix properties are important:

- $AI = A = IA$ (multiplicative identity)
- $AA^{-1} = I = A^{-1}A$ (multiplicative inverse)

Question

Let $A = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$ and $B = \frac{1}{3}\begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix}$.

Verify that

(a) $AI = A = IA$.

(b) $AB = I = BA$ where $B = A^{-1}$.

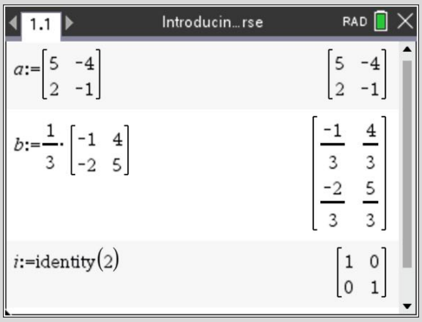
... continued

Solution

Parts **(a)** and **(b)** on a **Calculator** page:

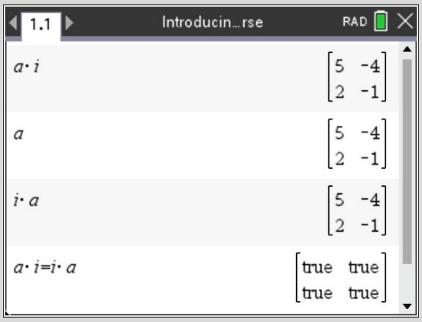
Assign A and B as follows:

- Press [ctrl] [template] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **2-by-2 Matrix** template and enter as shown.



To create I, where $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$:

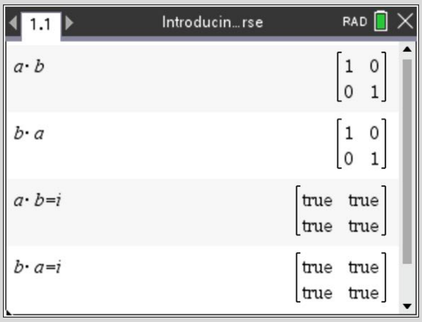
- Press [ctrl] [template] to access the **Assign** [:=] command.
- Press [menu] > **Matrix & Vector** > **Create** > **Identity**.
- Enter as shown.



Answer: (a) $AI=\begin{bmatrix}5 & -4\\ 2 & -1\end{bmatrix}$, $IA=\begin{bmatrix}5 & -4\\ 2 & -1\end{bmatrix}$ and $AI=A=IA$.

Note: *Entering* $AI=IA$ *gives the output* $\begin{bmatrix}\text{true} & \text{true}\\ \text{true} & \text{true}\end{bmatrix}$.

Enter AB and BA as shown.



Answer: (b) $AB=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $BA=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$.

Note: *Entering* $AB=I$ *and* $BA=I$ *both give the output* $\begin{bmatrix}\text{true} & \text{true}\\ \text{true} & \text{true}\end{bmatrix}$.

From **(b)**, it can be concluded that $B=A^{-1}$.

Introducing the determinant of 2 x 2 matrices

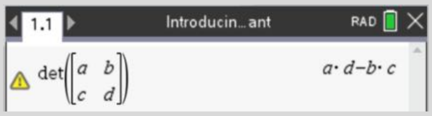
Question

Given that $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, verify that $\det(A)=ad-bc$.

Solution

On a **Calculator** page:

- Press [menu] > **Matrix & Vector** > **Determinant**.
- Press [catalog] [5], select the **2-by-2 Matrix** template and enter as shown.



Answer: $\det(A)=ad-bc$

Calculating the determinant of 2 x 2 Fibonacci matrices

Question

The Fibonacci sequence is defined by $F_n = F_{n-1} + F_{n-2}$ where $F_1 = F_2 = 1$, $F_0 = 0$ and $n \geq 3$.

F_n is the *n*th term of the sequence.

(a) Determine F_3, F_4, F_5 and F_6.

Consider $P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

(b) Find $\det(P)$.

(c) Find

(i) P^2 and $\det(P^2)$. **(ii)** P^3 and $\det(P^3)$. **(iii)** P^4 and $\det(P^4)$.

(d) Hence infer a general result for P^n and $\det(P^n)$.

(e) Use the result from part **(d)** to find an expression for $F_{n+1}F_{n-1} - F_n^{\,2}$ in terms of *n*.

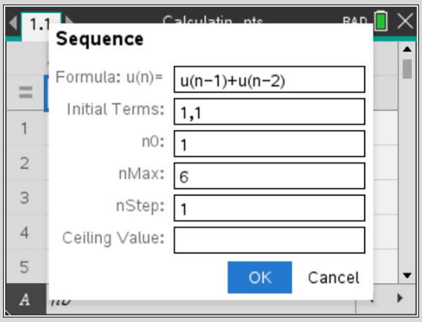
Solution

Part **(a)** on a **Lists & Spreadsheet** page:

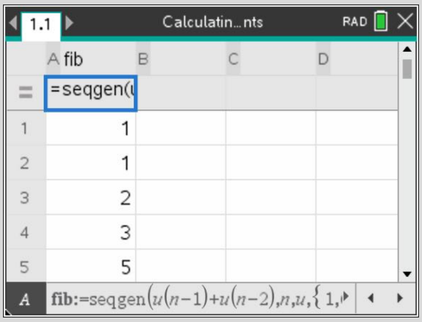
- In the column A heading cell, enter the variable ***fib***.

Generate terms of a Fibonacci sequence as follows:

- In the column A formula cell, press menu > **Data** > **Generate Sequence**.
- Complete the **Sequence** dialog box as shown.



Answer: (a) $F_3 = 2$, $F_4 = 3$, $F_5 = 5$ and $F_6 = 8$.

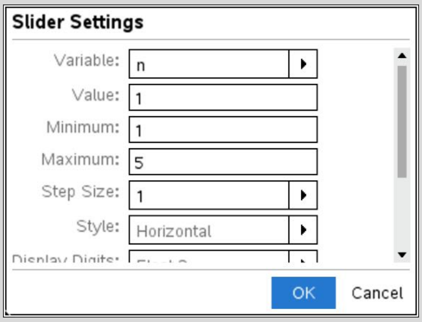


Answer: (b) $\det(P) = -1$.

Part **(c)** on a **Notes** page:

Insert a **Slider** to control the value of *n* as follows:

- Press menu > **Insert** > **Slider**.
- Set the **Slider Settings** as shown.
- Check the **Minimised** box.



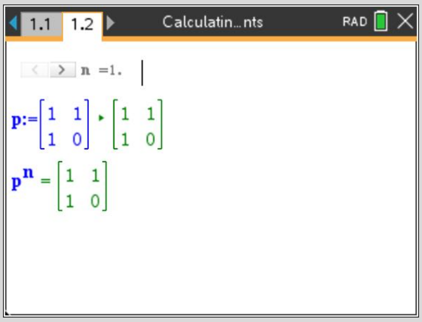
... continued

Solution (continued)

Insert a **Maths Box** as follows:

- Press [menu] **> Insert** > **Maths Box**.

Note: *Alternatively, to insert a* ***Maths Box****, press* [ctrl] [M].

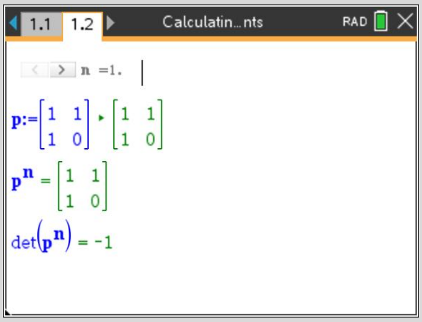


Assign P as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **2-by-2 Matrix** template and enter as shown.

Now:

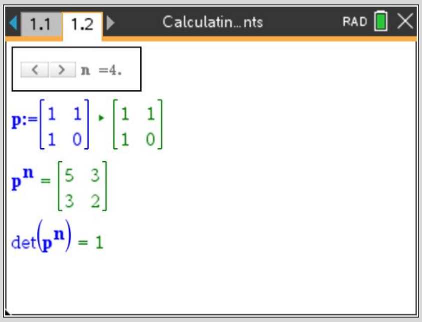
- Insert a **Maths Box** and enter the expression $\boldsymbol{p^n}$.



To display an equals sign in a **Maths Box**:

- Click on the **Maths Box**.
- Press [menu] **> Maths Box Options** > **Maths Box Attributes**.
- Press [tab] to highlight the **Insert Symbol** field.
- Press ▶ and select $=$.

Note: Maths Box Attributes *can also be accessed within a* ***Maths Box*** *by pressing* [ctrl] [menu].



To enter the expression $\mathbf{det}\left(p^n\right)$ in a **Maths Box**:

- Press [menu] **> Calculations** > **Matrix & Vector** > **Determinant**.
- Enter as shown and change the display to show an equals sign.

Click on the slider to change the value of n.

Answers: (c) (i) $P^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $\det\left(P^2\right) = 1$.

(ii) $P^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$ and $\det\left(P^3\right) = -1$.

(iii) $P^4 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ and $\det\left(P^4\right) = 1$.

(d) $P^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$ and $\det\left(P^n\right) = (-1)^n$.

(e) $F_{n+1}F_{n-1} - F_n^{\ 2} = (-1)^n$.

Showing that a given 2 x 2 matrix is non-singular

If $\det(A) \neq 0$, then A is a non-singular matrix.

Question

Let $M = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ where a and b are non-zero real numbers.

(a) Show that M is non-singular.

(b) Show that $\det(M^2)$ is always positive.

Solution

Parts **(a)** and **(b)** on a **Calculator** page:

Assign M as follows:

- Press [ctrl] to access the **Assign** $:=$ command.
- Press [5], select the **2-by-2 Matrix** template and enter as shown.

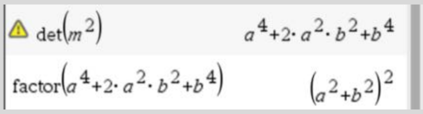
To find $\det(M)$ and $\det(M^2)$:

- Press [menu] > **Matrix & Vector** > **Determinant**.
- Enter as shown.

Answer: (a) $\det(M) = a^2 + b^2 \neq 0$ as $a, b \in \mathbb{R}, a, b \neq 0$ and so M is non-singular.

To confirm that $a^4 + 2a^2b^2 + b^4$ is a perfect square:

- Press [menu] > **Algebra** > **Factor**.
- Enter as shown.



Answer: (b) $\det(M^2) = (a^2 + b^2)^2 > 0$

Alternatively, $\det(M^2) = (\det(M))^2$ and since $\det(M) > 0$ so too is $\det(M^2)$.

Introducing the multiplicative inverse of a 2 x 2 matrix

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ where } \det(A) \neq 0 \, (ad - bc \neq 0).$$

If $\det(A) = 0$, then A is a singular matrix and A^{-1} does not exist.

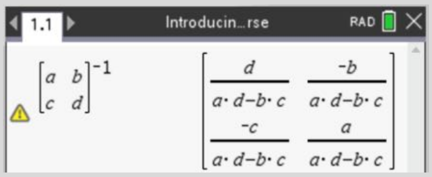
Question

Given that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, verify that $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Solution

On a **Calculator** page:

- Press [book][5], select the **2-by-2 Matrix** template and enter as shown.



Answer: $A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

***Note:** To find M^{-1} without using TI-Nspire CX II CAS, interchange the elements on the main diagonal, change the signs of the elements on the secondary diagonal and divide by* $\det(M)$.
To check your answer, calculate MM^{-1} or $M^{-1}M$. M and M^{-1} should satisfy $MM^{-1} = M^{-1}M = I$.

Calculating the determinant and multiplicative inverse of 2 x 2 matrices

Question

Consider $M = \begin{bmatrix} -2 & b \\ 3 & 4 \end{bmatrix}$ where $\det(M) = -14$.

(a) Find the value of b.

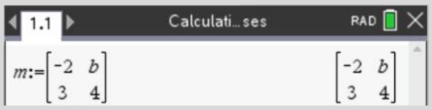
(b) Find M^{-1}.

Solution

Parts **(a)** and **(b)** on a **Calculator** page.

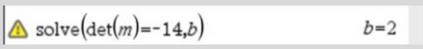
Assign M as follows:

- Press [ctrl][templates] to access the **Assign** [:=] command.
- Press [book][5], select the **2-by-2 Matrix** template and enter as shown.



To solve $\det(M) = -14$ for b:

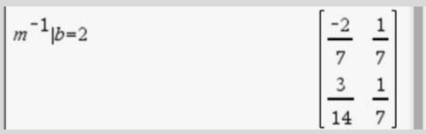
- Press [menu] > **Algebra** > **Solve**.
- Press [menu] > **Matrix & Vector** > **Determinant**.
- Enter as shown.



Answer: (a) Solving $\det(M) = -14$ for b gives $b = 2$.

To calculate M^{-1}, enter as shown.

- Press [ctrl][=] to access the 'with' or 'given' symbol |.



Answer: (b) $M^{-1} = -\frac{1}{14}\begin{bmatrix} 4 & -2 \\ -3 & -2 \end{bmatrix}$.

***Note:** Alternatively, to find an inverse using a step-by-step approach, press [book][6] > **linalgcas** > **inversestep**. The syntax for this command is **inversestep(Mat)**. Can you see how it works?*

Solving matrix equations involving matrices of up to dimension 2 x 2

If $AX = B$, where A is a square matrix and has inverse A^{-1} such that $A^{-1}A = I$, then the solution is $X = A^{-1}B$.

$\left(A^{-1}A\right)X = A^{-1}B$ (pre-multiplying both sides by A^{-1})

$X = A^{-1}B$ (since $A^{-1}A = I$)

Question

If $A = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 9 \\ 12 \end{bmatrix}$ and X is a matrix such that $AX = B$, find:

(a) A^{-1}.

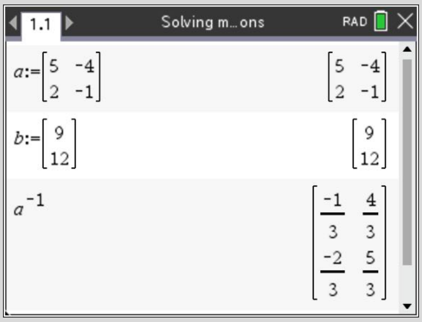
(b) X.

Solution

Parts **(a)** and **(b)** on a **Calculator** page:

Assign A and B as follows:

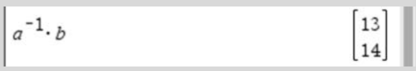
- Press ctrl [templates] to access the **Assign** [:=] command.
- Press [book] [5], select the **2-by-2 Matrix** template for A and enter as shown.
- Press [book] [5], select the **2-by-1 Matrix** template for B and enter as shown.



To calculate A^{-1} enter as shown.

Answer: (a) $A^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix}$.

Enter $A^{-1}B$ as shown.



Answer: (b) Since $AX = B$, then $X = A^{-1}B$.

$$X = \frac{1}{3}\begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 9 \\ 12 \end{bmatrix} = \begin{bmatrix} 13 \\ 14 \end{bmatrix}$$

***Note:** Alternatively, to find an inverse using a step-by-step approach, press [book] [6] **> linalgcas > inversestep**. The syntax for this command is **inversestep(Mat)**. Can you see how it works?*

Solving systems of linear equations involving matrices of up to dimension 2 x 2

Question

Solve the following system of linear equations

$$-2x+3y=-19$$
$$5x-2y=20$$

(a) using the **Reduced Row-Echelon Form** (**rref**) command.

(b) using the **Simultaneous** command.

(c) using the **Row Operations** menu.

Solution

In matrix form , the system of equations can be expressed as

$AX=B$ where $A=\begin{bmatrix}-2 & 3\\ 5 & -2\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$ and $B=\begin{bmatrix}-19\\ 20\end{bmatrix}$.

Parts **(a)**, **(b)** and **(c)** on a **Calculator** page:

Assign *A* and *B* as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **2-by-2 Matrix** template for *A* and enter as shown.
- Press [catalog] [5], select the **2-by-1 Matrix** template for *B* and enter as shown.

(a) Solve $AX=B$ using reduced row-echelon form:

- Press [menu] > **Matrix & Vector** > **Create** > **Augment**.
- Enter as shown.
- Press [menu] > **Matrix & Vector** > **Reduced Row-Echelon Form**.
- Enter as shown.

Notes: *The **Augment** command is used to combine A and B so that the **Reduced Row-Echelon Form** command can be used directly without the need to create a 2 × 3 matrix.*

*The **Reduced Row-Echelon Form** command instructs the TI-Nspire CX II CAS to solve the system of linear equations in the form of a* 2×3 *augmented matrix using the method of elimination.*

The new augmented matrix, $\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & -5\end{bmatrix}$, is a transformation of the original augmented matrix $\begin{bmatrix}-2 & 3 & -19\\ 5 & -2 & 20\end{bmatrix}$.

... continued

Solution (continued)

The matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \end{bmatrix}$ can be interpreted as the two equivalent transformed linear equations:

$1x + 0y = 2$

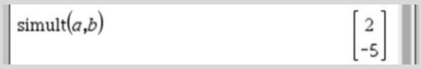
$0x + 1y = -5$

Answer: (a) Thus $x = 2$ and $y = -5$.

(b) Solve using the **Simultaneous** command:

- Press [menu] > **Matrix & Vector** > **Simultaneous** and enter as shown.

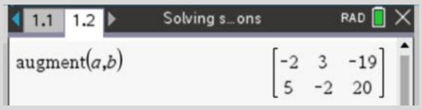
Answer: (b) Thus $x = 2$ and $y = -5$.



(c) Use the **Row Operations** menu of commands.

On a **Calculator** page with A and B assigned as before:

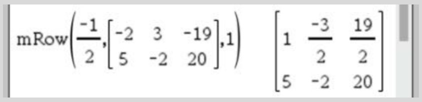
- Press [menu] > **Matrix & Vector** > **Create** > **Augment**.
- Enter as shown.



Step 1: $-\frac{1}{2}R_1 \rightarrow R_1$ (multiply row 1 by $-\frac{1}{2}$)

To perform **Step 1**:

- Press [menu] > **Matrix & Vector** > **Row Operations** > **Multiply Row**.
- Enter as shown.

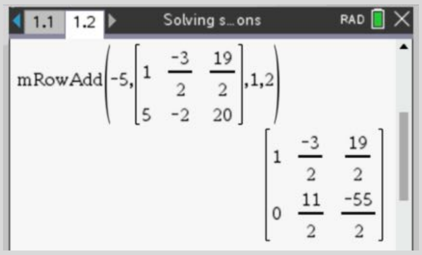


$$\begin{bmatrix} 1 & -\frac{3}{2} & \frac{19}{2} \\ 5 & -2 & 20 \end{bmatrix}$$

***Note:** The syntax for the **Multiply Row** command is **mRow(Value, Matrix, Index)**. **Value** is the multiplier and **Index** is the row number.*

... continued

Solution (continued)



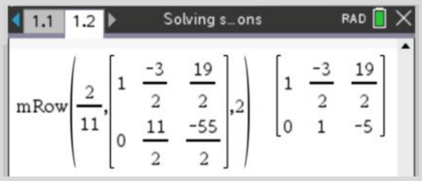
Step 2: $-5R_1+R_2 \to R_2$ (multiply row 1 by -5 and add it to row 2)

To perform **Step 2**:

- Press menu > **Matrix & Vector** > **Row Operations** > **Multiply Row & Add**.
- Enter as shown.

$$\begin{bmatrix} 1 & -\frac{3}{2} & \frac{19}{2} \\ 0 & \frac{11}{2} & -\frac{55}{2} \end{bmatrix}$$

Note: *The syntax for the* ***Multiply Row & Add*** *command is* ***mRowAdd(Value, Matrix, Index1, Index2)****.* ***Value*** *is the multiplier,* ***Index1*** *is the row being multiplied and* ***Index2*** *is the row being added to.*

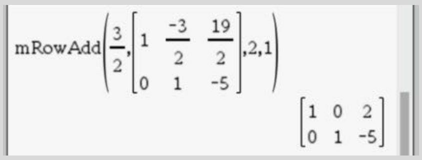


Step 3: $\frac{2}{11}R_2 \to R_2$ (multiply row 2 by $\frac{2}{11}$)

To perform **Step 3**:

- Press menu > **Matrix & Vector** > **Row Operations** > **Multiply Row**.
- Enter as shown.

$$\begin{bmatrix} 1 & -\frac{3}{2} & \frac{19}{2} \\ 0 & 1 & -5 \end{bmatrix}$$



Step 4: $\frac{3}{2}R_2+R_1 \to R_1$ (multiply row 2 by $\frac{3}{2}$ & add to row 1)

To perform **Step 4**:

- Press menu > **Matrix & Vector** > **Row Operations** > **Multiply Row & Add**.
- Enter as shown.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \end{bmatrix}$$

Answer: (c) Thus $x=2$ and $y=-5$.

Note: *Alternatively, to solve a system of linear equations, press* 📖 6 > ***linalgcas*** > ***simultstep****. The syntax for this step-by-step solving command is* ***simultstep(aMat,bVect)****. Can you see how it works?*

Verifying properties involving the multiplicative identity and inverse for 3 x 3 matrices

Question

Let $A=\begin{bmatrix}5 & 3 & 2\\ 2 & 2 & 4\\ 0 & 2 & 3\end{bmatrix}$ and $B=\frac{1}{20}\begin{bmatrix}2 & 5 & -8\\ 6 & -15 & 16\\ -4 & 10 & -4\end{bmatrix}$. Verify that

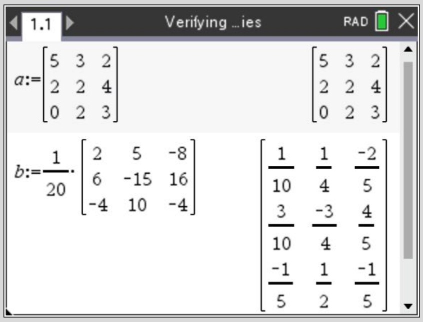
(a) $AI = A = IA$. **(b)** $AB = I = BA$ where $B = A^{-1}$.

Solution

Parts **(a)** and **(b)** on a **Calculator** page:

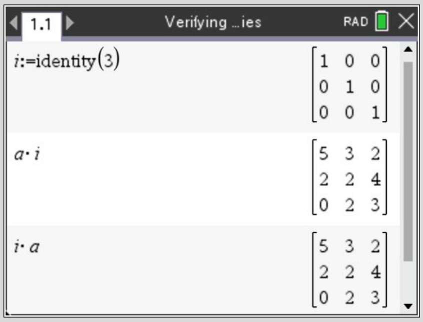
Assign A and B as follows:

- Press ctrl [template key] to access the **Assign** := command.
- Press [book key] 5, select the **m-by-n Matrix** template, fix the dimensions as 3-by-3 and enter as shown.



To create I:

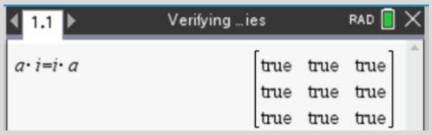
- Press ctrl [template key] to access the **Assign** := command.
- Press menu > **Matrix & Vector** > **Create** > **Identity**.
- Enter as shown.



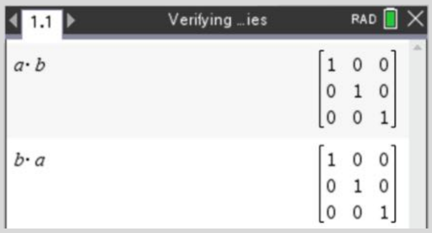
Answer: (a) $AI=\begin{bmatrix}5 & 3 & 2\\ 2 & 2 & 4\\ 0 & 2 & 3\end{bmatrix}$, $IA=\begin{bmatrix}5 & 3 & 2\\ 2 & 2 & 4\\ 0 & 2 & 3\end{bmatrix}$ and

$AI = A = IA$.

Note: *Entering* $AI = IA$ *gives the output* $\begin{bmatrix}\text{true} & \text{true} & \text{true}\\ \text{true} & \text{true} & \text{true}\\ \text{true} & \text{true} & \text{true}\end{bmatrix}$.

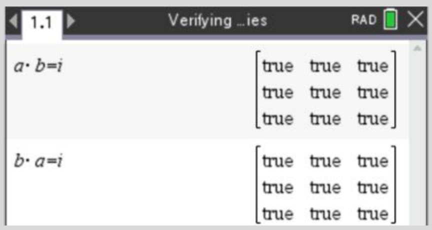


Enter AB and BA as shown.



Answer: (b) $AB=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ and $BA=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$.

Note: *Entering* $AB = I$ *and* $BA = I$ *both give the output* $\begin{bmatrix}\text{true} & \text{true} & \text{true}\\ \text{true} & \text{true} & \text{true}\\ \text{true} & \text{true} & \text{true}\end{bmatrix}$.



From part **(b)**, it can be concluded that $B = A^{-1}$.

Calculating the determinant and inverse of 3 x 3 matrices

Question

Consider $A = \begin{bmatrix} 1 & 2 & 4 \\ 5 & 3 & 0 \\ 3 & 2 & 1 \end{bmatrix}$.

(a) Find $\det(A)$.

Consider $M_1 = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$, $M_2 = \begin{bmatrix} 5 & 0 \\ 3 & 1 \end{bmatrix}$ and $M_3 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ where M_1, M_2 and M_3 are 2×2 matrices that form part of A.

(b) Find the value of $\det(M_1) - 2\det(M_2) + 4\det(M_3)$.

(c) What do you notice about the results obtained in parts **(a)** and **(b)**?

(d) Find A^{-1}.

Solution

Parts **(a)**, **(b)** and **(d)** on a **Calculator** page:

Assign A as follows:

- Press ctrl [:=] to access the **Assign** := command.
- Press [template] 5, select the **m-by-n Matrix** template, fix the dimensions as 3-by-3 and enter as shown.

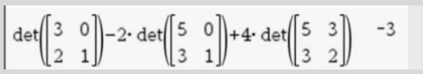
(a) To find $\det(A)$:

- Press menu > **Matrix & Vector** > **Determinant** and enter as shown.

Answer: (a) $\det(A) = -3$

(b) To find the value of $\det(M_1) - 2\det(M_2) + 4\det(M_3)$:

- Press menu > **Matrix & Vector** > **Determinant**.
- Press [template] 5 to select the **2-by-2 Matrix** template and enter as shown.



Answer: (b) $\det(M_1) - 2\det(M_2) + 4\det(M_3) = -3$

Answer: (c) $\det(A) = \det(M_1) - 2\det(M_2) + 4\det(M_3)$

Note: *In general,*

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

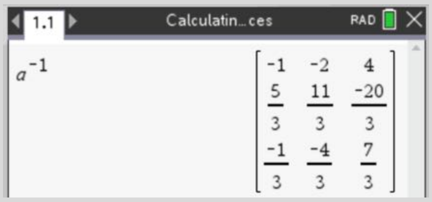
$$= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

... continued

Solution (continued)

(d) Calculate A^{-1} as shown.

Answer: (d) $A^{-1}=\begin{bmatrix}-1 & -2 & 4\\ \frac{5}{3} & \frac{11}{3} & -\frac{20}{3}\\ -\frac{1}{3} & -\frac{4}{3} & \frac{7}{3}\end{bmatrix}$



***Note:** Alternatively, to find an inverse using a step-by-step approach, press* 🕮 6 ***> linalgcas > inversestep**. The syntax for this command is **inversestep(Mat)**.*

Determining whether a 3 x 3 matrix is singular or non-singular

Question

Consider $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 3 & 3 & 8\end{bmatrix}$.

Find $\det(A)$ and hence determine whether A is singular or non-singular.

Solution

To find $\det(A)$:

- Press menu > **Matrix & Vector** > **Determinant**.
- Press 🕮 5, select the **m-by-n Matrix** template, fix the dimensions as 3-by-3 and enter as shown.

Answer: $\det(A)=0$ and so A is singular.

***Note:** Since 2 is a common factor of the elements of the second row, the determinant can be expressed as* $2\begin{vmatrix}1 & 2 & 3\\ 1 & 2 & 3\\ 3 & 3 & 8\end{vmatrix}$.

The resulting determinant has two identical rows. If the corresponding elements of two rows (or columns) of a square matrix A are equal, the determinant is zero. When the two equal rows of the matrix are interchanged, the matrix remains the same and hence the value of their determinants are equal.

Using matrices to encode and decode messages

Matrices can be used to encode and decode messages. In this coding method, assign each letter of the alphabet with its position number in the alphabet. So $A=1$, $B=2,\ldots,$ $Z=26$.

For example, to send the message *GO CATS*, write the letters in a 2×4 matrix *M*.

$$M=\begin{bmatrix} & G & O & \\ C & A & T & S\end{bmatrix}$$

Replace each letter of the alphabet with its position number in the alphabet and use a zero to represent a space.

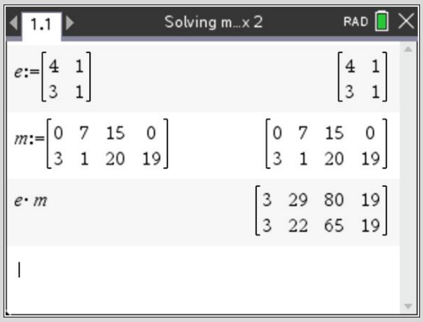
$$M=\begin{bmatrix}0 & 7 & 15 & 0\\ 3 & 1 & 20 & 19\end{bmatrix}$$

This code is fairly easy to crack. However, by multiplying *M* by a suitably sized encoding matrix, *E*, this message can be made more difficult to decode.

Let $E=\begin{bmatrix}4 & 1\\ 3 & 1\end{bmatrix}$ and form the product *EM*.

On a **Calculator** page, assign *E* and *M* as follows:

- Press ctrl to access the **Assign** := command.
- Press 5, select the **2-by-2 Matrix** template for *E* and enter as shown.
- Press 5, select the **m-by-n Matrix** template, fix the dimensions as 2-by-4 and enter as shown.



Calculate *EM* as shown.

$$EM=\begin{bmatrix}4 & 1\\ 3 & 1\end{bmatrix}\begin{bmatrix}0 & 7 & 15 & 0\\ 3 & 1 & 20 & 19\end{bmatrix}$$
$$=\begin{bmatrix}3 & 29 & 80 & 19\\ 3 & 22 & 65 & 19\end{bmatrix}$$

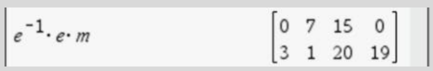
The encoded message sent to the recipient is the matrix product *EM*.

To decode this message, the recipient must pre-multiply the matrix product *EM* by E^{-1}.

$E^{-1}EM=M$ as $E^{-1}E=I$ and $IM=M$

Calculate $E^{-1}EM$ as shown.

$$E^{-1}EM=\begin{bmatrix}0 & 7 & 15 & 0\\ 3 & 1 & 20 & 19\end{bmatrix}$$



The recipient now replaces each letter's position number in the alphabet with the corresponding letter and inserts a space for the zero.

So $M=\begin{bmatrix} & G & O & \\ C & A & T & S\end{bmatrix}$ and the message received is *GO CATS*.

The trick to decoding messages of this type is to know E^{-1}, the inverse matrix of the encoding matrix E.

A good encoding matrix *E* is one that has $\det(E)=\pm1$ as this avoids the use of fractions when decoding a message.

You are encouraged to encode and decode various messages using the above approach. It is a good idea to vary *E*.

Solving systems of linear equations involving matrices beyond dimension 2 x 2

Question

Solve the following system of linear equations

$$\begin{aligned} 2u+4v+2z&=6\\ 3v+3w+z&=4\\ 2u+7v+9w+7z&=8\\ 6w+5z&=-4 \end{aligned}$$

using

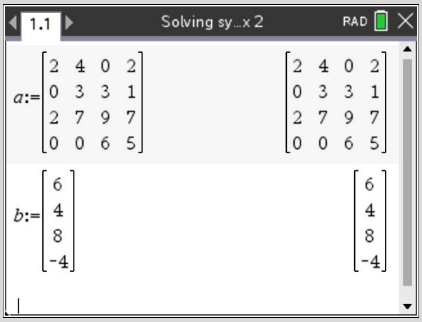
(a) the **Reduced Row-Echelon Form** (**rref**) command.

(b) the **Simultaneous** command.

Solution

In matrix form , the system of equations can be expressed as

$$AX=B \text{ where } A=\begin{bmatrix}2&4&0&2\\0&3&3&1\\2&7&9&7\\0&0&6&5\end{bmatrix},\ X=\begin{bmatrix}u\\v\\w\\z\end{bmatrix} \text{ and } B=\begin{bmatrix}6\\4\\8\\-4\end{bmatrix}.$$

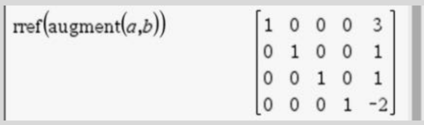


Parts **(a)** and **(b)** on a **Calculator** page:

Assign A and B as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [book] [5], select the **m-by-n Matrix** template, fix the dimensions as 4-by-4 and enter as shown.
- Press [book] [5], select the **m-by-n Matrix** template, fix the dimensions as 4-by-1 and enter as shown.

(a) Solve $AX=B$ using reduced row-echelon form:



- Press [menu] > **Matrix & Vector** > **Reduced Row-Echelon Form**.
- Press [menu] > **Matrix & Vector** > **Create** > **Augment**.
- Enter as shown.

Note: *The* ***Reduced Row-Echelon Form*** *command instructs the TI-Nspire CX II CAS to solve the system of linear equations in the form of a* 4×5 *augmented matrix using the method of elimination.*

... continued

Solution (continued)

The new augmented matrix, $\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$, is a transformation of the original augmented matrix $\begin{bmatrix} 2 & 4 & 0 & 2 & 6 \\ 0 & 3 & 3 & 1 & 4 \\ 2 & 7 & 9 & 7 & 8 \\ 0 & 0 & 6 & 5 & -4 \end{bmatrix}$.

The matrix $\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$ can be interpreted as the four equivalent transformed linear equations:

$1u + 0v + 0w + 0z = 3$

$0u + 1v + 0w + 0z = 1$

$0u + 0v + 1w + 0z = 1$

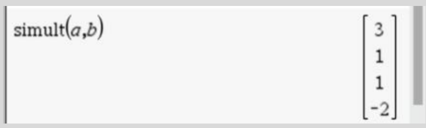
$0u + 0v + 0w + 1z = -2$

Answer: (a) Hence $u = 3$, $v = 1$, $w = 1$ and $z = -2$.

Note: *The **Augment** command is used to combine A and B so that the **Reduced Row-Echelon Form** command can be used directly without the need to create a 4 x 5 matrix.*

(b) Solve this system using the **Simultaneous** command:

- Press [menu] > **Matrix & Vector** > **Simultaneous** and enter as shown.



Answer: (b) Hence $u = 3$, $v = 1$, $w = 1$ and $z = -2$.

Note: *Pages 76–77 shows the use of the **Row Operations** menu which can be used to perform Gaussian techniques of elimination on augmented matrices. Alternatively, to solve a system of linear equations, press [book] [6] > **linalgcas** > **simultstep**. The syntax for this step-by-step solving command is **simultstep(aMat,bVect)**. Can you see how it works?*

Examining different cases for solutions of systems of linear equations

There are three cases for solutions of systems of equations:

- A unique solution.
- No solution.
- Infinitely many solutions.

Question

Consider the system of linear equations

$$6x - 3y - z = 3$$
$$3x - 9y + 2z = -6$$
$$2x + y - z = -1$$

Use the **Reduced Row-Echelon Form** (**rref**) command to show that this system of linear equations has no solution.

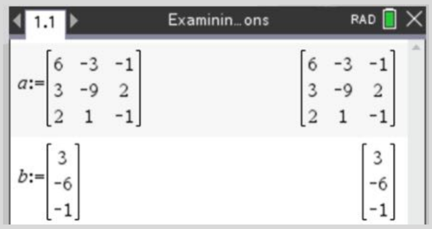
Solution

In matrix form , the system of equations can be expressed as

$$AX = B \text{ where } A = \begin{bmatrix} 6 & -3 & -1 \\ 3 & -9 & 2 \\ 2 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ -6 \\ -1 \end{bmatrix}.$$

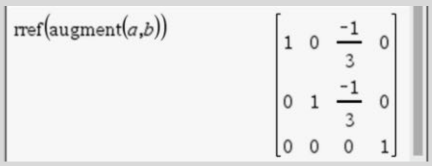
On a **Calculator** page, assign A and B as follows:

- Press [ctrl] [template key] to access the **Assign** [:=] command.
- Press [book key] [5], select the **m-by-n Matrix** template, fix the dimensions as 3-by-3 and enter as shown.
- Press [book key] [5], select the **m-by-n Matrix** template, fix the dimensions as 3-by-1 and enter as shown.



Attempt to solve $AX = B$ using reduced row-echelon form as follows:

- Press [menu] > **Matrix & Vector** > **Reduced Row-Echelon Form**.
- Press [menu] > **Matrix & Vector** > **Create** > **Augment**.
- Enter as shown.



Answer: The third row of the augmented matrix,

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ can be interpreted as the equation

$0x + 0y + 0z = 1$.

This system of equations has no solution.

VCE Specialist Mathematics Unit 2

2.1 Data analysis, probability and statistics

2.1.1 Simulation, sampling and sample distributions

Exploring the distribution of the sum of 3 Bernoulli random variables

Question

Three unbiased coins are thrown and the number of 'heads' appearing uppermost is noted.

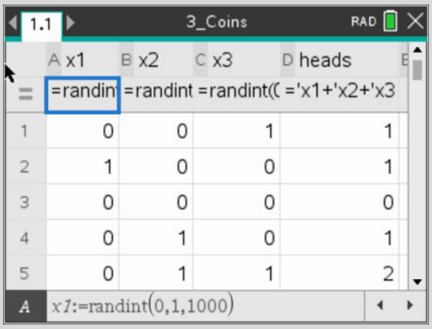
(a) Use simulation to explore the distribution of the random variable $S = X_1 + X_2 + X_3$, where X_i represents the number of 'heads' for coin i. Therefore $X_i = 0$ for 'tails' and $X_i = 1$ for 'heads'

(b) Estimate the mean and standard deviation of S from the simulation results and compare the results to the theoretical value.

Solution

(a) To simulate 1000 throws, on a **Lists & Spreadsheet** page:

- Enter the column headings as shown.
- In the formula cells for columns A, B and C, enter **= randInt(0,1,1000)** by pressing [📖] [1] [R] for **randInt**.
- In the column D formula cell enter **='*x*1+'*x*2+'*x*3** by pressing [var] to select ***x*1**, ***x*2** and ***x*3**.
- To recalculate the 1000 throws, press [ctrl] [R].

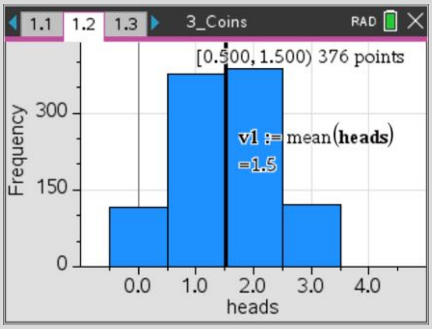


To obtain a graphical display, add a **Data & Statistics** page:

- Press [tab] and select **heads** on the horizontal axis.
- Press [menu] > **Plot Type** > **Histogram**
- Hover over the histogram and then press [ctrl] [menu] > **Bin Settings** and set **Width = 1** and **Alignment = –0.5**.

To display the mean number of heads for the 1000 trials:

- Press [menu] > **Analyse** > **Plot Value**. In the textbox that follows enter ***v*1:=mean(*heads*).**



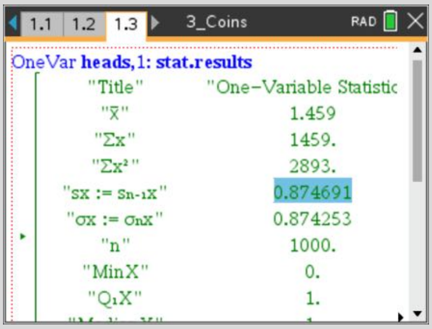
Note: *Hover over the histogram to see the bin frequencies.*

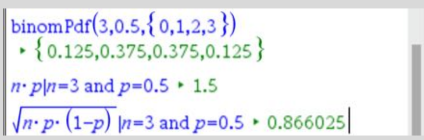
Answer: The relative frequencies are consistent with the theoretical values obtained using a tree diagram.

$S = X_1 + X_2 + X_3$	0	1	2	3
Success fraction	≈ 0.125	≈ 0.375	≈ 0.375	≈ 0.125

(b) To calculate E(*heads*) and sd(*heads*), on a **Notes** page:

- Press [ctrl] [M] to insert a **Maths Box**, then press [menu] > **Calculations** > **Statistics** > **Stat Calculations** > **One-variable Statistics**. In the dialog box, select **heads**.





Answer: The mean ≈ 0.15 and standard deviation ≈ 0.87.

The problem illustrates that $S \sim \text{Binomial}(n, p)$ is the sum of n jointly independent Bernoulli random variables with parameter p. This can be proved formally by induction.

Theoretical values: $E(S) = 1.5$, $sd(S) = \sqrt{0.75} \approx 0.866$

Implementing pseudocode for the sum of random variables in Python

Question

A version of the pseudocode that could be applied to simulating the distribution of the random variable $S = X_1 + X_2 + X_3$, where X_i represents the number of 'heads' for coin i, is shown below.

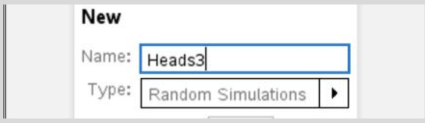
Define trials(n)
$h0 \leftarrow 0$, $h1 \leftarrow 0$, $h2 \leftarrow 0$, $h3 \leftarrow 0$
for k from 1 to n
s = randint(0,1) + randint(0,1) + randint(0,1)
	if $s = 0$ **then** $h0 \leftarrow h0 + 1$
	else if $s = 1$ **then** $h1 \leftarrow h1 + 1$
	else if $s = 2$ **then** $h2 \leftarrow h2 + 1$
	else if $s = 3$ **then** $h3 \leftarrow h3 + 1$
	end if
	end for
	return $h0/n$, $h1/n$, $h2/n$, $h3/n$

Implement the pseudocode in the **Python** application. Compare the results with theoretical values.

Solution

(a) To start coding, in a new **Document** (or a new **Problem**):

- Select **Add Python** > **New**.
- In the dialog box that follows, enter as shown.



To set up a function for n trials and initialise number of heads:

- Press [menu] > **Built-ins** > **Functions** > **def function():** and enter **def trials(*n*):** .
- Enter ***h0*** = ***h1*** = ***h2*** = ***h3*** = **0**, with indentation as shown.

To execute n repetitions of $S = X_1 + X_2 + X_3$:

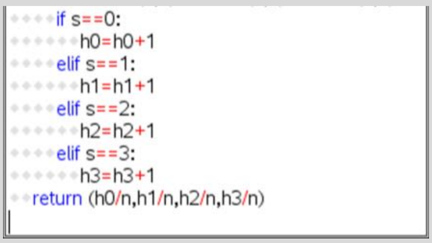
- Press [menu] > **Built-ins** > **Control**. Select **for index in range(size)** and enter **for *k* in range(*n*):** .
- Enter ***s*** = **randint(0,1)** + **randint(0,1)** + **randint(0,1)** by pressing [menu] > **Random** to select **randint(max,min)**.

To count and aggregate the trials with 0, 1, 2 or 3 heads:

- By pressing [menu] > **Built-ins** > **Control** to select **if…** and **elif:**, enter **if *s* = = 0:** then enter ***h0*** = ***h0*** + **1**, as shown.

Note: *= means* ***assign****, whereas = = means* ***is equal to****.*

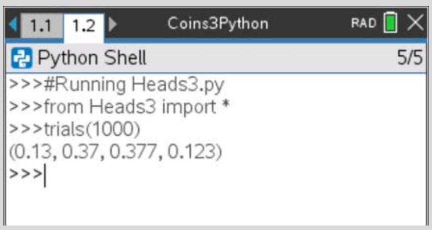
- Enter **elif *s* = = 1:** then enter ***h1*** = ***h1*** + **1** etc., as shown.
- Enter **return (*h0*/*n*, *h1*/*n*, *h2*/*n*, *h3*/*n*)**, by pressing [menu] > **Built-ins** > **Functions** to select **return**.



Note: *Press [tab] or [del] to increase/decrease indentation.*

To check syntax, save and run the program:

- Press [ctrl] [B] followed by [ctrl] [R] (or [menu] > **Run** …).
- In the **Python Shell** page that follows, press [var] and enter **trials(1000)** to run 1000 trials.



Answer: 0, 1, 2, 3 heads, approx. 0.125, 0.375, 0.375, 0.125.

Using the Programme Editor to implement the sum of random variables

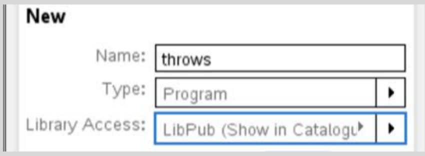
Question

Implement the pseudocode from the previous problem to simulate the distribution of the number of 'heads' when 3 unbiased coins are thrown. Compare the results with theoretical values.

Solution

To start coding, in a new **Problem** or a new **Document**:

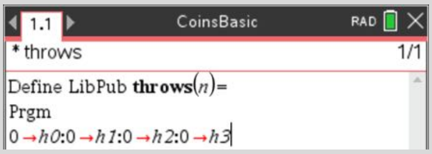
- Select **Add Programme Editor** > **New**.
- In the dialog box that follows, enter as shown.



The **Program Editor** will follow, ready to accept the code.

To input number of trials, n, and initialise number of heads:

- In line 0, enter ***n*** for **Define LibPub throws(*n*)=**
- In line 1, enter **0 → *h*0: 0 → *h*1: 0 → *h*2: 0 → *h*3** by pressing ctrl var (sto→) for **store** →, and ?!▸ for colon **:**

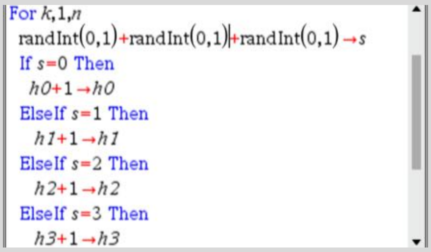


To execute n repetitions of $S = X_1 + X_2 + X_3$ using a **For** loop:

- Press menu > **Control** and select **For … End For**. In line 2 enter **For *k*,1,*n***

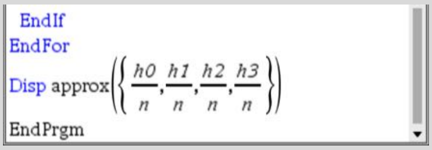
To count and aggregate the trials with 0, 1, 2 or 3 heads:

- Enter **randInt(0,1) + randInt(0,1) + randInt(0,1) → *s*** in line 3, pressing 📖 1 R to select **randint**.
- Press menu > **Control** and select **If…Then… End If**. Enter in line 4 **If *s* = 0 Then** and in line 5, ***h*0 + 1→ *h*0**
- Press menu > **Control** and select **ElseIf…Then.** Enter in line 6 **ElseIf *s* = 1 Then** and in line 7, ***h*1 + 1→ *h*1**
- Similarly, enter **ElseIf *s* = 2 Then** (line 9) ***h*2 + 1→ *h*2** and **ElseIf *s* = 3 Then** (line 11) ***h*3 + 1→ *h*3**, as shown.



To display the proportion of throws with 0, 1, 2 or 3 heads:

- Press menu > **I/O** > **Disp**. In line 14, enter **Disp approx({*h*0/*n*, *h*1/*n*, *h*2/*n*, *h*3/*n*})** by pressing 📖 1 A to select **approx**.



To check syntax, save and run the program:

- Press ctrl B followed by ctrl R (or menu > **Run** …).

To carry out 1000 trials of the simulation:

- In the **Calculator** page that follows, enter the values of n ***throws*(1000)**. Press var and select throws to repeat the simulation with n new trials.

Answer: 0, 1, 2, 3 heads, approx. 0.125, 0.375, 0.375, 0.125, consistent with the theoretical values.

Note: *Refer to section 3.6.1 for additional examples of combinations of random variables.*

Simulating a continuous uniform distribution

Question

Graphically display the results of 2000 randomly generated real numbers between 0 and 10. Calculate the mean and interpret the significance of key features of this distribution.

Solution

To generate 2000 random numbers, [0, 10], on a **Notes** page:

- Press [ctrl] [M] to insert a **Maths Box**. Press [📖] [1] [R], select **rand**([#Trials]) and enter **10 × rand(2000) → uniform** by pressing [ctrl] [var] ([sto→]) for **store →**.
- Press [enter] to select new sets of 2000 random numbers.

To display the results, add a **Data & Statistics** page, then:

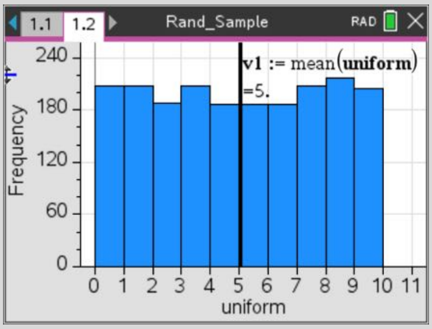
- Press [tab] and select **uniform** on the horizontal axis.
- Press [menu] > **Plot Type** > **Histogram**.
- Hover over the histogram, press [ctrl] [menu] > **Bin Settings** > **Equal Bin Width**. Set **Width = 1**.
- Press [menu] > **Window/Zoom** > **Zoom Data**.

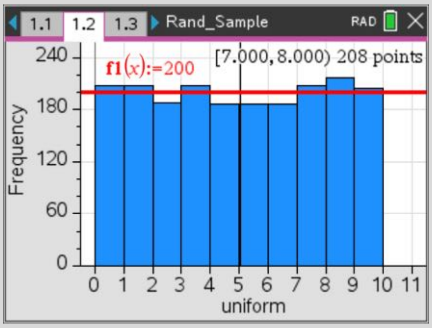
To display the mean and the function $f\,1(x) = 1/10 \times 2000$:

- Press [menu] > **Analyse** > **Plot Value**. In the textbox that follows, use the [var] key to enter ***v*1:=mean(*uniform*)**.
- Press [menu] > **Analyse** > **Plot Function**. In the text box that follows, enter: ***f* 1(*x*) := 1/10 × 2000**.

Answer: Shape: rectangular (uniform), $X \sim U(0,10)$.

E(X) = 5. Probability density function: $f(x) = \frac{1}{10}, 0 \le x \le 10$.





Sampling from the uniform distribution

Question

Random samples of size $n = 4$ are drawn from the continuous uniform distribution $U(0, 10)$ which was featured in the previous problem. Use simulation to repeat the sampling 500 times and explore a model of the sampling distribution of the sample mean. Interpret key features of the distribution.

Solution

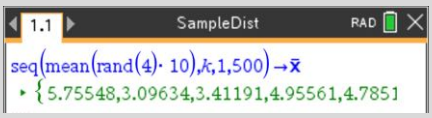
To simulate taking samples of size $n = 4$, on a **Notes** page:

- Press [ctrl] [M] to insert a **Maths Box**. Press [📖] [1] [R], select **rand** and enter **rand(4) × 10**.
- Edit to $\mathbf{mean(rand(4) \times 10)}$ for the sample mean.

To repeat the sampling 500 times and store the list as $\bar{x}$.

- Edit to $\mathbf{seq(mean(rand(4) \times 10), \mathit{k}, 1, 500) \to \bar{x}}$, pressing [ctrl] [var] ([sto→]) for **store →**, and [ctrl] [📖] ([∞β°]) for $\bar{x}$.

Note*: In the symbol list, $\bar{x}$ is located on the sixth row.*



... continued

Solution (continued)

To display the values of $\bar{x}$, add a **Data & Statistics** page:

- Press [tab] and select $\bar{x}$ on the horizontal axis.

To display the mean of $\bar{x}$:

- Press [menu] > **Analyse** > **Plot Value**. In the textbox that follows, use the [var] key to enter ***v*1:=mean(** $\bar{x}$ **)**.

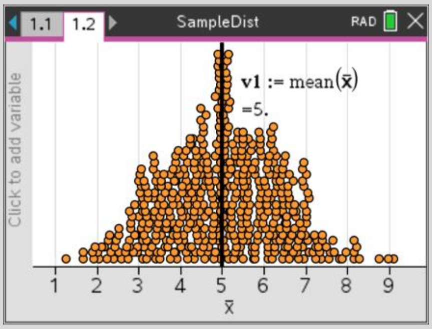
To display the data as a histogram with bin width of 0.5:

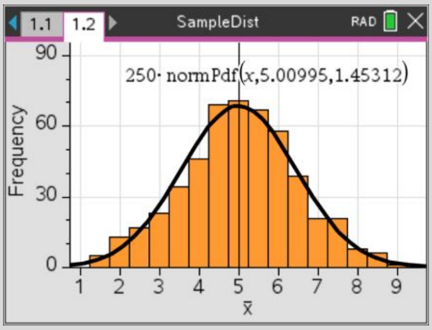
- Press [menu] > **Plot Type** > **Histogram**.

To visually compare the shape of a bell-shaped distribution:

- Press [menu] > **Analyse** and select **Show Normal PDF**. To hide the bell-shaped curve, select **Hide Normal PDF**.

Answer: When samples are drawn from the rectangular distribution, there is variability in the sample mean. Even with a sample size as small as 4, when the sampling is repeated many times, the distribution of sample means approximates a symmetrical bell-shaped distribution, and the expected value of $\bar{X}$ approaches the population mean, i.e., $\mathrm{E}\left(\bar{X}\right)=\mu=5$.





Exploring the effect of sample size on the distribution of sample means

Question

Modify the previous problem so that the sample size can be controlled using a slider.

Calculate the mean, standard deviation and spread of the distribution of sample means for a variety of sample sizes. Hence interpret the effect of sample size on the distribution.

Solution

To make a copy of the previous problem as a new problem:

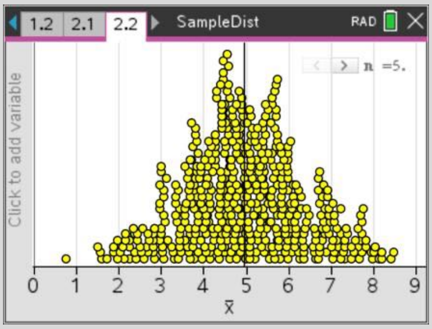
- Press [ctrl] ▲. In the thumbnail view that follows, click **Problem 1** and press [ctrl] [C] then [ctrl] [V].
- Open the second page of the copied **Problem 2**.
- Press [menu] > **Plot Type** and select **Dot Plot**.

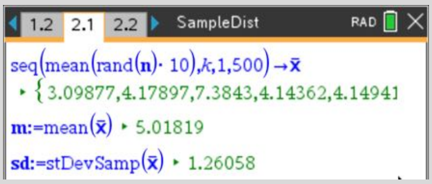
To add a slider that will control the sample size:

- Press [menu] > **Actions** > **Insert Slider**. Enter the settings: Var = ***n*** Value = **5** Min = **1** Max = **100** Step size = **5**.

To link sample size to n and calculate the standard deviation:

- On page 2.1, edit the **Maths Box** to replace **rand(4)** with **rand(*n*)**. That is, $\mathbf{seq(mean(rand}\left(\boldsymbol{n}\right)\mathbf{\times 10),}\boldsymbol{k}\mathbf{,1,500)}\rightarrow\bar{x}$.
- Add two **Maths Boxes** by pressing [ctrl] [M].
- In the **Maths Boxes**, enter ***m*:= mean(** $\bar{x}$ **)** and ***sd*:=stDevSamp(** $\bar{x}$ **)** by pressing [📖] [1] to select **mean** and **stDevSamp** and pressing [var] to select $\bar{x}$.



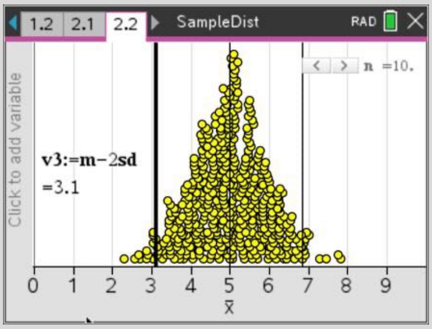


... continued

Solution (continued)

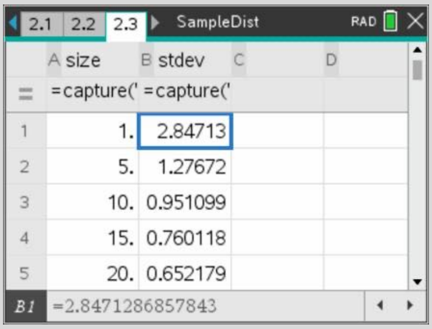
To quantify changes in the spread of the distribution, on page **2.2**, display two standard deviations either side of the mean:

- Enter ***v*2:= *m* + 2*sd*** and ***v*3:= *m* – 2*sd*** by pressing [menu] > **Analyse > Plot Value** for the variables ***v*2** and ***v*3**.
- Use the slider to vary the sample size, *n*.



To capture the standard deviation of $\bar{x}$ as *n* changes, edit the slider value to ***n* = 1** then add a **Lists & Spreadsheet** page:

- Enter the headings as shown. To enter the data capture:
- From the column A formula cell, press [menu] > **Data > Data Capture > Automatic**. Press [var] and select ***n***.
- Repeat for column B formula cell, but select ***sd*** from [var].
- On page 2.2, edit slider value to *n* = **5**. Use the slider to increase *n* up to ***n* = 100**. This will populate page 2.3.

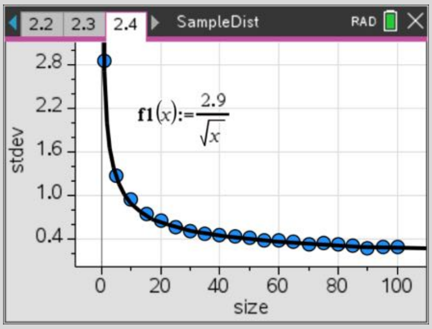


Note: ***To reset a data capture list, navigate to the formula cell press*** [ctrl] [menu] ***> Clear Data.***

To plot the standard deviation of $\bar{x}$ against sample size, *n*, add a **Data & Statistics** page, then:

- Press [tab] and select ***size*** on the horizontal axis.
- Press [tab] and select ***stdev*** on the vertical axis.

Note: *The standard deviation of $\bar{x}$ for n = 1 gives an estimate of σ, the population standard deviation of U(0, 10). Why?*

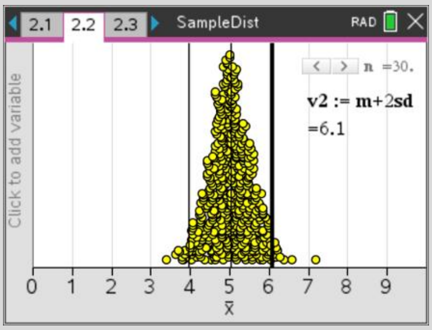


To fit a curve to the plotted values of $sd(\bar{x})$ against ***n***:

- Press [menu] > **Analyse > Plot Function**. In the textbox that follows, enter a function such as $f1(x) := 2.85/\sqrt{x}$, where the numerator is the value of cell B1**,** an estimate of the population standard deviation *σ,* and *x* is the sample size.

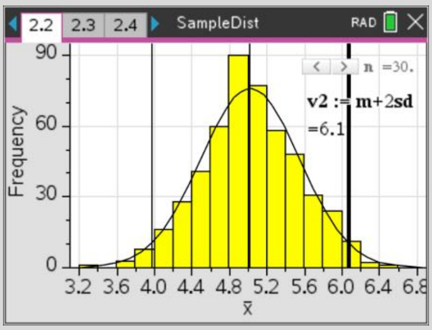
To visualise the distribution as a histogram, for a particular value of *n*, on page 2.2:

- Press [menu] > **Plot Type** and select **Histogram**.
- To adjust bin widths, hover over the histogram and press [ctrl] [menu] > **Bin Settings > Equal Bin Width**. Enter the desired bin width.



To compare the histogram with the bell-shaped curve:

- Press [menu] > **Analyse** and select **Show Normal PDF**. (To hide the curve, select **Hide Normal PDF** instead.)



Answer: Observations:
(1) As *n* increases, the distribution of sample means becomes more tightly clustered around the population mean.
(2) The spread of the sampling distribution decreases as sample size increases. (3) For sample size *n*, $\text{sd}(\bar{X}) = \sigma/\sqrt{n}$.

The graph on page 2.4 indicates that to halve the spread, the sample size needs to increase 4-fold.

2.2 Space and measurement

2.2.1 Trigonometry

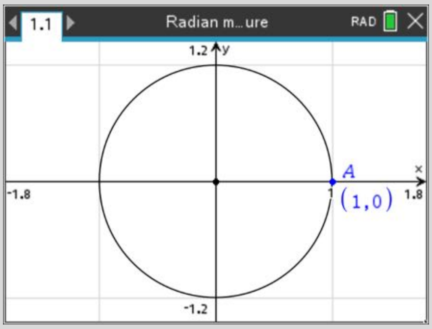
Understanding radian measure, arc length, and the unit circle

Question

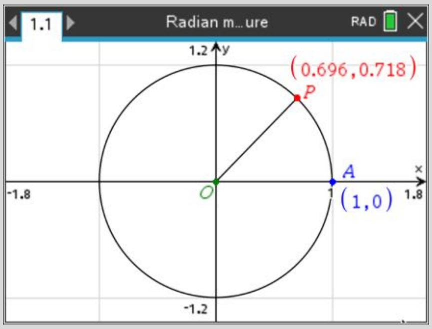
Use Geometry tools to construct an interactive model to display the arc length on a unit circle. Interpret the relationship between arc length and angle measurements in radians and degrees.

Solution

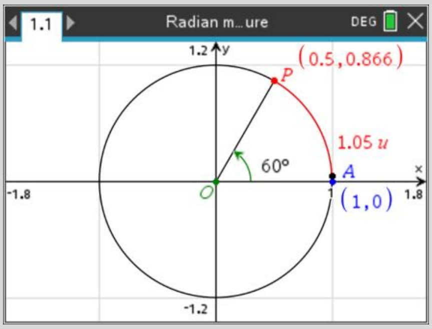
To construct a model for measuring the angle subtended at the centre by an arc in a unit circle, on a **Graphs** page:



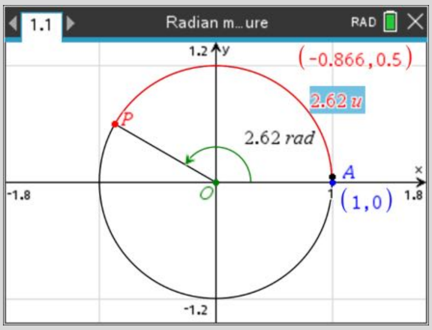
- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows enter the following values:
  XMin: – 1.8 XMax: 1.8 XScale: 1
  YMin: – 1.2 YMax: 1.2 YScale: 1
- Press [P] > **Point by Coordinates** and enter (**1**, **0**). Hover over this point, press [ctrl] [menu] > **Label** and enter label ***A***.
- Similarly, label the origin (point at (0, 0)) as ***O***.
- Press [menu] > **Geometry** > **Shapes** > **Circle**.
- Click (i.e. press) the origin, ***O***, then point ***A***.
- Press [menu] > **Geometry** > **Points & Lines** > **Segment**.
- Click the centre of the circle, then a point on the circumference (in the first quadrant) and press [esc]. Hover over this point, press [ctrl] [menu] > **Label** and enter label ***P***.
- Hover over point ***P*** and press [ctrl] [menu] > **Coordinates …**



To measure the arc length ***AP*** and the angle ***AOP***:



- Press [menu] > **Geometry** > **Points & Lines** > **Circle arc**. Click point ***A***, then a point on the circumference between ***A*** and ***P***, then click point ***P***. Press [esc] to exit the tool.
- To measure arc length ***AP***, hover over the arc and press [ctrl] [menu] > **Measurement** > **Length**.
- To toggle angle settings, click on the **DEG** or **RAD** setting at top right of the screen. Select **DEG**.
- To measure angle ***AOP***, press [menu] > **Measurement** > **Directed Angle** then click points ***A***, ***O*** and ***P*** in that order.
- Press [esc] to exit the tool.
- Move point ***P*** around the circle by hovering over the point and pressing [ctrl] to grab and [esc] to release the point.
- After moving ***P*** to a new position, toggle the **DEG** or **RAD** setting and observe the arc length.



Answer: The magnitude of angle AOP in radians is numerically equal to the arc length AP. If arc length $AP = \theta$ units, then (by definition) angle $AOP = \theta^c$ (θ radians).

Exploring circle mensuration interactively using the Geometry application

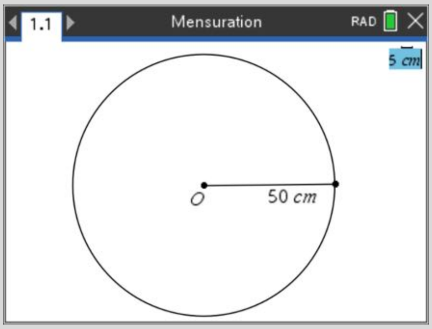
Question

For a circle of radius, $r = 50$ cm, use tools from the Geometry application to interactively determine the **(a)** arc length, **(b)** area of a minor sector, and **(c)** area of a minor segment. Consider cases where the angle subtended at the centre by the arc has magnitude **(i)** 70° **(ii)** 110°.
Give the answer in cm for arc length or cm^2 for areas, correct to the nearest integer.

Solution

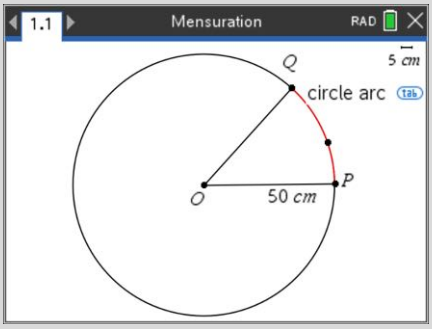
To draw a circle scaled to $r = 50$ cm, on a **Geometry** page:

- Press menu > **Points & Lines** > **Segment**.
- Click two points on the workspace, then press esc to exit.
- Hover over the **segment** and press ctrl menu > **Measurement** > **Length**. Edit the length to 10 cm.
- Press ctrl menu > **Pin** to lock-in the **segment** length.
- Press menu > **Shapes** > **Circle**, click each endpoint of the **segment**. Press esc to exit the tool.
- Edit the scale (top right-hand corner) to **5 cm**.



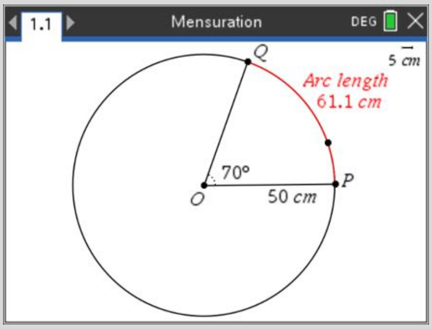
To draw a minor arc that subtends an angle *POQ*:

- Press menu > **Points & Lines** > **Segment**.
- Click the centre of the circle followed by a point on the circumference. Press esc to exit the tool.
- Label the points ***P***, ***O***, ***Q***, as shown, by hovering over a point, pressing ctrl menu > **Label** and entering the label.
- Press menu > **Points & Lines** > **Circle arc**.
- Click the point ***P***, followed by a point on the circle between ***P*** and ***Q***. To complete the arc, click point ***Q***.



To measure angle *POQ* and the associated minor arc:

- Select **DEG** by clicking top right to toggle **RAD**/**DEG**.
- Press menu > **Measurement** > **Angle**. Click points ***P***, ***O***, ***Q***.
- Press menu > **Measurement** > **Length**. Hover over the minor arc and press tab until the label **circle arc** appears.
- Click the arc, then press esc to exit the tool.
- Grab point ***Q*** by hovering over it and pressing ctrl [grab].
- Drag point ***Q*** so that angle ***POQ*** is **(i) 70.0°**, **(ii) 110.0°**.



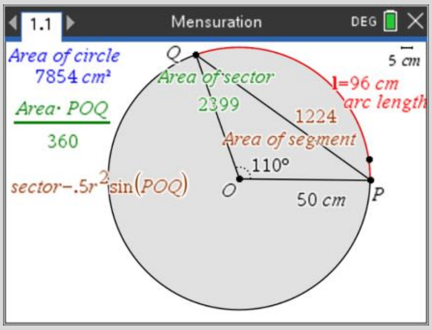
Answer: (a) (i) 70°, $l \approx 61$ cm **(ii)** 110°, $l \approx 96$ cm

To measure the area of the whole circle:

- Press menu > **Settings.** Set Display Digits: **Float 4.** Hover over circle and press ctrl menu > **Measurement** > **Area**.

(b), (c) To determine the area of the sector and segment:

- Press ctrl menu > **Text** then enter ***Area***×***POQ***/**360**.
- Draw the segment *PQ*, press ctrl menu > **Text** then enter ***sector*** **– 0.5*r*²sin(*POQ*)**.
- For each textbox expression, press menu > **Actions** > **Calculate**, click the expression and follow the prompts.
- Drag point ***Q*** to recalculate the area of the minor sector.



Note: *To increase/decrease display digits, hover over a value and press* + *or* –.

Answer: (b) (i) 70°, $A \approx 1526\ \text{cm}^2$ **(ii)** 110°, $A \approx 2400\ \text{cm}^2$
(c) (segment) **(i)** 70°, $A \approx 352\ \text{cm}^2$ **(ii)** 110°, $A \approx 1224\ \text{cm}^2$

Setting up a Notes page to solve triangles using the sine rule

Question

Set up a Notes page to perform calculations related to the sine rule. Test the page using:

(a) In ΔABC, $a = 5$, $A = 45°$ and $B = 28°$. Determine b, correct to two decimal places.

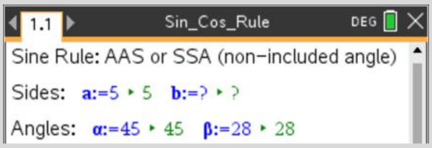
(b) In ΔABC, $a = 12$ and $b = 8$. **(i)** If $A = 52°$, find possible values of B in degrees and minutes. **(ii)** If $B = 36°20'$, find possible values of A correct to the nearest minute.

(c) Use the sine rule to solve $A = 42°$, $a = 3$, $b = 8$, $B = ?$, and interpret the result.

Solution

(a) To set up the sine rule for $a = 5.0$, $A = 45°$ and $B = 28°$, on a **Notes** page click to select **DEG** at top right corner, then:

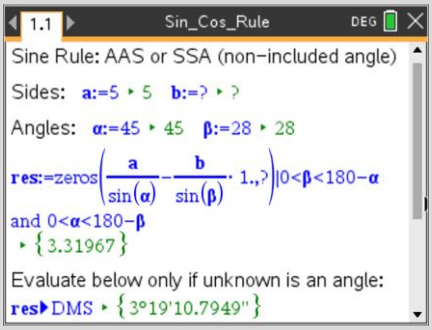
- Enter the text as shown in black font.
- Next to the heading **Sides:**, insert two **Maths Boxes** by pressing [ctrl] [M]. In the first **Maths Box**, enter ***a*:=5** and in the second, enter ***b*:= ?**, pressing [?!▸] to select **?** symbol.
- Similarly, in **Maths Boxes** next to the heading **Angles:**, enter ***α*:=45** and ***β*:=28**, pressing [ctrl] [📖] ($\infty\beta°$) for α and β.



Note: *To avoid ambiguity, use **a** and **b** for the side lengths, and **α** and **β** for the angles. TI-Nspire treats uppercase and lowercase letters as the same symbol.*

To determine the unknown value, ?, add a **Maths Box**, then:

- Key in $\boldsymbol{res} := \textbf{zeros}\left(\frac{a}{\sin(\alpha)} - \frac{b}{\sin(\beta)} \times 1.0, ?\right)$, pressing [📖] [1] [Z] to select **zeros**. The '×1.0' forces a decimal result.
- Press [ctrl] [=] to select |, the **given** symbol. Add restriction **|0 < *β* < 180 – *α* and 0 < *α* < 180 – *β***, then press [enter].
- In a **Maths Box** enter **round(*res*,2)**, pressing [📖] [1] [R] to select **round**. Edit to change number of decimal places.

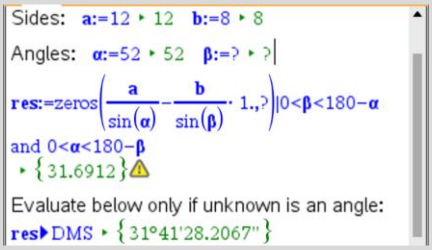


To find angle values in DMS, add a **Maths Box** then:

- Enter **res▶DMS**, press [📖] [1] [D] to select **▶DM'×1.0'S**.

Answer: Side $b = 3.32$ (2 decimal places).

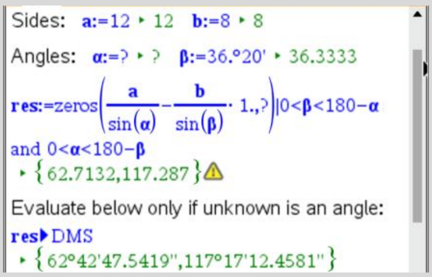
(b) (i) To find B if $a = 12$, $b = 8$, $A = 52°$, edit as shown:
Answer: $B = 31.69…° = 31°41'$ (correct to nearest minute).



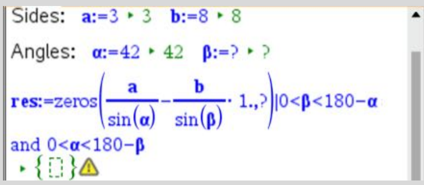
(ii) To find A if $a = 12$, $b = 8$, $B = 36°20'$, edit as shown using:

- Press [template] to select ***dd°mm'ss"*** template to enter $36°20'$.

Answer: This illustrates the *ambiguous case* of the sine rule. There are two possible triangles that meet the measurements:
Either $a = 12$, $b = 8$, $B = 36°20'$ and $A = 62°43'$.
Or $a = 12$, $b = 8$, $B = 36°20'$ and $A = 117°17'$.



(c) For $A = 42°$, $a = 3$, $b = 8$, $B = ?$, edit as shown.
Answer: No triangle is possible with measurements $A = 42°$, $a = 3$, $b = 8$. The result is therefore an empty list.



Note: *Save in MyWidgets folder to insert into other documents.*

Setting up a widget to solve triangles using the cosine rule

Question

Create a widget to perform cosine rule calculations on a Notes page. Test the page using:

(a) In ΔABC with $a = 3$, $b = 5$ and $c = 7$, determine the magnitude of the largest angle.

(b) A triangle has side lengths of 5 and 10. If the included angle is 26°24′, determine

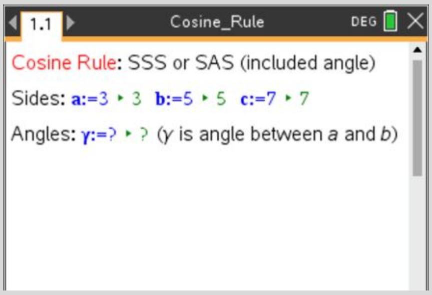
(i) the length of the third side, correct to two decimal places,

(ii) the magnitude of the other two angles, correct to the nearest minute.

Solution

(a) To set up the cosine rule for $a = 3$, $b = 5$ and $c = 7$, on a **Notes** page click to select **DEG** at top right corner, then:

- Enter the text as shown.
- By pressing [ctrl][M], insert three Maths Boxes next to the heading **Sides:**, and one **Maths Box** next to **Angle:**.
- In the first three **Maths Boxes**, enter ***a*:=3**, ***b*:= 5** and ***c*:= 7**.
- In the last **Maths Box**, enter ***γ*:=?** by pressing [ctrl][📖]([∞β°]) for *γ* and [?!▸] to select the **?** symbol.

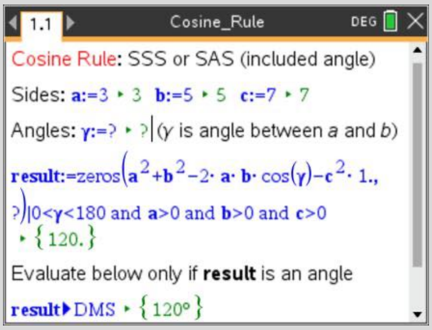


To evaluate the unknown, add a **Maths Box** and key in:

- **result:= zeros($a^2 + b^2 - 2a \times b \times \cos(\gamma) - c^2 \times 1.0$,?)** by pressing [📖][1][Z] to select **zeros**.
- Press [ctrl][=] to select |, the **given** symbol. Add restriction **|0 < *γ* < 180 and *a*>0 and *b*>0 and c>0**, then press [enter].

To find angle values in DMS, add a **Maths Box** then:

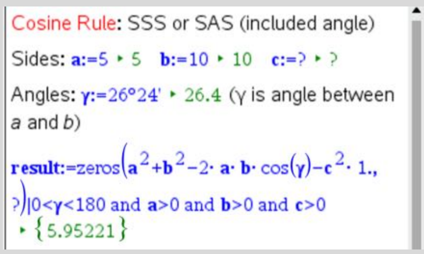
- Enter **result▶DMS**, pressing [📖][1][D] to select **▶DMS**.



Answer: For $a = 3$, $b = 5$ and $c = 7$, largest angle $C = 120°$.

(b)(i) To find c if $a = 5$ and $b = 10$, edit the following values:

- In the **Maths Boxes**, enter ***a*:=5**, ***b*:=10**, ***c*:=?**, ***γ*:= 26°24′**, pressing [template] to select ***dd*°*mm*′*ss*″** template to enter 26°24′.

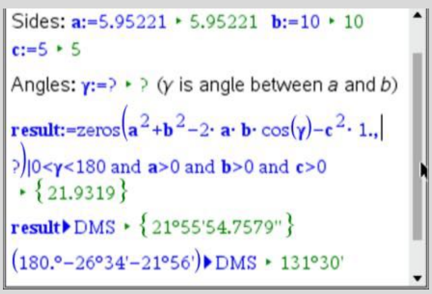


Answer: Third side, $c = 5.95$ (2 decimal places).

(ii) To find the angle opposite the side with length 5, edit:

- In the **Maths Boxes**, enter ***a*:=5.95221**, ***b*:=10**, ***c*:=5**, ***γ*:=?**

Note: *In the formula, γ is always the angle opposite side c.*



Answer: The angles are: 21°56′ and 131°30′. (Total = 180°)

To save this page as a **Widget**, with file name **Cosine_rule**:

- Press [doc▾] > **File** > **Save As**. Save in **MyWidgets** folder.

To open the **Widget** in any document to apply the cosine rule:

- Press [ctrl][doc▾][+page] > **Add Widget.** Select **Cosine_Rule**
- To reuse this page, open it in any document, press [ctrl][A] then [menu] > **Actions** > **Evaluate**. Edit the values of ***a***, ***b***, ***c*** and *γ*, as necessary.

Note: *A **Widget** is a reusable interactive page that can be inserted into other documents.*

Visualising a geometric proof for double angle identities

Question

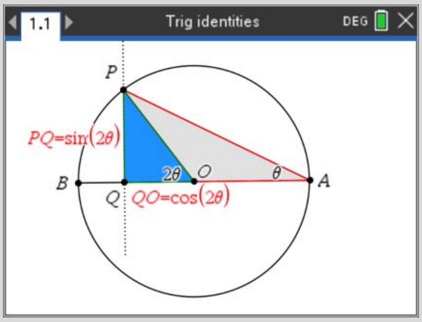
Use the Geometry application to aid in visualising a geometric proof that:

(a) $\cos(2\theta) = 2\cos^2(\theta) - 1$ **(b)** $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$

Solution

To set up the first part of the proof, on a **Geometry** page:

- Press menu > **Shapes** > **Segment**. Click at two points.
- Press menu > **Construction** > **Midpoint**. Click the segment. Press esc to exit the tool.
- Hover over the rightmost point, press ctrl menu > **Label** and enter the label ***A***. Similarly, label the leftmost point as ***B*** and the midpoint as ***O***.
- Press menu > **Shapes** > **Circle**. Click point ***O*** then point ***A***.
- To construct *PQ* as shown, press P and click the circumference of the circle in the 2nd quadrant. Press esc .
- Label this point as ***P***.
- Press menu > **Construction** > **Perpendicular**. Click the line *AB* then point *P*.
- Press P, click the intersection of the perpendicular and line *AB*. Press esc, then label the point as ***Q***.
- Press menu > **Points & Lines** > **Segment** and draw segments *AP, OA* and *OP*. Label all points as shown.



***Note:** Press esc to exit any Geometry tool before proceeding.*

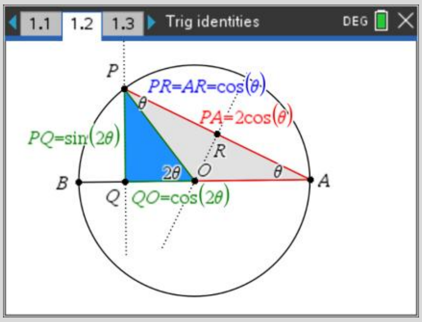
Answer: *Part 1 of the proof.*

For the angles on arc *PB*, if $\measuredangle PAB = \theta$, then $\measuredangle POB = 2\theta$.

Hence for ΔPOQ, $PO = 1$, $QO = 1 \cdot \cos(2\theta) = \cos(2\theta)$,
$PQ = 1 \cdot \sin(2\theta) = \sin(2\theta)$.

For the second part of the proof, on page 1.1 above:

- Press ctrl ▲. In thumbnail view, press ctrl C then ctrl V to obtain a copy of page 1.1 on page 1.2.
- On page 1.2, construct *OR*. Press menu > **Construction** > **Perpendicular**. Click segment *PA* and then click point *O*.
- Construct and label the point *R*, as shown, by pressing menu > **Points and Lines** > **Intersection**.



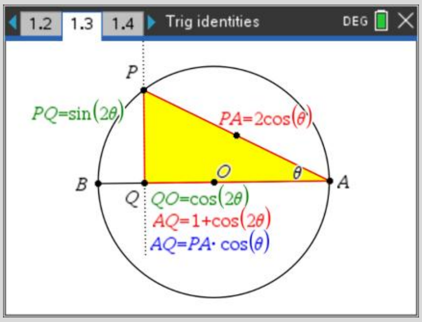
Answer: *Part 2 of the proof.* $PO = AO = 1$, hence $\Delta PRO \cong \Delta ARO$ and $PR = AR = 1 \times \cos(\theta)$.

Therefore, $PA = 2 \times \cos(\theta)$.

***Note:** The symbol $\cong$ is used to denote **congruence** of shapes.*

... continued

Solution (continued)



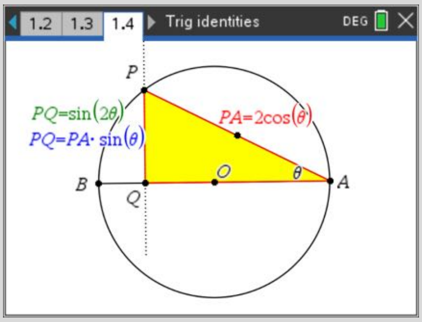
For the third part of the proof, on page 1.2 above:

- Press [ctrl] ▲. In thumbnail view, press [ctrl] [C] then [ctrl] [V] to obtain a copy of page 1.2 on page 1.3.
- On page 1.3, press [menu] > **Actions** > **Hide/Show**. Click to hide objects, except those for ΔPAQ, as shown.

Answer: *Part 3 of proof.* $AQ = 1 + \cos(2\theta) = PA\cos(\theta)$

(a) Substituting $PA = 2\cos(\theta)$ in the above equation,

$1 + \cos(2\theta) = 2\cos(\theta)\cos(\theta) \Leftrightarrow \cos(2\theta) = 2\cos^2(\theta) - 1$.



For the fourth part of the proof, on page 1.3 above:

- Press [ctrl] ▲. In thumbnail view, press [ctrl] [C] then [ctrl] [V] to obtain a 'copy' of page 1.3 on page 1.4.
- On page 1.4, press [menu] > **Actions** > **Hide/Show**. Click to hide objects, except for those shown.

Answer: *Part 4 of proof.* $PQ = \sin(2\theta) = PA\sin(\theta)$

(b) Substituting $PA = 2\cos(\theta)$ in equation above,

$\sin(2\theta) = 2\cos(\theta)\sin(\theta)$, as required.

Verifying and proving some trigonometric identities

Question

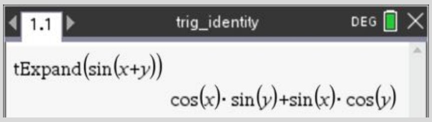
(a) Use the *tExpand* command to verify that sin($x + y$) = sin(x) cos(y) + cos(x) sin(y).

(b) **(i)** Verify and **(ii)** prove the identity sin($x + y$) sin($x - y$) = $\sin^2(x) - \sin^2(y)$.

Solution

(a) To verify sin($x + y$) = sin(x) cos(y) + cos(x) sin(y), on a **Calculator** page:

- Press [menu] > **Algebra** > **Trigonometry** > **Expand**. Enter **tExpand(sin(*x* + *y*))**. The result verifies the identity.

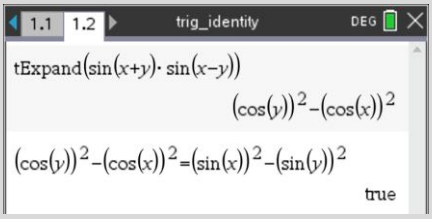


(b)(i) To verify that sin($x + y$) sin($x - y$) = $\sin^2(x) - \sin^2(y)$:

- Enter **tExpand(sin(*x* + *y*) × sin(*x* – *y*))**.

To establish the equivalence of the output and the identity:

- Press [ctrl] [(–)] (ans) and enter **Ans = (sin(*x*))^{2} – (sin(*y*))^{2}** .

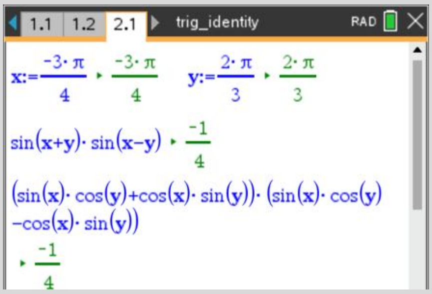


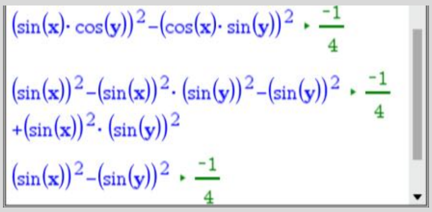
Answer: (i) It is verified that:

sin($x + y$) sin($x - y$) = $\cos^2(y) - \cos^2(x) = \sin^2(x) - \sin^2(y)$.

(b)(ii) Proof that sin($x + y$) sin($x - y$) = $\sin^2(x) - \sin^2(y)$.

To test the proof, in **Maths Boxes** on a **Notes** page, substitute editable values for x and y, as shown.





Answer: (ii) Using the identity from **(a)** and sin($-y$) = $-$sin(y),

LHS = (sin(x)cos(y)+cos(x)sin(y))×(sin(x)cos(y)−cos(x)sin(y))

Applying $(a + b)(a - b) = a^2 - b^2$ to the previous line:

LHS = $(\sin(x)\cos(y))^2 - (\cos(x)\sin(y))^2$.

Expand, apply the identity $\cos^2(\theta) = 1 - \sin^2(\theta)$ and simplify.

LHS = $\sin^2(x) - \sin^2(x)\sin^2(y) - \sin^2(y) + \sin^2(x)\sin^2(y)$
LHS = $\sin^2(x) - \sin^2(y)$ = RHS, as required.

Exploring equivalent forms of $a\cos(x) + b\sin(x)$

Question

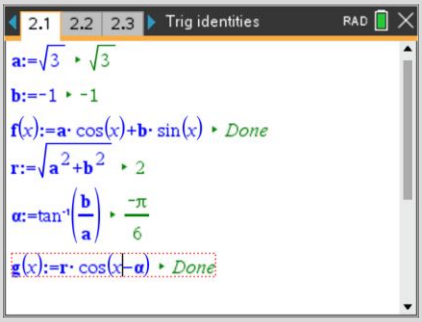
Show that the graph of $f(x)=\sqrt{3}\cos(x)-\sin(x)$ is equivalent to the graph of a function of the form $g(x)=r\cos(x-\alpha)$, with appropriate amplitude and phase shift.

Hence solve the equation $f(x)=-1, x\in[-2\pi,2\pi]$.

Solution

To determine the values of r and α, on a **Notes** page:

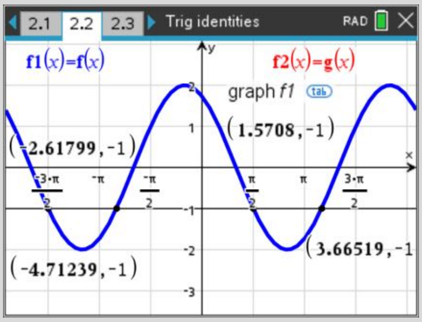
- Insert **Maths Boxes** by pressing [ctrl] [M] and enter values for the coefficients ***a*** and ***b***, and the formulas to determine ***r*** and $\boldsymbol{\alpha}$, as shown. Select $\boldsymbol{\alpha}$ by pressing [ctrl] [📖] (∞β°).



Answer: $r=2,\ \alpha=-\frac{\pi}{6},\ \sqrt{3}\cos(x)-\sin(x)=2\cos\left(x+\frac{\pi}{6}\right)$

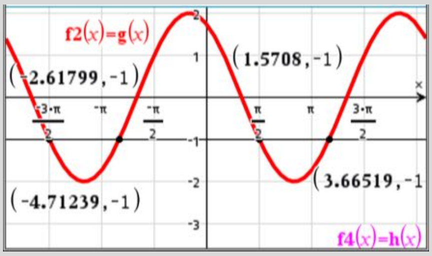
To compare the graphs of $f(x)=\sqrt{3}\cos(x)-\sin(x)$ and $g(x)=2\cos\left(x+\frac{\pi}{6}\right)$, on a **Graphs** page:

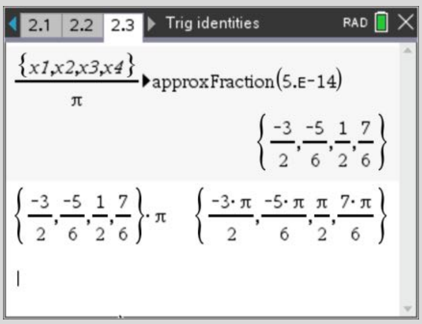
- Enter $\boldsymbol{f1(x)=f(x)}$, $\boldsymbol{f2(x)=g(x)}$ and $\boldsymbol{f3(x)=-1}$
  Press [menu] > **Window Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin: -2π XMax: 2π XScale: $\pi/2$
  YMin: –3.33 YMax: 3.33 YScale: 1



To solve $f(x)=-1, x\in[-2\pi,2\pi]$, on the **Graphs** page:

- Press [menu] > **Geometry** > **Points & Lines** > **Intersection Point(s)**. Click on graph $\boldsymbol{f1}$ then on graph $\boldsymbol{f3}$.
  Hover over the x-coordinate of the leftmost intersection point, press [var] and store as **x1**. Similarly, store the other x-coordinates: ***x*2**, ***x*3**, ***x*4**.
- For solutions in terms of π, on a **Calculator** page enter $\frac{\{\mathbf{x1,x2,x3,x4}\}}{\boldsymbol{\pi}}$►**approxFraction**, then enter $\mathbf{ans}\cdot\boldsymbol{\pi}$.





Note: ►*approxFraction is found in the* ***Number*** *menu.*

Answer: The graphs are identical, verifying their equivalence.

$f(x)=-1, x\in[-2\pi,2\pi]$ for $x=-\frac{3\pi}{2},-\frac{5\pi}{6},\frac{\pi}{2},\frac{7\pi}{6}$.

2.2.2 Transformations

Representing translations as column vectors

Question

The lines $y=-\frac{7}{3}$ and $x=7$ intersect at point R. These lines intersect the line with equation $3x+y=7$ at points P and Q, respectively.

(a) Show on the coordinate plane the image of triangle PQR under the transformation $\begin{bmatrix}-6\\4\end{bmatrix}$.

Hence find the coordinates of the vertices of the image of triangle PQR under the transformation.

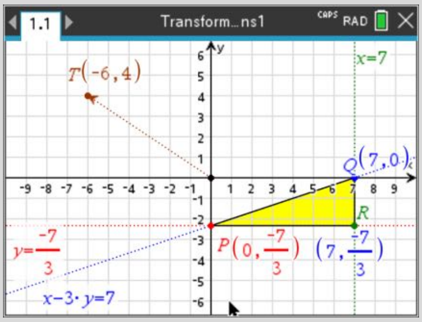
(b) Determine the equation of the line that passes through the image of points P and Q.

(c) Explore the image of triangle PQR under the transformation $\begin{bmatrix}a\\b\end{bmatrix}$ for various values of a and b.

Solution

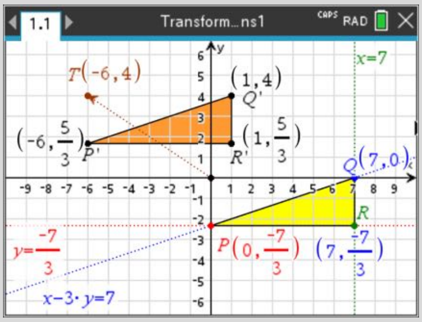
To construct ΔPQR and the translation vector, on a **Graphs** page:

- Press menu > **Graph Entry/Edit** > **Relation** and enter the relations $\boldsymbol{x-3y=7}$, $\boldsymbol{y=-\frac{7}{3}}$ and $\boldsymbol{x=7}$.
- Press P > **Point.** Click each intersection point. Press esc.
- Hover over the leftmost point, press ctrl menu > **Label.** Enter label ***P***. Similarly, label points ***Q*** and ***R***, as shown.
- Press menu > **Geometry** > **Shapes** > **Triangle**. Click points P, Q and R to form ΔPQR.
- Edit the coordinates of the vertices to exact values.
- Press P > **Point by Coordinates**. Enter $\boldsymbol{(-6,4)}$ and label as shown.
- Press menu > **Geometry** > **Points & Lines** > **Vector**. Click on the origin and then on the point at $T(-6,4)$.



(a) To obtain the image of ΔPQR under the transformation:

- Press menu > **Geometry** > **Transformation** > **Translation**.
- Click on ΔPQR and then click on the vector to $T(-6,4)$.
- Hover over point P', press ctrl menu > **Coordinates and Equations**. Repeat for Q' and R'. Edit these coordinates to exact form. If you get a dialog box with the message 'Cannot accept change, invalid input', click OK and the exact value will be locked in.



Answer: Coordinates of the image are $P'\left(-6,\frac{5}{3}\right), Q'(1,4), R'\left(1,\frac{5}{3}\right)$.

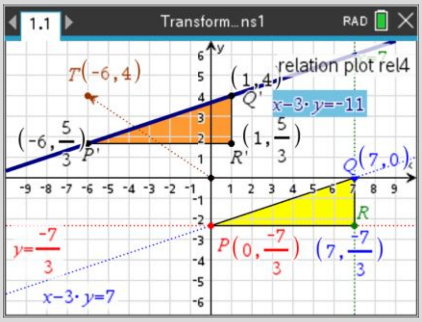
... continued

Solution (continued)

(b) To find the equation of the line through $P'Q'$:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -6 \\ 4 \end{bmatrix} \Rightarrow \begin{matrix} x = x' + 6 \\ y = y' - 4 \end{matrix}.$$

$x - 3y = 7 \to (x' + 6) - 3(y' - 4) = 7$, or $x' - 3y' = -11$



- Enter the relation with equation $\boldsymbol{x - 3y = -11}$.

Answer: Equation of the line $P'Q'$ is $x - 3y = -11$.

(c) To explore the image of ΔPQR under the transformation $\begin{bmatrix} a \\ b \end{bmatrix}$, edit/drag the vector endpoint coordinates to $(\boldsymbol{a}, \boldsymbol{b})$.

Answer: The area is invariant under the transformation, T.

Representing dilations of the form (x,y) → (ax,by) as matrices

Question

The triangle PQR from the previous problem is transformed according to $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1.5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

(a) Determine the coordinates of the image of points P, Q and R under this transformation.

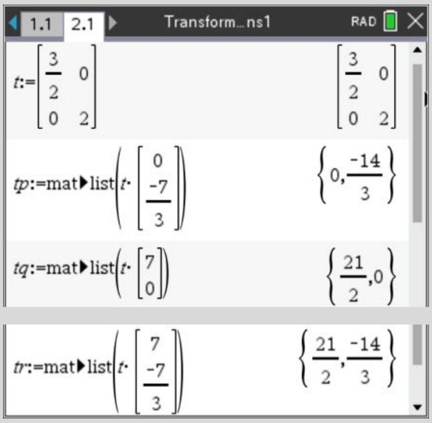
(b) Plot the image of triangle PQR and compare the area of PQR and the area of its image.

(c) Calculate the determinant of the dilation matrix and interpret its geometric significance.

(d) Use the Dilation tool to explore transformations of the form $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

Solution

(a) To determine the coordinates of the image of $P(0, -7/3), Q(7, 0)$ and $R(7, -7/3)$, in the document from the previous problem, add a **Calculator** page to a **New Problem** (via doc▾ > **Insert** >**Problem**).



- Enter $\boldsymbol{t} := \begin{bmatrix} \mathbf{3/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} \end{bmatrix}$, then $\boldsymbol{tp}$ **:= mat▶list** $\left(\boldsymbol{t} \cdot \begin{bmatrix} \mathbf{0} \\ \mathbf{-7/3} \end{bmatrix}\right)$.
- Copy and paste the previous entry and edit it to $\boldsymbol{tq}$ **:= mat▶list** $\left(\boldsymbol{t} \cdot \begin{bmatrix} \mathbf{7} \\ \mathbf{0} \end{bmatrix}\right)$ and $\boldsymbol{tr}$ **:= mat▶list** $\left(\boldsymbol{t} \cdot \begin{bmatrix} \mathbf{7} \\ \mathbf{-7/3} \end{bmatrix}\right)$

Note: *To select the **mat ▶list** command, press* 📖 1 M.

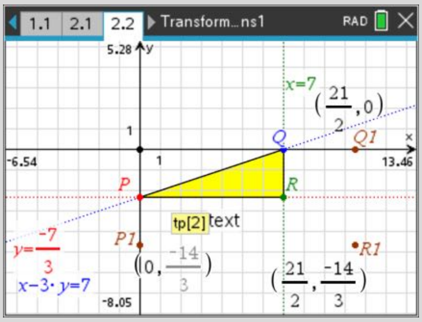
Answer: Coordinate: $P'\left(0, -\frac{14}{3}\right), Q'\left(\frac{21}{2}, 0\right), R'\left(\frac{21}{2}, -\frac{14}{3}\right)$

... continued

Solution (continued)

(b) To plot the image of the vertices of ΔPQR:

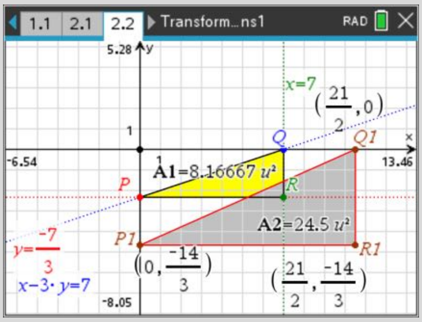
- Press [ctrl] [▲]. In thumbnail view, select page 1.1, press [ctrl] [C], select **Problem 2** then press [ctrl] [V] to obtain a copy of page 1.1 on page 2.2.
- On page 2.2, delete all objects except those for ΔPQR.
- Press [P] > **Point by coordinates**. Enter coordinates $(\boldsymbol{tp}[1], \boldsymbol{tp}[2])$. Repeat for $(\boldsymbol{tq}[1], \boldsymbol{tq}[2])$ and $(\boldsymbol{tr}[1], \boldsymbol{tr}[2])$.
- Press [ctrl] [✋]. Drag the workspace to reveal all points.



***Note: tp[1]** and **tp[2]** are the first and second elements, respectively, of the list **tp**.*

To find the area of ΔPQR and its image, on page 2.2:

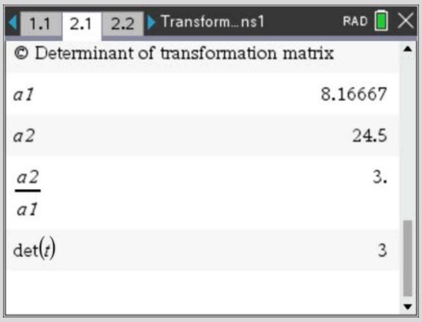
- Label the image of the vertices: ***P*1**, ***Q*1** and ***R*1**, as shown.
- Press [menu] > **Geometry** > **Shapes** > **Triangle**. Construct $\Delta P1Q1R1$.
- Press [menu] > **Geometry** > **Measurement** > **Area**. Click on ΔPQR then click again to position the value of the area on the page. Repeat these steps for $\Delta P1Q1R1$.
- Hover over the area measure for ΔPQR, press [var] > **Store** and enter ***A*1** as the variable name. Repeat for $\Delta P1Q1R1$, and enter ***A*2** as the variable name.



Answer: Area of $\Delta PQR = 8.1\dot{6} = 8\frac{1}{6}$ and $\Delta P1Q1R1 = 24.5$

(c) To explore the geometric meaning of the determinant:

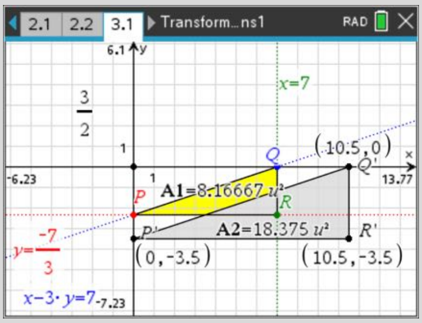
- On page 2.1, enter the ratio of areas: $\dfrac{A2}{A1}$.
- To calculate the determinant of matrix T, enter $\mathbf{det}(\boldsymbol{t})$.

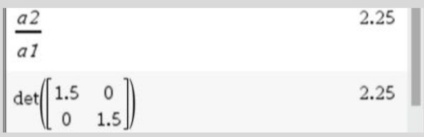


Answer: In this case, the determinant is equal to the ratio of the area of the object, triangle *PQR*, and its image. This can be explored further with other transformation matrices to confirm the relationship between the determinant of a transformation matrix and the effect of the linear transformation on the area of a bounded region.

(d) To explore the **Dilation** tool, on a copy of page 1.1:

- Delete all objects except those for ΔPQR.
- For a dilation of $a = 1.5$ parallel to both x and y axes, press [menu] > **Geometry** > **Transformation** > **Dilation**.
- Click on ΔPQR, then click on the origin and enter **3/2**.
- Hover over point $\boldsymbol{P'}$, press [ctrl] [menu] > **Coordinates and Equations**. Repeat for points $\boldsymbol{Q'}$ and $\boldsymbol{R'}$.





Answer: $A2 = \left(\frac{3}{2}\right)^2 \times A1$, hence $A2 / A1 = \left(\frac{3}{2}\right)^2 = 2.25$. As in in previous case, $A2 / A1 = \det(T)$.

Exploring rotation of angle θ anticlockwise about the origin

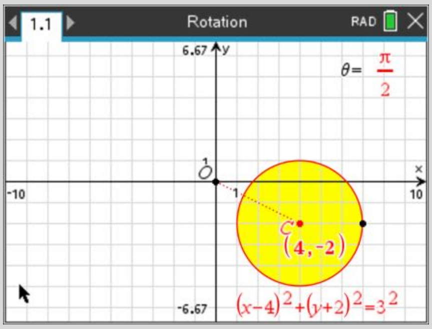
Question

Use the **Rotation** transformation tool, together with rotation matrices, to explore the coordinates of the image of the centre of circles with equation $(x-h)^2+(y-k)^2=r^2$ under different rotation angles, including $\theta=\frac{\pi}{2},\pi,\frac{3\pi}{2}$ and $\theta=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4}$.

Solution

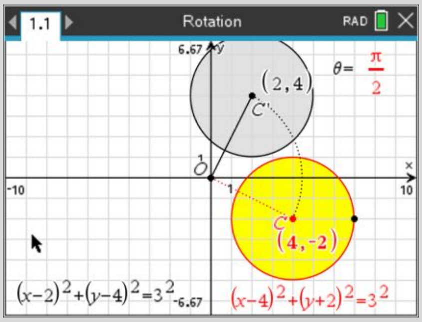
To set up the **Rotation** tool exploration, on a **Graphs** page:

- Press [menu] > **View** > **Grid**. Select **Lined Grid**.
- Press [menu] > **Geometry** > **Shapes** > **Circle**. To draw the circle, click on the point at $(\mathbf{4},-\mathbf{2})$ and then on the point at $(\mathbf{7},-\mathbf{2})$.
- Label centre ***C***. Press [menu] > **Actions** > **Coordinates and Equations**. Click the circle circumference and point *C*.
- To input the rotation angle, with the cursor near the top right corner, press [ctrl] [menu] > **Text**. Enter $\boldsymbol{\pi}$ **/ 2** .
- Press [P] > Point. Click the origin, then press [esc]. Hover over the origin, press [ctrl] [menu] > **Label**. Enter the label, ***O***.
- Draw a line segment ***OC*** by pressing [menu] > **Geometry** > **Points & Lines** > **Segment** and clicking points *O* and *C*.



To apply the rotation to the circle and segment:

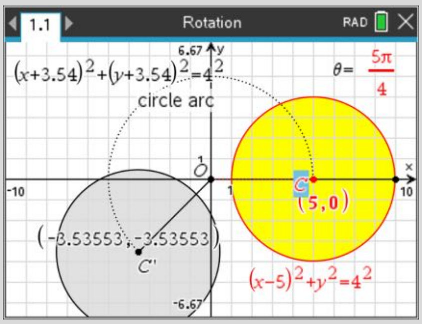
- Press [menu] > **Geometry** > **Transformation** > **Rotation**. Click on the circle, then the origin, then the angle textbox.
- Click in turn on: the segment *OC*, the origin, the angle textbox.
- Press [menu] > **Actions** > **Coordinates and Equations**. Double-click on the images, C' and on the circle centred at C' to set the equations and coordinates. Press [esc] to exit.



Note: *The image of the circle and segment will now interactively update if a change is made to the circle centre, C, or to the circle radius, or to the rotation angle textbox.*

To explore changing the circle centre and rotation angle:

- Drag point *C* or the point on the circle to vary the circle centre or radius. Edit the textbox to vary the rotation angle.



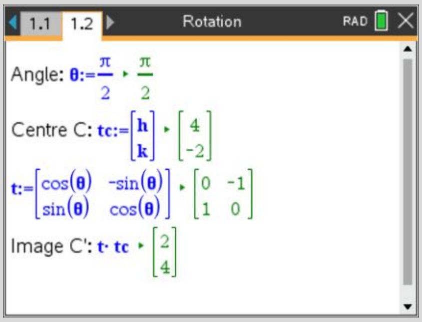
... continued

Solution (continued)

To confirm geometric results using the transformation matrix:

- Store the coordinates of C as variables h and k. (Hover over the coordinate, press [var] and enter variable h or k).
- Add a **Notes** page, and in **Maths Boxes**, enter the variables, matrices and calculations as shown.

Note: *To insert a* ***Maths Box****, press* [ctrl] [M].



If the coordinates of point C change on page 1.1, this will automatically update in vector ***tc*** on page 1.2. However, changes to θ need to be manually changed on both pages.

Answer: The radius and shape of the circles are invariant under the rotation. However, the coordinates of $C(h,k)$ are transformed to $C'(h',k')$, where $h' = h\cos(\theta) - k\sin(\theta)$ and $k' = h\sin(\theta) + k\cos(\theta)$.

Exploring reflection in the x and y axes, geometrically and with matrices

Question

Let R_1 be the region bounded by the lines $x - 5y = 16$, $x + 5y = -14$ and $x = 6$. Region R_1 is reflected in the x-axis followed by a reflection in the y-axis.

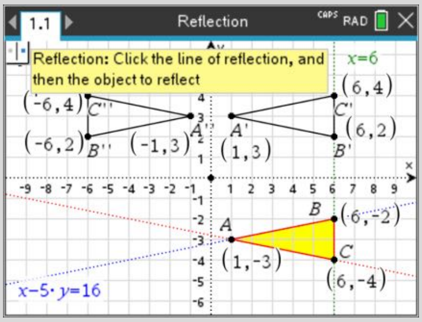
(a) Use the Reflection transformation tool to explore the image of R_1.

(b) Use the Rotation transformation tool and matrix multiplication to show that the combined effect of the two reflections is a rotation.

Solution

(a) To construct region R_1 and its image, on a **Graphs** page:

- Press [menu] > **Graph Entry/Edit > Relation** and enter the relations $\boldsymbol{x - 5y = 16}$, $\boldsymbol{x + 5y = -14}$ and $\boldsymbol{x = 6}$.
- Press [menu] > **Geometry > Shape > Triangle**, then click the vertices of the region. Label the vertices A, B, C by hovering over a vertex, pressing [ctrl] [menu] > **Label** and entering the label.
- Press [menu] > **Geometry > Transformation > Reflection**. Click on the x-axis, then ΔABC.
  Click on the y-axis, then $\Delta A'B'C'$.
- Press [menu] > **Actions > Coordinates and Equations**. To show coordinates, click on the points $A, B, C, A', B', C', A'', B'', C''$.

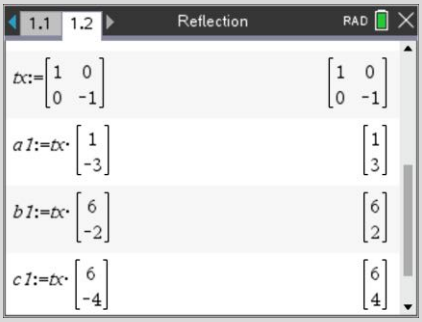


Answer: The image of points $A(1, –3)$, $B(6, –2)$, and $C(6, –4)$, under the reflection in the x-axis followed by the y-axis is $A''(–1, 3)$, $B''(–6, 2)$, and $C(–6, 4)$.

... continued

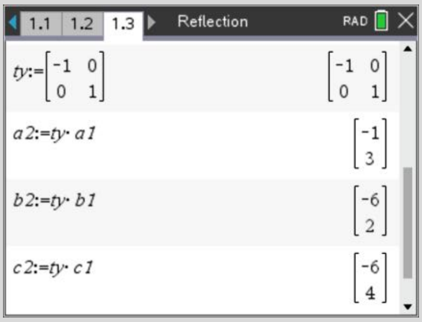
Solution (continued)

(b) To find the coordinates of the image of the vertices of the triangle *ABC*, using matrices, on a **Calculator** page:



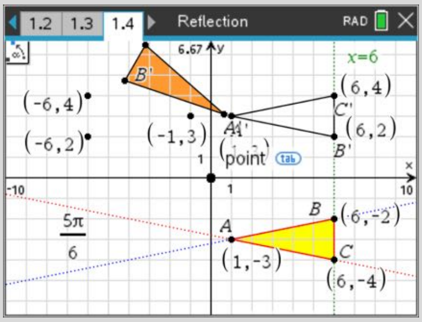
- Enter $\boldsymbol{tx} := \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{bmatrix}$, where $\boldsymbol{tx}$ is the matrix for the reflection mapping in the *x*-axis, such that $(x_1, y_1) \to (x_1, -y_1)$.
- Enter the matrix products as shown, where the numbers in the column vectors are the coordinates of *A*, *B* and *C*.

To find the coordinates of the vertices for the combined reflections in the *x* and *y* axes, on a **Calculator** page:



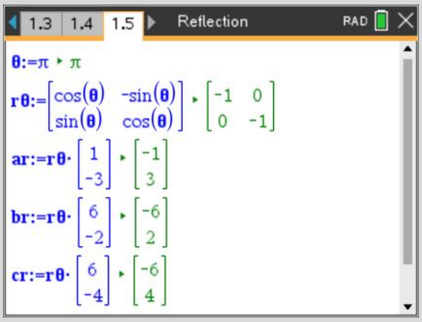
- Enter $\boldsymbol{ty} := \begin{bmatrix} \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$, where $\boldsymbol{ty}$ is the matrix for the reflection mapping in the *y*-axis, such that $(x_1, y_1) \to (-x_1, y_1)$.
- Enter the matrix products, as shown, where $a1$, $b1$ and $c1$ are column vectors of the coordinates of A', B' and C'.

To explore geometrically the equivalent rotation:



- Copy **page 1.1** (press ctrl ▲, then ctrl C, and ctrl V).
- On copy of **page 1.1**, hide $\Delta A''B''C''$ by pressing menu > **Actions** > **Hide/Show** and clicking the triangle (show only vertices).
- Input the rotation angle by pressing ctrl menu > **Text**. In the textbox that follows, enter, say, $\mathbf{3\pi / 2}$.
- Press menu > **Geometry** > **Transformation** > **Rotation**. Click on ΔABC, then the origin, then the rotation angle text box.
- Edit the rotation angle until the rotation image fits the vertices of the double reflection. The rotation $\boldsymbol{\theta = 5\pi / 6}$ is shown.

To explore the problem using matrices, and confirm the geometric result that the combined reflections in the *x* and *y* axes is equivalent to a rotation of $\theta = \pi$, on a **Notes** page:



- Press ctrl menu to insert a **Maths Box**, then enter a rotation angle e.g. $\theta := \pi$, as shown.
- Insert additional **Maths Boxes** and enter the matrix operations, as shown. Explore the value of θ for which the coordinates of the image of *A*, *B* and *C* correspond to those of the combined reflections in the *x* and *y* axes.

Answer: The use of the geometric Rotation tool and multiplication using the rotation matrix both confirm that the combined effect of the two reflections is an anticlockwise rotation of magnitude π.

Representing reflection in the line $y = m \cdot x = \tan(\theta) \cdot x$ as a matrix

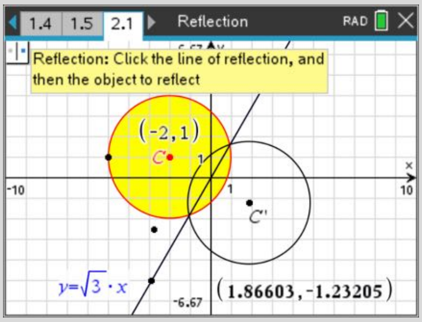
Question

Let the point C with coordinates $(-2,1)$ be the centre of a circle of radius $r = 3$. Using the Reflection transformation tool, together with matrices calculations, show that a reflection in the line $y = \sqrt{3}x$, corresponds to the transformation $\frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.

Solution

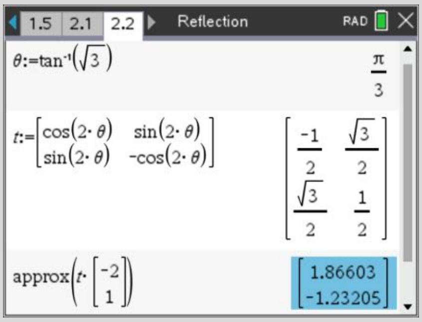
To set up the geometric transformation, on a **Graphs** page:

- Press menu > **Graph Entry/Edit** > **Relation** and enter the relation $\boldsymbol{y = \sqrt{3}x}$. Press menu > **Geometry** > **Points and Lines** > **Line**. Click the graph of $\boldsymbol{y = \sqrt{3}x}$ at two points.
- Press menu > **View** > **Grid**. Select **Lined Grid**.
- Press menu > **Geometry** > **Shapes** > **Circle**. Click on point $(-2,1)$ then click on $(-5,1)$ for a circle with $r = 3$.
- Hover over centre point at $(-2,1)$. Press ctrl menu > **Label**. Enter the label, ***C***.
- Press menu > **Geometry** > **Transformation** > **Reflection**. Click on the line along $\boldsymbol{y = \sqrt{3}x}$, then click on the circle.
- Press menu > **Actions** > **Coordinates and Equations**. Click on the point C' to obtain its coordinates.



To confirm the transformation matrix, on a **Calculator** page:

- Enter $\boldsymbol{\theta := \tan^{-1}(\sqrt{3})}$, pressing ctrl (∞β°) to select ***θ***.
- Enter the transformation matrix, as shown.
- To obtain a decimal approximation of the coordinates of the image of C, enter $\textbf{approx}\left(\boldsymbol{t} \cdot \begin{bmatrix} \mathbf{-2} \\ \mathbf{1} \end{bmatrix}\right)$.



Answer: The coordinates of C' indicate consistency between the transformation matrix and a reflection in the line $y = \sqrt{3}x$.

2.2.3 Vectors in the plane

Representing vector addition and subtraction with the triangle rule

A vector is a set of equivalent directed line segments.

- If $\underset{\sim}{u} = \overrightarrow{AB}$ and $\underset{\sim}{v} = \overrightarrow{BC}$, then $\underset{\sim}{u} + \underset{\sim}{v} = \overrightarrow{AB} + \overrightarrow{BC}$.
- Subtraction of vectors: $\underset{\sim}{u} - \underset{\sim}{v} = \underset{\sim}{u} + \left(-\underset{\sim}{v}\right)$.

The triangle rule represents the resultant vector from the sum and difference of two vectors.

Question

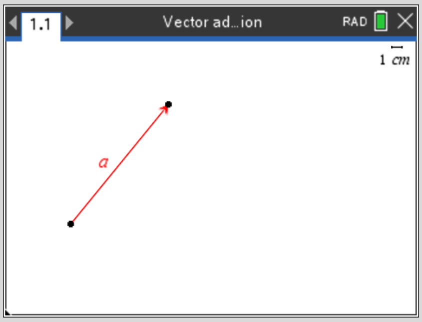
Construct a simple geometric demonstration of the triangle rule.

Solution

***Note:** This construction is best attempted using the TI-Nspire CX-II CAS Teacher Software rather than on the handheld device.*

On a **Geometry** page, add a vector $\underset{\sim}{a}$ as follows:

- Press [menu] > **Points & Lines** > **Vector**.
- Decide on the vector's starting point and press [enter].
- Decide on the vector's end point and press [enter].
- Complete as shown and press [esc].

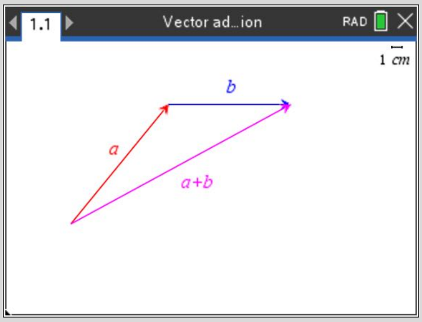


To give a vector a label:

- Hover over the vector.
- Press [ctrl] [menu] > **Label**.
- Label as shown.

***Note:** To change the line colour of a vector, hover over the vector, press [ctrl] [menu] > **Colour** > **Line Colour** and change the colour of the vector as desired.*

***Note:** To change the attributes of a vector, hover over the vector, press [ctrl] [menu] > **Attributes** and change the appearance of the vector as desired.*



Repeat the above instructions to add a vector $\underset{\sim}{b}$ from the end of $\underset{\sim}{a}$ and a vector $\underset{\sim}{a} + \underset{\sim}{b}$ with labels and line colours as shown.

***Note:** Press [tab] when you want to select an object from a set of objects that are close to each other on a page. A [tab] icon will appear next to the cursor in these situations and acts as a fine motor control.*

To show the triangle rule for addition dynamically, a slider with conditional attributes can be used.

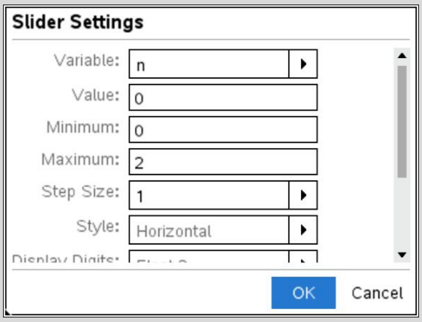
This slider, set up for values of a variable ***n***, controls when vector $\underset{\sim}{b}$ and vector $\underset{\sim}{a} + \underset{\sim}{b}$ are displayed on the page.

... continued

Solution (continued)

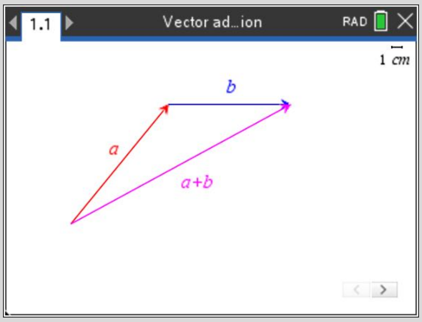
Insert a **Slider** to control the value of n as follows:

- Press [menu] > **Actions** > **Insert Slider**.
- Set the **Slider Settings** as shown.
- Check the **Minimised** box.
- Uncheck the **Show Variable** and **Show Scale** boxes.



To move the **Slider**:

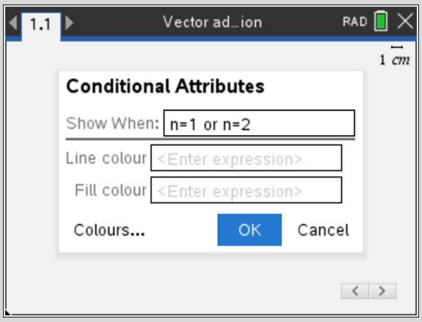
- Press [ctrl] [menu] and move it to the bottom right-hand corner as shown.



To set vector $\underset{\sim}{b}$ to display when $n = 1$ or $n = 2$, complete as follows:

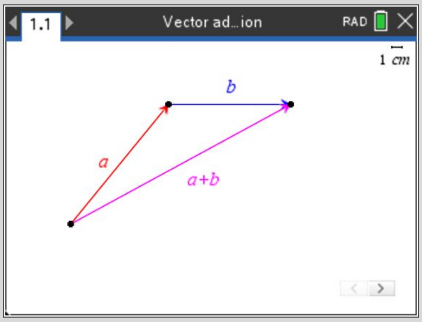
- Hover over the vector $\underset{\sim}{b}$ and press [tab] to select the vector.
- Press [ctrl] [menu] > **Conditions**.
- Complete the **Conditional Attributes** settings as shown.

***Note:** Alternatively, to set a condition press* [menu] ***> Actions > Set Conditions**.*



To set vector $\underset{\sim}{a} + \underset{\sim}{b}$ to display when $n = 2$, complete as follows:

- Hover over the vector $\underset{\sim}{a} + \underset{\sim}{b}$ and press [tab] to select the vector.
- Press [ctrl] [menu] > **Conditions**.
- Complete the **Conditional Attributes** settings.



To hide all points except the starting point for vector $\underset{\sim}{a}$:

- Hover over each point and press [ctrl] [menu] > **Hide.**

The screenshots right show the page when $n = 2$ (for $\underset{\sim}{a} + \underset{\sim}{b}$).

Note: The above construction can be adapted to show vector $\underset{\sim}{a}$, vector $-\underset{\sim}{b}$ and the resulting vector $\underset{\sim}{a} - \underset{\sim}{b}$.

Representing scalar multiplication

Multiplication by a real number (scalar) changes the length of a vector.

- The vector $k\underset{\sim}{u}$, where $k \in \mathbb{R}^+$, has the same direction as $\underset{\sim}{u}$, but its length is multiplied by a factor of k.
- If $\underset{\sim}{u} = \overrightarrow{AB}$, then $-\underset{\sim}{u} = -\overrightarrow{AB} = \overrightarrow{BA}$.
- Two non-zero vectors, $\underset{\sim}{u}$ and $\underset{\sim}{v}$, are parallel if $\underset{\sim}{u} = k\underset{\sim}{v}$ where $k \in \mathbb{R} \setminus \{0\}$.

Question

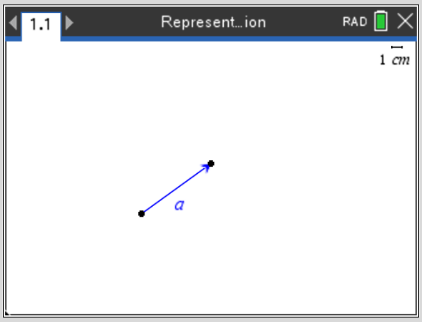
Construct a simple geometric demonstration of scalar multiplication.

Solution

Note: *This construction is best attempted using the TI-Nspire CX-II CAS Teacher Software rather than on the handheld device.*

On a **Geometry** page, add a vector $\underset{\sim}{a}$ as follows:

- Press [menu] > **Points & Lines** > **Vector**.
- Decide on the vector's starting point and press [enter].
- Decide on the vector's end point and press [enter].
- Complete as shown and press [esc].

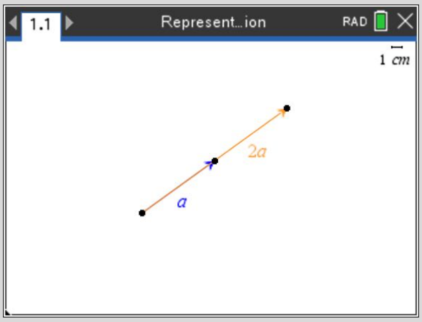


To give a vector a label:

- Hover over the vector.
- Press [ctrl] [menu] > **Label**.
- Label as shown.

Note: *To change the line colour of a vector, hover over the vector, press* [ctrl] [menu] *>* ***Colour*** *>* ***Line Colour*** *and change the colour of the vector as desired.*

Note: *To change the attributes of a vector, hover over the vector, press* [ctrl] [menu] *>* ***Attributes*** *and change the appearance of the vector as desired.*

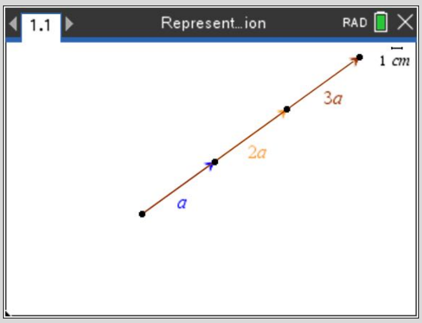


The **Transformation** menu provides a dilation tool.

The **Dilation** command provides the image of an object with a point that is the centre of dilation and a number specifying the dilation factor.

Add a vector $2\underset{\sim}{a}$ with label and line colour as follows:

- Press [menu] > **Transformation** > **Dilation**.
- Click (press [click]) on the vector's starting point.
- Click (press [click]) on the vector.
- Press [2] to set the dilation factor and press [enter] [esc].
- Set the vector with label and line colour as shown.
- Hover over the 2 and press [ctrl] [menu] > **Hide**.



Repeat the above to display the vector $3\underset{\sim}{a}$ with label and line colour as shown.

Note: *For vector* $3\underset{\sim}{a}$*, press* [3] *to set the dilation factor.*

... continued

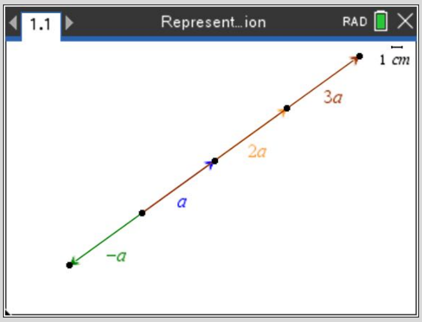
Solution (continued)

Repeat the above to display the vector $-\underset{\sim}{a}$ with label and line colour as shown.

***Note:** For vector $-\underset{\sim}{a}$, press* [(–)] [1] *to set the dilation factor.*

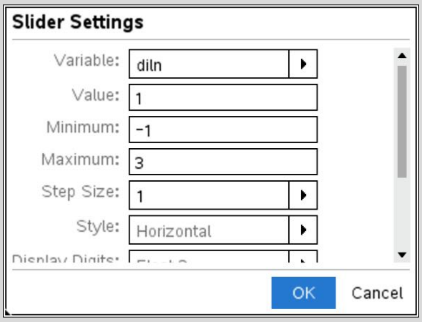
To show scalar multiplication of a vector dynamically, a slider with conditional attributes can be used.

This slider, set up for values of a variable ***diln***, controls when vector $\underset{\sim}{a}$, vector $2\underset{\sim}{a}$, vector $3\underset{\sim}{a}$ and vector $-\underset{\sim}{a}$ are displayed on the page.



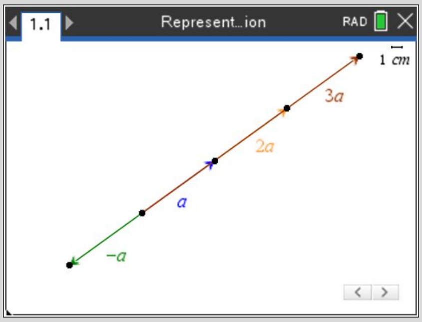
Insert a **Slider** to control the value of ***diln*** as follows:

- Press [menu] **> Actions** > **Insert Slider**.
- Set the **Slider Settings** as shown.
- Check the **Minimised** box.
- Uncheck the **Show Variable** and **Show Scale** boxes.



To move the **Slider**:

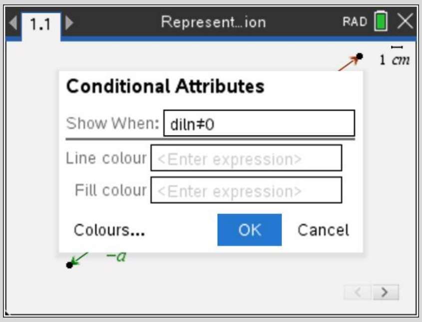
- Press [ctrl] [menu] and move it to the bottom right-hand corner as shown.



To set vector $\underset{\sim}{a}$ to display when $\boldsymbol{diln \neq 0}$, complete as follows:

- Hover over the vector $\underset{\sim}{a}$ and press [tab] to select the vector.
- Press [ctrl] [menu] > **Conditions**.
- Complete the **Conditional Attributes** settings as shown.

Notes:** To access ≠, press* [ctrl] [=] ([≠≥▸]) *and select as required. Alternatively, to set a condition press* [menu] *>* ***Actions > Set Conditions*.*

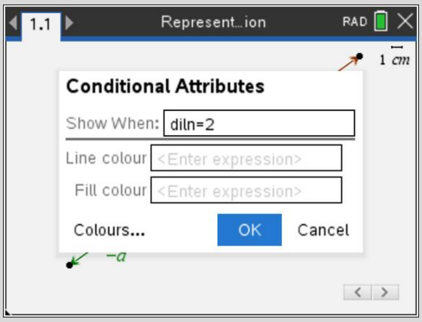


... continued

Solution (continued)

To set vector $2\underset{\sim}{a}$ to display when $\boldsymbol{diln} = \mathbf{2}$, complete as follows:

- Hover over the vector $2\underset{\sim}{a}$ and press tab to select the vector.
- Press ctrl menu > **Conditions**.
- Complete the **Conditional Attributes** to display vector $2\underset{\sim}{a}$ when $\boldsymbol{diln} = \mathbf{2}$.



Repeat the above instructions to display vector $3\underset{\sim}{a}$ with the following conditional attribute:

- Display vector $3\underset{\sim}{a}$ when $\boldsymbol{diln} = \mathbf{3}$.

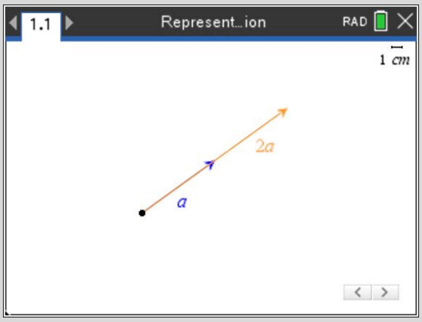
Repeat the above instructions for vector $-\underset{\sim}{a}$ with the following conditional attribute:

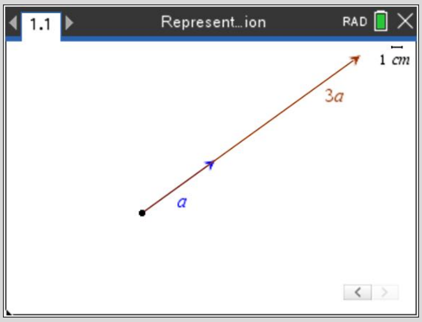
- Display vector $-\underset{\sim}{a}$ when $\boldsymbol{diln} = \mathbf{-1}$.

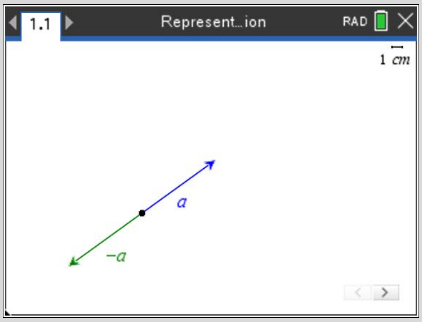
To hide all points except the starting point for $\underset{\sim}{a}$:

- Hover over each point and press ctrl menu > **Hide**.

The following screenshots show the page when $\boldsymbol{diln} = \mathbf{2}$, $\boldsymbol{diln} = \mathbf{3}$ and $\boldsymbol{diln} = \mathbf{-1}$ respectively.







Calculating the magnitude and direction of a vector

A position vector in two dimensions can be represented using ordered pair notation (x, y) and column vector notation $\begin{pmatrix} x \\ y \end{pmatrix}$. The magnitude and direction of a vector are defined respectively as:

$$|\underset{\sim}{a}| = \left|\begin{pmatrix} x \\ y \end{pmatrix}\right| = \sqrt{x^2 + y^2}. \qquad \tan(\theta) = \frac{y}{x},\ x \neq 0.$$

Question

Raghu cycled 28 km north from A to B, then 19 km east from B to C and finally 12 km south from C to D.

(a) Find $\overrightarrow{AD}$.

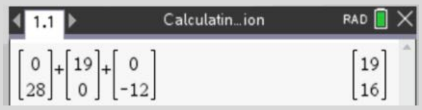
(b) Find $\left|\overrightarrow{AD}\right|$, giving your answer correct to the nearest tenth of a km.

(c) Find the direction of $\overrightarrow{AD}$, giving your answer as a true bearing correct to the nearest degree.

Solution

Parts **(a)**, **(b)** and **(c)** on a **Calculator** page:

- Press [ ] [5], select the **2-by-1 Matrix** template and enter as shown.



Note: *Alternatively, press [ ] to access this template.*

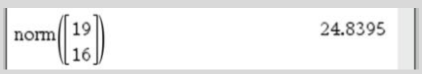
Answer: (a) $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$.

$$\overrightarrow{AD} = \begin{pmatrix} 0 \\ 28 \end{pmatrix} + \begin{pmatrix} 19 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -12 \end{pmatrix} = \begin{pmatrix} 19 \\ 16 \end{pmatrix}$$

Note: *Here, column vectors are used because there is enough space on the screen to display all the calculations. In any subsequent examples where this is not possible, row vectors will be used.*

(b) To find $\left|\overrightarrow{AD}\right|$:

- Press [menu] > **Matrix & Vector** > **Norms** > **Norm**.
- Press ▲ to select the column vector and press [enter].
- Press [ctrl] [enter] to obtain a decimal magnitude.



Answer: (b) $\left|\overrightarrow{AD}\right| = \left|\begin{pmatrix} 19 \\ 16 \end{pmatrix}\right| = \sqrt{19^2 + 16^2} = 24.8394...$

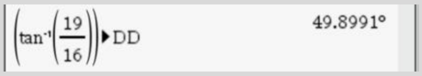
$\left|\overrightarrow{AD}\right| = 24.8$ (km)

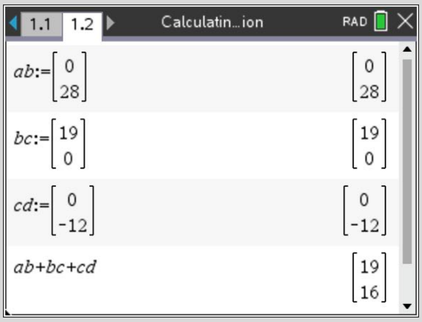
... continued

Solution (continued)

(c) To find the direction of $\overrightarrow{AD}$:

- Press [trig] and enter as shown.
- Press [📖] [1] [D], scroll down and select ►DD.





Answer: (c) $\tan^{-1}\left(\frac{19}{16}\right) = 49.8990...°$ and so the bearing is $050°T$.

Note: The **Assign** command can be used to assign vectors as shown at right.

Using vectors in Cartesian form

A unit vector, $\hat{\underset{\sim}{n}}$, in the plane is given by $\hat{\underset{\sim}{n}} = \frac{\underset{\sim}{n}}{|\underset{\sim}{n}|}$.

Vectors in Cartesian form are expressed using the unit perpendicular vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

Question

A unit vector in the direction of $3\underset{\sim}{i} - 2\underset{\sim}{j}$ is equal to

A. $\frac{1}{13}\left(3\underset{\sim}{i} - 2\underset{\sim}{j}\right)$

B. $\frac{1}{5}\left(3\underset{\sim}{i} - 2\underset{\sim}{j}\right)$

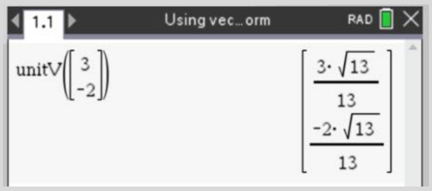
C. $\frac{1}{\sqrt{13}}\left(3\underset{\sim}{i} - 2\underset{\sim}{j}\right)$

D. $-\frac{1}{\sqrt{13}}\left(3\underset{\sim}{i} - 2\underset{\sim}{j}\right)$

Solution

On a **Calculator** page:

- Press [menu] > **Matrix & Vector** > **Vector** > **Unit Vector**.
- Press [📖] [5], select the **2-by-1 Matrix** template and enter as shown.



A unit vector in the direction of $3\underset{\sim}{i} - 2\underset{\sim}{j}$ is $\frac{1}{\sqrt{13}}\left(3\underset{\sim}{i} - 2\underset{\sim}{j}\right)$.

Answer: Option **C**.

Option **D**, $-\frac{1}{\sqrt{13}}\left(3\underset{\sim}{i} - 2\underset{\sim}{j}\right)$ is a unit vector in the opposite direction.

Using vectors in Cartesian form and polar form

In polar form, a vector in the plane is expressed using the notation (r,θ).

- $\underset{\sim}{u} = x\underset{\sim}{i} + y\underset{\sim}{j}$ where $x = r\cos(\theta)$ and $y = r\sin(\theta)$.
- $r = \sqrt{x^2 + y^2}$ and $\tan(\theta) = \dfrac{y}{x}$, $x \neq 0$.

Question

(a) Convert $\underset{\sim}{u} = (5, 30°)$ to Cartesian form.

(b) Convert $\underset{\sim}{v} = \underset{\sim}{i} - 3\underset{\sim}{j}$ to polar form, giving the angle correct to the nearest tenth of a degree.

Solution

Note: *In* ***Document Settings > Real or Complex*** *(accessed by pressing* [on]*), there is a choice to set the TI-Nspire CX II CAS to either* ***Real*** *or* ***Rectangular*** *or* ***Polar*** *mode. In this example, TI-Nspire CX II CAS was set to* ***Rectangular*** *mode and* ***Radian*** *mode.*

Parts **(a)** and **(b)** on a **Calculator** page:

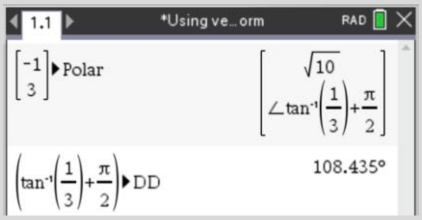
- Press [templates] [5] to select the **2-by-1 Matrix** template.
- Press [catalog] [1] [Z], scroll down and select the angle symbol.
- Press [π▸] to access the degree symbol.
- Enter as shown.

Answer: (a) $\underset{\sim}{u} = (5, 30°)$.

$$\underset{\sim}{u} = 5\cos(30°)\underset{\sim}{i} + 5\sin(30°)\underset{\sim}{j} = \frac{5\sqrt{3}}{2}\underset{\sim}{i} + \frac{5}{2}\underset{\sim}{j}.$$

On a **Calculator** page:

- Press [templates] [5], select the **2-by-1 Matrix** template and enter as shown.
- Press [menu] > **Number > Complex Number Tools > Convert to Polar**.
- Copy and paste the exact angle (in radians) to a new entry line.
- Press [catalog] [1] [D], scroll down and select ►DD.
- Press [ctrl] [enter] to obtain a decimal angle in degrees.



Answer: (b) $r = \sqrt{1^2 + (-3)^2} = \sqrt{10}$.

$$\begin{aligned}\theta &= \tan^{-1}(-3)\\ &= 180° - \tan^{-1}(3)\\ &= 180° - 71.5650...°\\ &= 108.434...°\end{aligned}$$

In polar form, $\underset{\sim}{v} = (\sqrt{10}, 108.4°)$, where θ is correct to the nearest tenth of a degree.

Using position vectors in Cartesian form

Question

The position vectors of points A and B are given by $\overrightarrow{OA} = \underset{\sim}{i} + 3\underset{\sim}{j}$ and $\overrightarrow{OB} = 5\underset{\sim}{i} - \underset{\sim}{j}$.

Find the exact distance between points A and B.

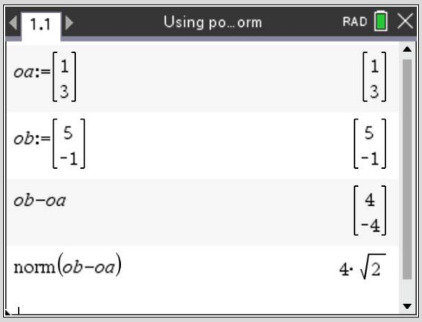
Solution

On a **Calculator** page, assign $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as follows:

- Press [ctrl] [template] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **2-by-1 Matrix** template and enter as shown.

To find the exact distance between points A and B:

- Press [menu] > **Matrix & Vector** > **Norms** > **Norm**.
- Enter as shown.



$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \left(5\underset{\sim}{i} - \underset{\sim}{j}\right) - \left(\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

$= 4\underset{\sim}{i} - 4\underset{\sim}{j}$

$\left|\overrightarrow{AB}\right| = \sqrt{4^2 + (-4)^2}$

$= 4\sqrt{2}$

Answer: The exact distance between A and B is $4\sqrt{2}$.

***Note:** Press [var] to access assigned/stored variables.*

Using the scalar (dot) product to find the angle between two vectors

The scalar (dot) product is defined as:

- $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$
- $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2$

Question

Points A, B and C are defined by the position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively, where

$\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j}$, $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{c} = \underset{\sim}{i} - 2\underset{\sim}{j}$.

Find the angle in radians between $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

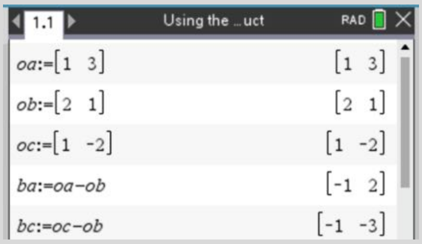
Solution

On a **Calculator** page, assign $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ as row vectors as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **1-by-2 Matrix** template and enter as shown.

Assign $\overrightarrow{BA}$ and $\overrightarrow{BC}$ as row vectors as follows:

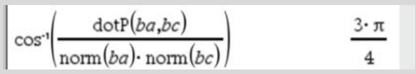
- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **1-by-2 Matrix** template and enter as shown.



***Note:** Press [var] to access assigned/stored variables.*

To determine the angle, θ, between $\overrightarrow{BA}$ and $\overrightarrow{BC}$:

- Press [trig] and select **cos^{-1}**.
- Press [ctrl] [÷] to access the **Fraction** template.
- Press [menu] > **Matrix & Vector** > **Vector** > **Dot Product**.
- Enter the numerator as shown.
- Press [menu] > **Matrix & Vector** > **Norms** > **Norm**.
- Enter the denominator as shown.

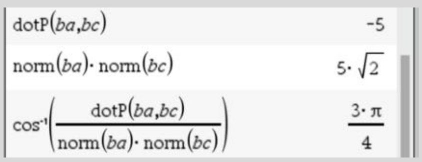


Answer: The angle between $\overrightarrow{BA}$ and $\overrightarrow{BC}$ is $\frac{3\pi}{4}$.

$$\overrightarrow{BA}\cdot\overrightarrow{BC}=\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|\cos(\theta)$$

$$\begin{aligned}\theta&=\cos^{-1}\left(\frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|}\right)\\&=\cos^{-1}\left(\frac{\left(-\underset{\sim}{i}+2\underset{\sim}{j}\right)\cdot\left(-\underset{\sim}{i}-3\underset{\sim}{j}\right)}{\left|-\underset{\sim}{i}+2\underset{\sim}{i}\right|\left|-\underset{\sim}{i}-3\underset{\sim}{j}\right|}\right)\\&=\cos^{-1}\left(\frac{1-6}{\sqrt{5}\times\sqrt{10}}\right)\\&=\cos^{-1}\left(-\frac{5}{5\sqrt{2}}\right)\\&=\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)\\&=\frac{3\pi}{4}\end{aligned}$$

***Note:** Instead of performing all the steps at once on TI-Nspire CX II CAS, it is a good teaching idea to show the required steps one at a time as shown at right.*



Using the scalar (dot) product to determine when two vectors are perpendicular

- $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$ if and only if . $\underset{\sim}{a} = \underset{\sim}{0}$. or $\underset{\sim}{b} = \underset{\sim}{0}$ or $\underset{\sim}{a} \perp \underset{\sim}{b}$.

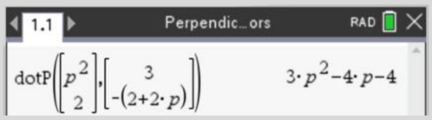
Question

Find the value(s) of p for which the vectors $\underset{\sim}{u} = p^2\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{v} = 3\underset{\sim}{i} - (2+2p)\underset{\sim}{j}$ are perpendicular.

Solution

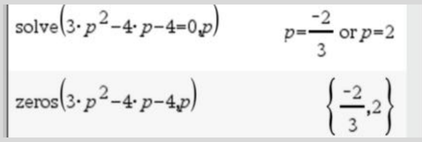
To determine the scalar (dot) product on a **Calculator** page:

- Press [menu] > **Matrix & Vector** > **Vector** > **Dot Product**.
- Press [📖][5], select the **2-by-1 Matrix** template.
- Enter as shown.



To solve $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$ for p:

- Press [menu] > **Algebra** > **Solve**.
- Enter as shown.



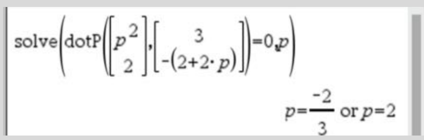
Note: *Alternatively, to obtain the values of p as a list, press* [menu] *>* ***Algebra*** *>* ***Zeros****.*

Answer: $p = -\frac{2}{3}$ or 2.

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left(p^2\underset{\sim}{i} + 2\underset{\sim}{j}\right) \cdot \left(3\underset{\sim}{i} - (2+2p)\underset{\sim}{j}\right)$$
$$= 3p^2 - 2(2+2p)$$
$$= 3p^2 - 4p - 4$$

$$3p^2 - 4p - 4 = 0$$
$$(3p+2)(p-2) = 0$$
$$p = -\frac{2}{3}, 2$$

Note: *The* ***Dot Product*** *and* ***Solve*** *commands can be combined or 'nested' as shown. Make clear to students that the inner command is performed before the outer command. It is a good idea to show this approach after having shown the required steps one at a time.*



Finding the vector projection of one vector onto another

The scalar projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ is defined as:

- $|\underset{\sim}{a}|\cos(\theta) = \underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$

The vector projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ is defined as:

- $|\underset{\sim}{a}|\cos(\theta)\hat{\underset{\sim}{b}} = \left(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}} = \left(\dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\right)\underset{\sim}{b}$

Question

Find the vector projection of $\underset{\sim}{a} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$ onto $\underset{\sim}{b} = -\underset{\sim}{i} + 3\underset{\sim}{j}$.

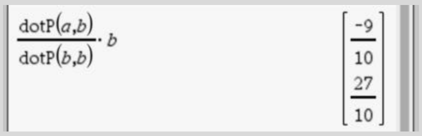
Solution

On a **Calculator** page, assign $\underset{\sim}{a}$ and $\underset{\sim}{b}$ as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [book] [5], select the **2-by-1 Matrix** template and enter as shown.

To determine the vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$:

- Press [ctrl] [÷] to access the **Fraction** template.
- Press [menu] > **Matrix & Vector** > **Vector** > **Dot Product**.
- Enter the numerator and denominator as shown.
- Press ▶ [×] and enter as shown.



Answer: The vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $\begin{pmatrix} -\dfrac{9}{10} \\ \dfrac{27}{10} \end{pmatrix}$.

Let $\underset{\sim}{u}$ be the vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

$$\begin{aligned}
\underset{\sim}{u} &= \left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\right)\underset{\sim}{b} \\
&= \left(\frac{-3+12}{1+9}\right)\left(-\underset{\sim}{i} + 3\underset{\sim}{j}\right) \\
&= \frac{9}{10}\left(-\underset{\sim}{i} + 3\underset{\sim}{j}\right) \\
&= -\frac{9}{10}\underset{\sim}{i} + \frac{27}{10}\underset{\sim}{j}
\end{aligned}$$

Modelling and solving problems with vectors

Question

Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors in the east and north directions respectively.

Po leaves her base camp at point O and walks on flat terrain for 4 km in a NE direction to point A.
She then walks a further 6 km on a true bearing of $300°$ to point B.
If Po then walks directly back to O, find an expression in exact form in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$ for the vector that describes her final path.

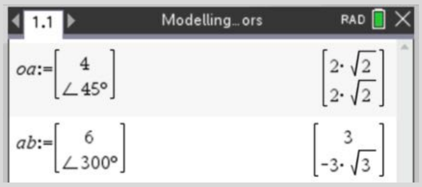
Solution

***Note:** In **Document Settings > Real or Complex** (accessed by pressing [on]), there is a choice to set the TI-Nspire CX II CAS to either **Real** or **Rectangular** or **Polar** mode.*
*In this example, TI-Nspire CX II CAS was set to **Rectangular** mode and **Radian** mode.*

In polar form, $\overrightarrow{OA} = (4, 45°)$ and $\overrightarrow{AB} = (6, 300°)$.

On a **Calculator** page, assign $\overrightarrow{OA}$ and $\overrightarrow{AB}$ as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [book] [5], select the **2-by-1 Matrix** template.
- Press [book] [1] [Z], scroll down and select the angle symbol.
- Press [π▸] to access the degree symbol.
- Enter as shown.



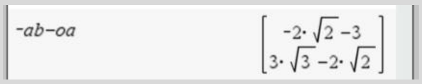
$\overrightarrow{OA} = \begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix}$ and $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -3\sqrt{3} \end{pmatrix}$.

***Note:** Press [var] to access assigned/stored variables.*

Po's final path is described by the vector $\overrightarrow{BO}$.

$\overrightarrow{BO} = \overrightarrow{BA} + \overrightarrow{AO} = -\overrightarrow{AB} - \overrightarrow{OA}$

Enter $-\overrightarrow{AB} - \overrightarrow{OA}$ as shown:



Answer: $\overrightarrow{BO} = \left(-3 - 2\sqrt{2}\right)\underset{\sim}{i} + \left(3\sqrt{3} - 2\sqrt{2}\right)\underset{\sim}{j}$

Resolving into $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components:

$\overrightarrow{OA} = 4\cos(45°)\underset{\sim}{i} + 4\sin(45°)\underset{\sim}{j} = 2\sqrt{2}\left(\underset{\sim}{i} + \underset{\sim}{j}\right)$

$\overrightarrow{AB} = 6\cos(300°)\underset{\sim}{i} + 6\sin(300°)\underset{\sim}{j} = 3\left(\underset{\sim}{i} - \sqrt{3}\underset{\sim}{j}\right)$

Po's final path is described by the vector $\overrightarrow{BO}$.

$\overrightarrow{BO} = \overrightarrow{BA} + \overrightarrow{AO} = -\overrightarrow{AB} - \overrightarrow{OA}$

$$\overrightarrow{BO} = -3\left(\underset{\sim}{i} - \sqrt{3}\underset{\sim}{j}\right) - 2\sqrt{2}\left(\underset{\sim}{i} + \underset{\sim}{j}\right)$$
$$= \left(-3 - 2\sqrt{2}\right)\underset{\sim}{i} + \left(3\sqrt{3} - 2\sqrt{2}\right)\underset{\sim}{j}$$

Proving the midsegment theorem for a triangle using vectors

Question

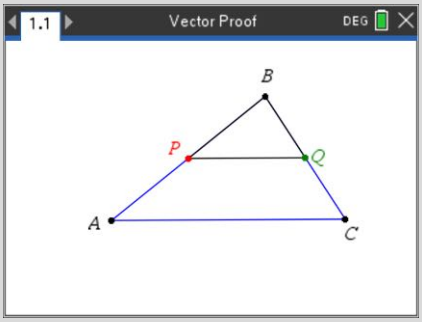
(a) Use vectors to prove that the line segment connecting the midpoints of two sides of an arbitrary triangle is parallel to the third side and half its length.

(b) Use appropriate Geometry tools to visually verify the theorem.

Solution

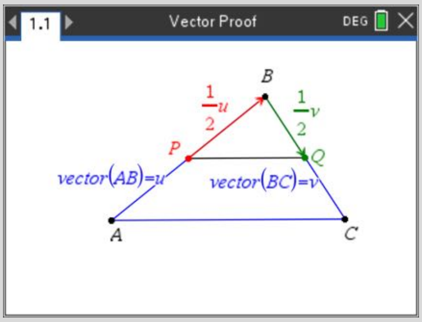
To visually illustrate this context, on a **Geometry** page.

- Press [menu] > **Settings** > **Automatically label points** and enable this feature.
- Press [menu] > **Shapes** > **Triangle**. Click three points on the workspace to form triangle *ABC*.
- Press [menu] > **Construction** > **Midpoint**. Click on each side *AB* and *BC*, then press [esc] to exit the tool.
- Label the midpoints as *P* and *Q* by hovering over the point, press [ctrl] [menu] > **Label**, enter label ***P*** then ***Q***.
- Press [menu] > **Points & Lines** > **Segment**. Construct $\overline{PQ}$.



To complete the diagram:

- Press [menu] > **Points & Lines** > **Vector**. Construct $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{PB}$ and $\overrightarrow{BQ}$. Press [esc] to exit the tool.
- Hover over $\overrightarrow{PB}$, press [ctrl] [menu] > **Label**. Enter label $\frac{1}{2}\boldsymbol{u}$.
- Similarly, label $\overrightarrow{BQ}$ as $\frac{1}{2}\boldsymbol{v}$. Label $\overrightarrow{AB}$ and $\overrightarrow{BC}$ as shown.



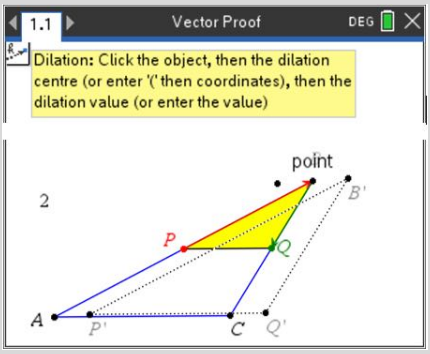
(a) Answer: To prove that $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AC}$:

$$\overrightarrow{PQ} = \overrightarrow{PB} + \overrightarrow{BQ} = \frac{1}{2}(\underset{\sim}{u} + \underset{\sim}{v}) \text{ and } \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = (\underset{\sim}{u} + \underset{\sim}{v})$$

$\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AC}$ as required.

To use the **Dilation** tool to visually verify that triangle *ABC* is a dilation of triangle *PBQ* by a factor of 2:

- Press [menu] > **Shapes** > **Triangle**. Construct ΔPBQ, colour as required and hide unwanted labels.
- Press [menu] > **Transformation** > **Dilation**.
- Click on triangle *PBQ*, then press [2] [enter] (dilation factor of 2), and then move the cursor to point *B*.
- Press [enter] to superimpose the dilation of triangle *PBQ* over triangle *ABC*. The image of *PBQ* is shown as $P'B'Q'$.



Answer: Dragging any vertex shows that the property holds.

Proving Thales' inscribed semicircle theorem using vectors

Question

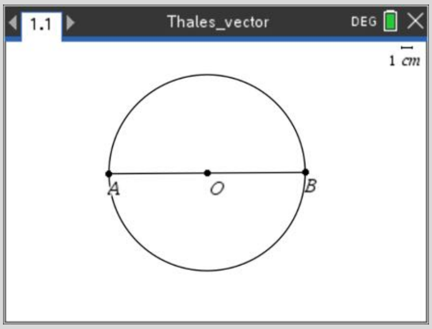
Thales' theorem states that an angle inscribed in a semicircle is a right angle.

(a) Visually verify this theorem in the Geometry application.

(b) Use vector properties to formally prove the theorem.

Solution

(a) To draw a diameter and circle, on a **Geometry** page:

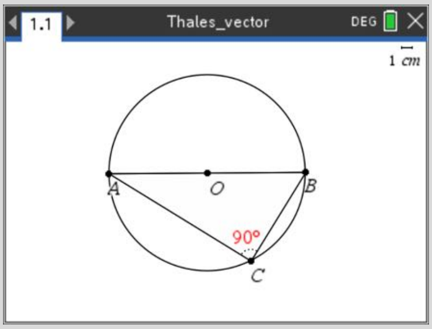


- Press [menu] > **Settings** > **Automatically label points** and enable this feature.
- Press [menu] > **Points & Lines** > **Segment**. Click at two distinct points. The labels ***A***, ***B*** will appear automatically.
- Press [menu] > **Construction** > **Midpoint**. Click $\overline{AB}$ **.**

Note: *To exit any **Geometry** tool, press [esc] and continue.*

- Hover over midpoint, press [ctrl][menu] > **Label**. Enter ***O***.
- Press [menu] > **Shapes > Circle**. Click point ***O*** then point ***A***.

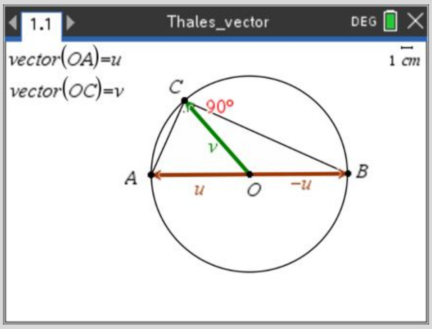
To draw the inscribed triangle and measure the angle:



- Press [menu] > **Shapes > Triangle**. Click points ***A*** and ***B***, then a third point on the circumference. Press [esc] to exit.
- Press [menu] > **Settings**. Set measurement angle to **Degree**.
- Press [menu] > **Measurement > Angle**. Click in turn points ***A***, ***C*** and ***B*** to measure $\measuredangle ACB$.

Answer: Dragging point C around the circle shows that $\measuredangle ACB$ is invariable at 90°.

(b) To construct vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ with the vector tool:



- Press [menu] > **Points and Lines** > **Vector**.
- Use the **Label** tool to label the vectors as shown.

Answer: To prove $\overrightarrow{AC} \perp \overrightarrow{BC}$:

$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = (-\underset{\sim}{u} + \underset{\sim}{v})$ and $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = (\underset{\sim}{u} + \underset{\sim}{v})$

$\overrightarrow{AC} \cdot \overrightarrow{BC} = (-\underset{\sim}{u} + \underset{\sim}{v}) \cdot (\underset{\sim}{u} + \underset{\sim}{v}) = -\underset{\sim}{u} \cdot \underset{\sim}{u} - \underset{\sim}{u} \cdot \underset{\sim}{v} + \underset{\sim}{v} \cdot \underset{\sim}{u} + \underset{\sim}{v} \cdot \underset{\sim}{v}$

$\overrightarrow{AC} \cdot \overrightarrow{BC} = |\underset{\sim}{v}|^2 - |\underset{\sim}{u}|^2 = 0$ because $|\underset{\sim}{v}| = |\underset{\sim}{u}|$ =radius.

Therefore $\overrightarrow{AC} \perp \overrightarrow{BC}$. Angle ACB is a right angle, as required.

2.3 Algebra, number and structure

2.3.1 Complex numbers

Introducing the number i with the property $i^2 = -1$

Question

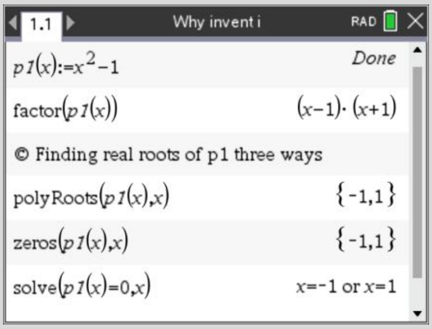
Find the linear factors, roots and discriminants of the following quadratic polynomials.

(a) $p_1(x) = x^2 - 1$, **(b)** $p_2(x) = x^2 + 1$. Interpret the results in each case.

Solution

(a) To find the linear factors of $p_1(x)$, on a **Calculator** page:

- Enter $\boldsymbol{p1(x) := x^2 - 1}$, then press menu > **Algebra** > **Factor** and enter **factor(*p*1(*x*), *x*).**



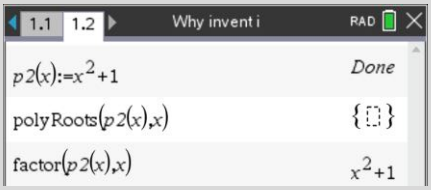
To find real roots of $p_1(x)$ using three different approaches:

- Press menu > **Algebra** > **Polynomial Tools** > **Real Roots of Polynomial.** Enter **polyRoots(*p*1(*x*), *x*).**
- Press menu > **Algebra** > **Zeros**. Enter **zeros(*p*1(*x*), *x*).**
- Press menu > **Algebra** > **Solve**. Enter **solve(*p*1(*x*) = 0, *x*).**

Answer: Real factors: $(x-1)(x+1)$ and roots $x \in \{-1, 1\}$

(b) To find **real** roots of $p_2(x)$, assign $\boldsymbol{p2(x) := x^2 + 1}$ then:

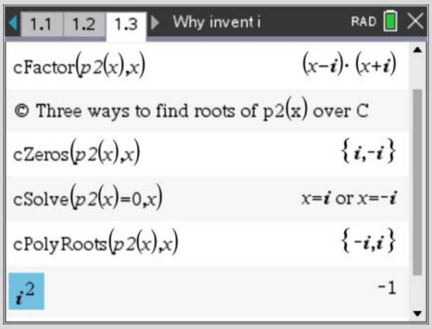
- Press menu > **Algebra** > **Polynomial Tools** > **Real Roots of Polynomial**. Enter **polyRoots(*p*2(*x*), *x*)**.
- Press menu > **Algebra** > **Factor** and enter **factor(*p*2(*x*), *x*).**



Answer: $p_2(x) = x^2 + 1$ has no roots or linear factors over *R*.

To find linear factors of $p_2(x)$ over the complex field, *C*:

- Press menu > **Algebra** > **Complex** > **Factor** and enter **cFactor(*p*2(*x*), *x*).**



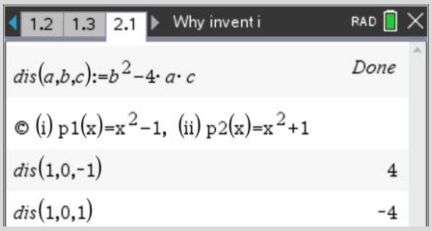
To find roots of $p_2(x)$ over the complex field, *C*:

- Press menu > **Algebra** > **Complex** > **Zeros** or **…** > **Solve** and enter **cZeros(*p*2(*x*), *x*)** or **cSolve(*p*2(*x*) = 0, *x*).**
- Press menu > **Algebra** > **Polynomial Tools** > **Complex Roots of Polynomial**. Enter **cPolyRoots(*p*2(*x*), *x*)**.
- Enter $\boldsymbol{i^2}$, pressing π▸ to select the special number ***i***.

Answer: $x^2 + 1 = (x-i)(x+i)$. Roots $x \in \{-i, i\}$, $i^2 = -1$.

Introducing *i* extended the number system, allowing polynomial equations with no real solutions to be solved.
To find the discriminant of **(a)** $p1(x)$ and **(b)** $p2(x)$:

- Enter $\mathbf{dis}(\boldsymbol{a,b,c}) := \boldsymbol{b^2 - 4a \cdot c}$, then enter as shown.



Answer: (a) $\Delta_{p1(x)} = 4 > 0$, real roots. **(b)** $\Delta_{p2(x)} = -4 < 0$.

Applying complex conjugates

Question

Let $a, b \in R$ and $i^2 = -1$. If $z = a + bi$, then the conjugate, $\overline{z}$, is given by $\overline{z} = a - bi$.

(a) Given $z = -3 + 2i$ find the multiplicative inverse $z^{-1} = \dfrac{1}{z}$ and verify that $z^{-1} = \dfrac{\overline{z}}{z\overline{z}}$.

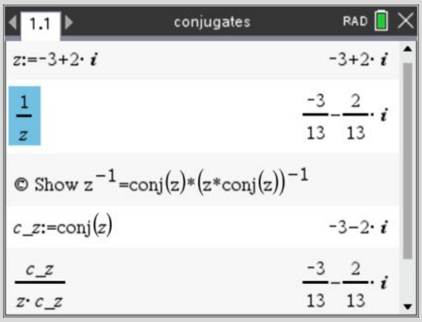
(b) Given $z_1 = 2 - 3i$ and $z_2 = 3 + 4i$ express the following in the form $x + yi$.

(i) $z_1 + \overline{z}_2$ **(ii)** $\overline{z}_1\overline{z}_2$ **(iii)** $\overline{z_1 z_2}$

(c) Verify that $z_2\overline{z}_2$ is a real number and show that $\dfrac{z_1}{z_2} = \dfrac{z_1\overline{z}_2}{z_2\overline{z}_2}$.

Solution

(a) To find the multiplicative inverse of $z = -3 + 2i$ and verify that $z^{-1} = \dfrac{\overline{z}}{z\overline{z}}$, on a **Calculator** page:



- Enter $\boldsymbol{z := -3 + 2i}$ (using [π▸] to select $\boldsymbol{i}$), then enter $\dfrac{\boldsymbol{1}}{\boldsymbol{z}}$.
- For the conjugate command, press [menu] > **Number > Complex Number Tools > Complex Conjugate.**
- Using this command, enter $\boldsymbol{c\_z := \text{conj}(z)}$.
- Enter $\dfrac{\boldsymbol{c\_z}}{\boldsymbol{z \cdot c\_z}}$.

Answer: $\dfrac{1}{z} = \dfrac{\overline{z}}{z\overline{z}} = \dfrac{-(3+2i)}{13}$, hence the result is verified.

(b) To express these complex numbers in the form $x + iy$, open a new **Problem** and add a **Calculator** page.

- Enter $\boldsymbol{z1 := 2 - 3i}$, then enter $\boldsymbol{z2 := 3 + 4i}$.

(i) To evaluate $z_1 + \overline{z}_2$, enter $\boldsymbol{z1 + \text{conj}(z2)}$

Answer: $z_1 + \overline{z}_2 = 5 - 7i.$

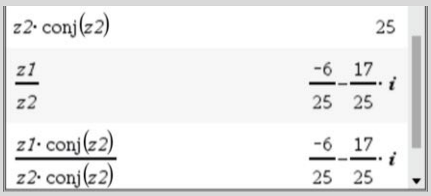
To evaluate **(ii)** $\overline{z}_1\overline{z}_2$ and **(iii)** $\overline{z_1 z_2}$:

- Enter $\boldsymbol{\text{conj}(z1) \times \text{conj}(z2)}$ and $\boldsymbol{\text{conj}(z1 \times z2)}$.

Answer: $\overline{z}_1\overline{z}_2 = \overline{z_1 z_2} = 18 + i$.

(iv) To evaluate $z_2\overline{z}_2$, $\dfrac{z_1}{z_2}$ and $\dfrac{z_1\overline{z}_2}{z_2\overline{z}_2}$:

- Enter the following $\boldsymbol{z2 \cdot \text{conj}(z2)}$, $\dfrac{\boldsymbol{z1}}{\boldsymbol{z2}}$ and $\dfrac{\boldsymbol{z1 \cdot \text{conj}(z2)}}{\boldsymbol{z2 \cdot \text{conj}(z2)}}$.



Answer: $(z_2\overline{z}_2 = 25) \in R$, $\dfrac{z_1}{z_2} = \dfrac{z_1\overline{z}_2}{z_2\overline{z}_2} = \dfrac{-(6+17i)}{25}$.

Solving quadratic equations over C and the conjugate root theorem

Question

(a) Set up an editable Notes page to find the discriminant and roots of quadratic polynomials over C. Test the page using **(i)** $p_1(z)=z^2-4z+8$ **(ii)** $p_2(z)=2z^2-2z+1$ **(iii)** $p_3(z)=3z^2+3z+5$ and interpret how the pairs of roots in each case are related to one another.

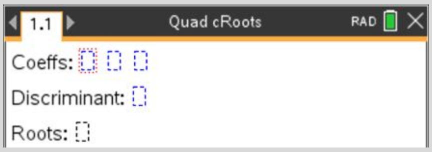
(b) Determine the value of k, where $k>0$, if: **(i)** $\frac{k}{2}-3i$ is a root of $p_4(z)=z^2-kz+21$,

(ii) $1-ki$ is a root of $p_5(z)=z^2-bz+3b$, where $3b=k^2+1$.

Solution

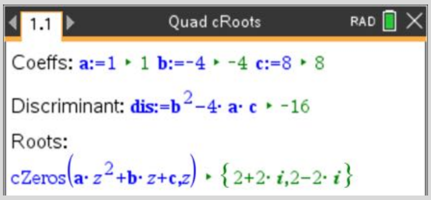
(a) (i) To set up the interactive template, on a **Notes** page:

- Enter the labels, **Coeffs:**, **Discriminant:** and **Roots:**.
- Press [ctrl] [M] to insert **Maths Boxes**, as shown.



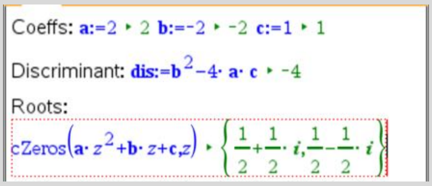
- In the **Maths Boxes** next to **Coeffs:**, enter ***a*:= 1**, ***b*:= –4**, ***c*:= 8**. Similarly, beside the other two labels, enter ***dis*:= *b*²** **– 4×*a*×*c*** and **cZeros(*a*×*z*² + *b*×*z* + *c*, *z*)**, as shown.

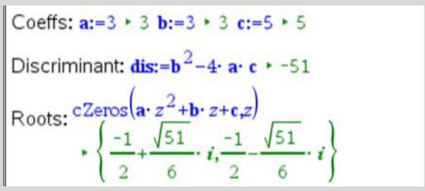
Note: *For a quick way to select **cZeros**, press* [📖] [1] [D].



To find the discriminant and roots of **(ii)** $p(z)=2z^2-2z+1$ and **(iii)** $p(z)=3z^2+3z+5$:

- Edit the values of parameters ***a***, ***b*** and ***c***, as shown, ensuring that [enter] is pressed after each value is changed.





Answer: (i) $p_1(z)=z^2-4z+8$. $\Delta_{p_1}=-16$, roots: $2\pm 2i$

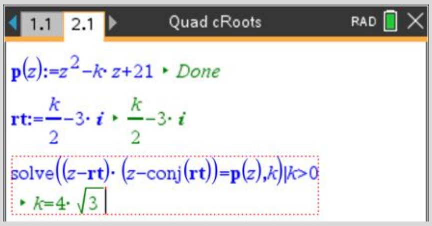
(ii) $p_2(z)=2z^2-2z+1$. $\Delta_{p_2}=-4$, roots: $\frac{1}{2}\pm\frac{1}{2}i$

(iii) $p_3(z)=3z^2+3z+5$. $\Delta_{p_3}=-51$, roots: $-\frac{1}{2}\pm\frac{\sqrt{51}}{6}i$.

In each case, the roots are conjugates of each other.

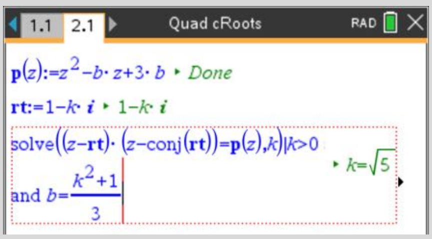
(b)(i) To find k if $(k/2)-3i$ is a root of $p_4(z)=z^2-kz+21$, in **Maths Boxes** on a **Notes** (or on a **Calculator**) page:

- Enter $\boldsymbol{p(z):=z^2-k\times z+21}$ and $\boldsymbol{rt:=k/2-3i}$.
- Enter **solve((*z* – *rt*)×(*z* – conj(*rt*)) = *p*(*z*), *k*)|*k* > 0**, pressing [menu] **> Calculations** then … **> Algebra** to select **Solve** and **… > Number > Complex Number Tools** for **conj**.



(ii) Edit to $\boldsymbol{p(z):=z^2-b\times z+3b}$ and $\boldsymbol{rt:=1-k\times i}$, then:

- **solve((*z* – *rt*)×(*z* – conj(*rt*))=*p*(*z*),*k*)|*k*>*0* and *b* = (*k*²+1)/3.**



Answer: (i) $k=4\sqrt{3}$, **(ii)** $k=\sqrt{5}$

Representing addition of complex numbers on the complex plane

Question

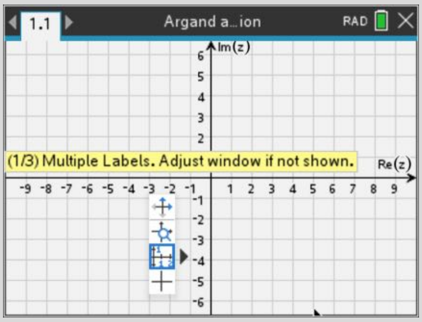
Consider $z_1 = 5 + 2i$ and $z_2 = 1 + 2i$. Show the following complex numbers on the complex plane:

(a) $z_1 + z_2$ **(b)** $z_1 + (-2z_2) = z_1 - 2z_2$

Solution

To set up the complex plane (Argand diagram) workspace, on a **Graphs** page:

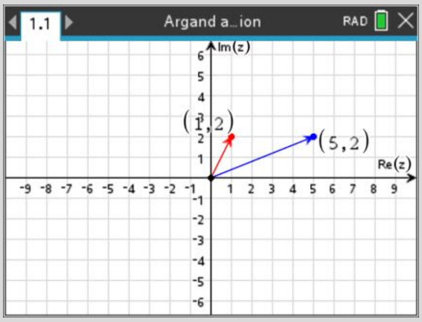
- Click on the axis label 'y' until it is editable. Edit label to $\mathbf{Im}(z)$. Similarly, edit the axis label 'x' to $\mathbf{Re}(z)$.
- Press [menu] > **View** > **Grid** > **Lined Grid**.
- Hover over an axis, press [ctrl][menu] > **Attributes**. Select the axes labels icon and then select **Multiple Labels**.



To illustrate $z_1 = 5 + 2i$ and $z_2 = 1 + 2i$, on a **Graphs** page:

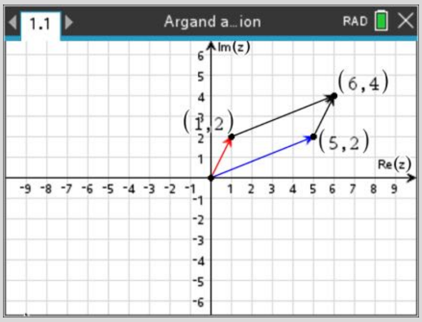
- Press [menu] > **Geometry** > **Points & Lines** > **Vector**.
- For z_1, click the origin then the grid point $(5, 2)$.
- Repeat for z_2, with grid point $(1, 2)$, then press [esc].
- Hover over the vector for z_1, press [ctrl][menu] > **Colour** and select **Blue**. Likewise, for the point z_1 also select **Blue**.
- Repeat as above, selecting **Red** for the z_2 vector and point.
- Hover over the point at $(5, 2)$, press [ctrl][menu] > **Coordinates and Equations**. Repeat for point at $(1, 2)$.

$z_1 + \bar{z}_2 = 5 - 7i$



(a) To illustrate $z_1 + z_2$, proceed as follows:

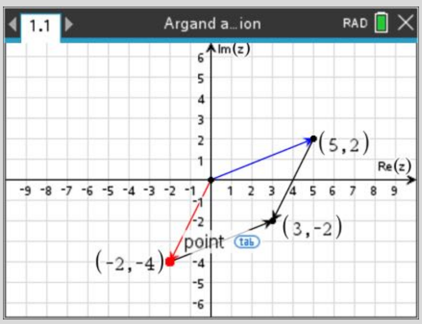
- Press [menu] > **Geometry** > **Transformation** > **Translation**.
- Click the blue vector followed by the red vector. Click the red vector followed by the blue vector and press [esc].
- Hover over the new point where the translated vectors intersect. Press [ctrl][menu] > **Coordinates and Equations**.



Answer: The translated vectors intersect at the point with coordinates $(6, 4)$, confirming that $z_1 + z_2 = 6 + 4i$.

(b) To illustrate $z_1 + (-2z_2) = z_1 - 2z_2$, proceed as follows:

- Hover over the point at the end of the red vector. Press [tab] to select the point. Grab ([ctrl][grab]) and move the point to coordinates $(-2, -4)$, then press [esc].



Answer: The vectors intersect at the point with coordinates $(3, -2)$, confirming that $z_1 + (-2z_2) = z_1 - 2z_2 = 3 - 2i$.

***Note**: Save this document in **MyWidgets** folder for future use.*

Illustrating complex conjugates on the complex plane

Question

(a) Using the saved 'Argand' widget from the previous section, represent the complex numbers $z_1 = 4+2i$, $\overline{z}_1 = 4-2i$, $z_2 = -4-3i$ and $\overline{z}_2 = -4+3i$ on the complex plane. Interpret the relationship between the location of a complex number and its complex conjugate.

(b) Illustrate $\overline{z}_1 + \overline{z}_2$ on the complex plane.

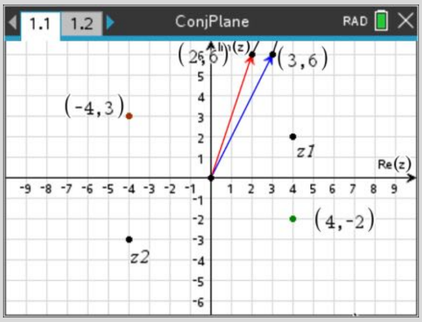
Solution

To open the **Widget** 'Argand' from the previous problem:

- Open a **New** document or press ctrl +page in an opened document. Select **Add Widget** then 'Argand'.
- Move the red and blue vectors out of the way.

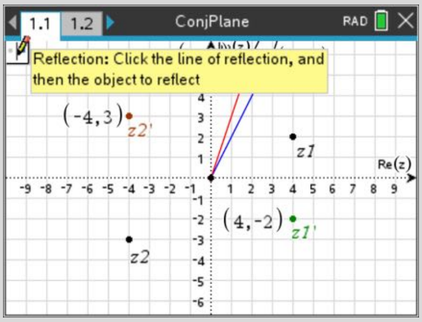
(a) To plot $z_1 = 4+2i$, $\overline{z}_1$ and $z_2 = -4+3i$, $\overline{z}_2$:

- Press P > **Point.** Click the grid points with coordinates $(4,2),(4,-2),(-4,-3)$ and $(-4,3)$. Press esc to exit.
- Hover over point at $(4,2)$, press ctrl menu > **Label** and enter the label $\boldsymbol{z1}$. Label the point at $(-4,-3)$ as $\boldsymbol{z2}$.



To confirm the relationship between the location of a complex number and its complex conjugate on the complex plane:

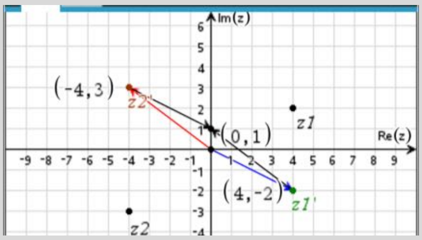
- Press menu > **Geometry** > **Transformation** > **Reflection**.
- Click on the horizontal axis, then click on point $\boldsymbol{z1}$. Repeat for point $\boldsymbol{z2}$, then press esc to exit this tool.



Answer: This illustrates that the complex conjugates, $\overline{z}_1$ and $\overline{z}_2$ are the images, $z_1{}'$ and $z_2{}'$, of z_1 and z_2 under a reflection in the horizontal axis.

(b) To Illustrate $\overline{z}_1 + \overline{z}_2$ on the complex plane:

- Move the tip of the blue vector to point $z_1{}'$ and the tip of the red vector to point $z_2{}'$.



Answer: The vectors intersect at $(0,1)$, $\overline{z}_1 + \overline{z}_2 = 0+1i = i$.

Visualising multiplication by i as a rotation in the complex plane

Question

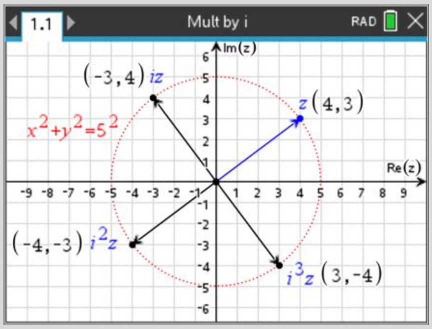
Consider $z=4+3i$. Illustrate on the complex plane the complex numbers z, iz, i^2z, i^3z, i^4z. Hence describe a linear transformation in the complex plane that is equivalent to multiplying a complex number by i.

Solution

Note: *This construction is best attempted using the TI-Nspire CX II CAS Teacher Software rather than on the handheld device.*

To illustrate $z=i^4z=4+3i$, $iz=-3+4i$, $i^2z=-4-3i$ and $i^3z=3-4i$ on the complex plane, set up a **Graphs** page as previously described, then proceed as follows:

- Press [menu] > **Geometry** > **Points & Lines** > **Vector**.
- To represent z, click on the origin, and then on the grid point $(4,3)$.
- Hover over the point at the tip of the vector, press [ctrl] [menu] > **Label** and enter the label z.
- Repeat the above for the complex numbers iz, i^2z, i^3z.

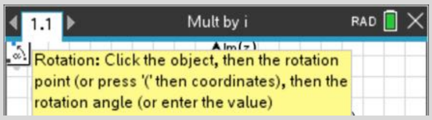


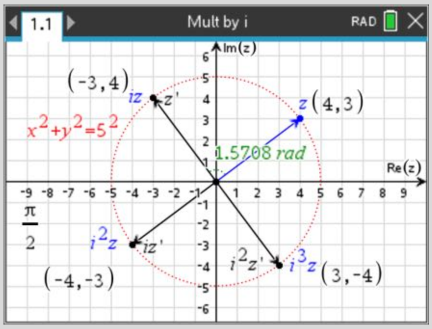
To show that these vectors are all of equal magnitude:

- Press [menu] > **Geometry** > **Shapes** > **Circle**.
- Click the origin then click the point z. Then press [esc].
- Hover over the circle, press [ctrl] [menu] > **Coordinates and Equations**; [ctrl] [menu] > **Attributes** > **Line style is dotted**.

To confirm the transformation that maps z to iz, iz to i^2z etc:

- Place the cursor in an empty part of the workspace, press [ctrl] [menu] > **Text** and enter $\pi/2$.
- Press [menu] > **Geometry** > **Transformation** > **Rotation**.
- Click on the vector to point z, then the origin (the rotation point), then the text $\frac{\pi}{2}$ (the rotation angle in radians).
- Repeat the previous procedure for the image of vectors to points iz, i^2z, i^3z under a rotation of $\frac{\pi}{2}^c$ or 90°.





Answer: Under the 90° anticlockwise rotation about O, iz is the image of z, i^2z is the image of iz, etc. This illustrates that multiplying a complex number by i is equivalent to a 90° anticlockwise rotation about the origin. This is consistent with the imaginary axis being set at right angles to the real axis. An important conclusion is that although i was introduced for algebraic reasons, it is evident that its true nature is geometric: it encodes rotation, something the real numbers cannot do. This makes complex numbers natural and powerful.

Introducing the modulus and polar angle of a complex number

Question

Let $z_1 = 2 + 2\sqrt{3}i$ and $z_2 = -5 - 5i$.

(a) Illustrate z_1 and z_2 on the complex plane. Measure the modulus and polar angle in each case.

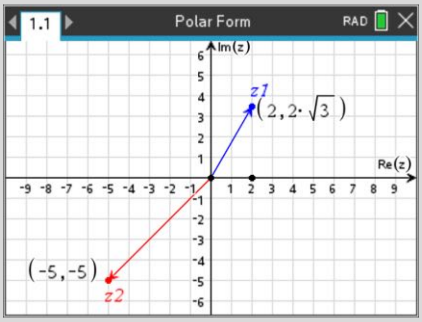
(b) Use complex number tools to find: **(i)** the modulus of z_1 and z_2, $|z_1| = \sqrt{z_1\bar{z}_1}$ and $|z_2| = \sqrt{z_2\bar{z}_2}$, **(ii)** $\text{Arg}(z_1)$ and $\text{Arg}(z_2)$, the polar angles. Compare these results with part **(a)** above.

Solution

(a) To illustrate z_1, z_2 on the complex plane, set up a **Graphs** page as shown and previously described, then:

- Press [menu] > **Geometry** > **Points & Lines** > **Vector**.
- For z_1, click the origin and then the workspace, but *not* on a grid point. Repeat for z_2 and press [esc] to exit.

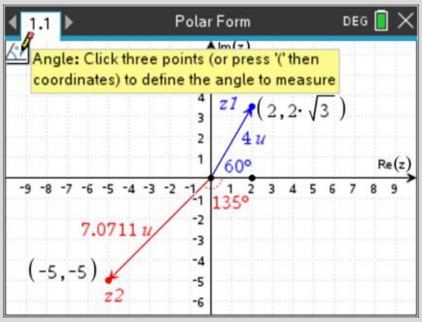
Note: To make the document fully editable for future use, avoid locking onto an integer grid point in the first instance.



- Hover over the point at the tip of the vector, press [ctrl][menu] > **Coordinates and Equations**. Edit the coordinates to $\left(2, 2\sqrt{3}\right)$ for z_1 and $(-3, -3)$ for z_2.
- Hover over a point and press [ctrl][menu] > **Label**. Enter the labels $\mathbf{z1}$ and $\mathbf{z2}$, as shown.

To measure the magnitudes of the vectors joining the origin to z_1, z_2, and the angles, in degrees, with the positive real axis:

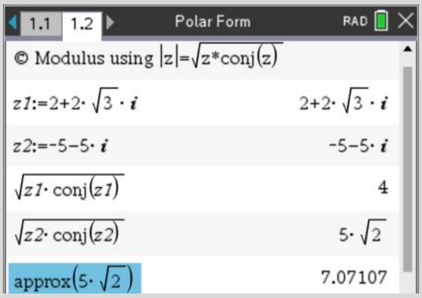
- Hover over the vector to z_1, press [ctrl][menu] > **Measurement** and select **Length**.
- Repeat for the vector to z_2.
- Select **DEG** by clicking the top right corner (**RAD/DEG**).
- Press [menu] > **Geometry** > **Measurement** > **Angle**.
- Click on a point on the positive real axis, then on the origin, then on the point z_1. Repeat for angle to z_2.



Answer: $|z_1| = 4,\ \theta_{z_1} = 60^\circ,\ |z_2| \approx 7.0711 = 5\sqrt{2},\ \theta_{z_2} = -135^\circ$.

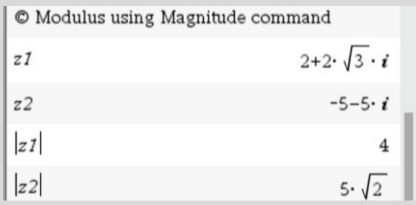
(b) To find $|z_1| = \sqrt{z_1\bar{z}_1}$, $|z_2| = \sqrt{z_2\bar{z}_2}$, on a **Calculator** page:

- Enter $\mathbf{z1 := 2 + 2\sqrt{3}i}$ and $\mathbf{z2 := -5 - 5i}$.
- Enter $\sqrt{\mathbf{z1 \cdot conj(z1)}}$ and $\sqrt{\mathbf{z2 \cdot conj(z2)}}$, pressing [menu] > **Number** > **Complex Number Tools** for **conj**.



To find $|z_1|$ and $|z_2|$ using the inbuilt **Magnitude** command, on a **Calculator** page with z_1, z_2 defined:

- Press [template key] and select the $|\square|$ template (alternatively, select **Magnitude** from the **Complex Number Tools** list).
- Enter $|\mathbf{z1}|$, then $|\mathbf{z2}|$.

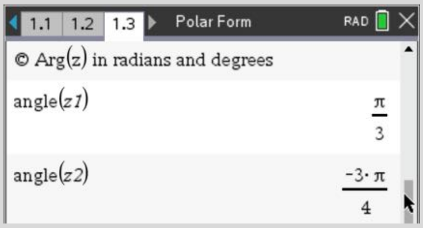


... continued

Solution (continued)

(ii) To find $\text{Arg}(z_1)$ and $\text{Arg}(z_2)$ in **RAD** and **DEG** modes:

- Click to toggle **RAD/DEG** at the top right. Select **RAD**.
- Press [menu] > **Number** > **Complex Number Tools** > **Polar Angle**, enter **angle**$(z1)$ then **angle**$(z2)$.
- Select **DEG**, press ▲ to select **angle**$(z1)$, then [enter] (or [ctrl][enter] when required). Repeat for **angle**$(z2)$.



To convert to DMS in any mode:

- Press ▲ to select **angle**$(z1)$. Press [📖][1][D], select ▶DMS then press [enter]. Repeat for **angle**$(z2)$.

Answer: (i) $|z_1| = 4, |z_2| = 5\sqrt{2} \approx 7.0711$, **(ii)** $\text{Arg}(z_1) = \frac{\pi}{3} = 60^\circ$
$\text{Arg}(z_1) = -\frac{3\pi}{4} = -135^\circ$, consistent with part **(a)** results.

Converting between rectangular and polar forms of a complex number

Question

Convert the following using various 'Real or Complex' document setting.

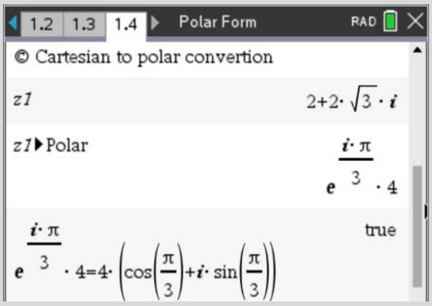
(a) $z_1 = 2+2\sqrt{3}i$ from rectangular to polar form, $z_1 = r(\cos(\theta)+i\sin(\theta)) = r\text{cis}(\theta)$.

(b) $z_2 = 5\sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right)$ from polar form to rectangular form, $z_2 = x+yi$.

Solution

(a) To convert z_1 to polar form, on a **Calculator** page, with z_1 defined and default (**Real**) document settings:

- Input $z1$, then press [📖][1][P]. Select ▶**Polar** and enter **z1▶Polar**. Verify the equivalence of the output, $4e^{i\pi/3}$, and $4\text{cis}(\pi/3)$, as shown.



To convert z_1 to polar form with **Polar** document setting:

- Click the battery icon (top right). Select **Document Settings** > **Real or Complex: Polar** and click **OK**.
- Enter $z1$ (if previously defined) or $2+2\sqrt{3}i$. In this setting, the output will be in polar form.
- Return the **Document Settings** to **Real** mode for part **(b)**.

Answer: $z_1 = 2+2\sqrt{3}i = 4\text{cis}\left(\frac{\pi}{3}\right)$ (or alternatively, $4e^{\frac{i\pi}{3}}$).

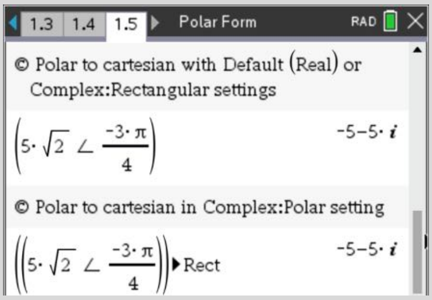
(b) To convert $z_2 = 5\sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right)$ to rectangular form, on a **Calculator** page, with either default **Real** or **Complex: Rectangular** document settings:

Note: $\left(\mathbf{5\sqrt{2}\angle -\frac{3\pi}{4}}\right)$ *is equivalent to entering* $5\sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right)$.

- Enter $\left(\mathbf{5\sqrt{2}\angle -3\pi/4}\right)$ by pressing [ctrl][📖]([∞β°]) to select ∠ symbol. Ensure the expression is in brackets, as shown.

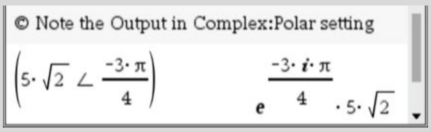
If the **Document Settings** are **Complex: Polar**, then:

- Input $\left(\mathbf{5\sqrt{2}\angle -3\pi/4}\right)$, then press [menu] > **Number** > **Complex Number Tools** > **Convert to Rectangular**.



Answer: $z_2 = 5\sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right) = -5-5i$.

***Note:** In **Complex: Polar** setting, the output of* $\left(\mathbf{5\sqrt{2}\angle -3\pi/4}\right)$ *is of the form* $\mathbf{5\sqrt{2}\,e^{-\frac{3\pi \cdot i}{4}}}$.



Performing arithmetic operations on complex numbers expressed in polar form

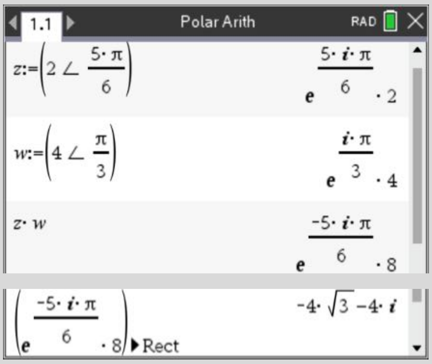
Question

If $z = 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$ and $w = 4\text{cis}\left(\frac{\pi}{6}\right)$, find expressions for the following in both polar and rectangular forms: **(a)** zw **(b)** $z \div w$ **(c)** z^3.

Solution

To assign z and w using the shorthand notation $(r\angle\theta)$, on a **Calculator** page with **Document Settings - Complex: Polar**:

Enter $\mathbf{z} := \left(2\angle\frac{5\pi}{6}\right)$, then $\mathbf{w} := \left(4\angle\frac{\pi}{6}\right)$, pressing [ctrl] [📖] ($\infty\beta°$) to select the $\angle$ symbol.

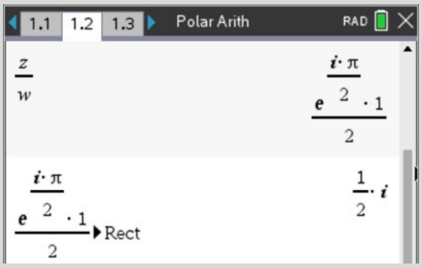


(a) To find an expression for zw in polar form:

- Enter $z \times w$. Then, to convert to rectangular form:
- Press [menu] > **Number** > **Complex Number Tools** > **Convert to Rectangular**, then press [enter].

Answer: $2\text{cis}\left(\frac{5\pi}{6}\right) \times 4\text{cis}\left(\frac{\pi}{3}\right) = 8\text{cis}\left(\frac{-5\pi}{6}\right) = -4\sqrt{3} - 4i$

(b) To find an expression for $z \div w$ in polar form:



- Enter z / w. Then, to convert to rectangular form:
- Press [menu] > **Number** > **Complex Number Tools** > **Convert to Rectangular**, then press [enter].

Answer: $\left(2\text{cis}\left(\frac{5\pi}{6}\right)\right) \div \left(4\text{cis}\left(\frac{\pi}{3}\right)\right) = \frac{1}{2}\text{cis}\left(\frac{\pi}{2}\right) = \frac{1}{2}i$

(c) To find an expression for z^3 in polar form:

- Enter z^3. Then, to convert to rectangular form:
- Press [menu] > **Number** > **Complex Number Tools** > **Convert to Rectangular**, then press [enter].

Answer: $\left(2\text{cis}\left(\frac{5\pi}{6}\right)\right)^3 = 8\text{cis}\left(\frac{\pi}{2}\right) = 8i$

Introducing subsets of the complex plane

Question

(a) Set up a radial grid with lines spaced at $\frac{\pi}{6}^c$.

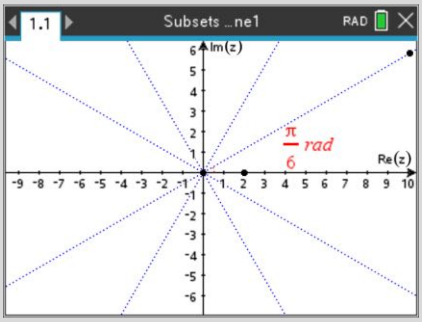
(b) On the radial grid from part **(a)**, illustrate the subsets of the complex plane defined by:

(i) $\text{Arg}(z) = \frac{\pi}{3}$ **(ii)** $\text{Arg}(z) \geq \frac{5\pi}{6}$ **(iii)** $|z| = 4$ **(iv)** $|z| = 2$

Solution

(a) To set up radial lines spaced at $\pi/6$, on a **Graphs** page:

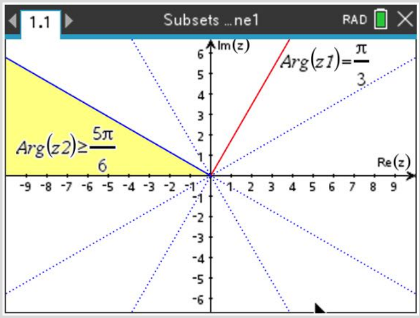
- Enter $\boldsymbol{f1(x) = \tan(n \cdot \pi / 6) \cdot x \mid n = \{1,2,3,4,5\}}$. If prompted to create a slider for ***n***, click **cancel** to decline.
- Hover over a radial line, press ctrl menu > **Attributes** > **Line style is dotted**. Repeat for the other radial lines.
- Press menu > **Actions** > **Hide/Show**. Click on unwanted labels to hide them. **Save** the grid in **MyWidgets** folder.



(b)(i) To illustrate $\text{Arg}(z) = \pi/3$ on the **Graphs** page:

- Enter $\boldsymbol{f2(x) = \tan(\pi/3) \cdot x \mid x > 0}$.

Answer: The open ray along the line $y = \sqrt{3}x$ in the first quadrant, excluding the origin because Arg(0) is undefined.



(ii) To illustrate $\text{Arg}(z_2) \geq 5\pi/6$:

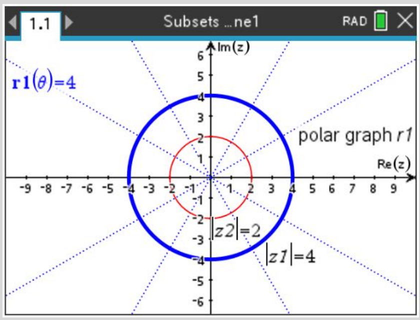
- Enter $\boldsymbol{f3(x) = \tan(5\pi/6) \cdot x \mid x < 0}$

Answer: The region is the set $\{r\text{cis}(\theta) : r > 0, 5\pi/6 \leq \theta \leq \pi\}$

That is, the sector with $5\pi/6 \leq \theta \leq \pi$, excluding the origin.

To illustrate **(iii)** $|z_1| = 4$ and **(iv)** $|z_2| = 2$:

- Press menu > **Graph Entry/Edit** > **Polar**.
- Enter $\boldsymbol{r1(\theta) = 4}$, then $\boldsymbol{r2(\theta) = 2}$.
- Accept the default settings of $0 \leq \theta \leq 6.28$ $\theta step = 0.13$.



Answer: Circles, **(iii)** $|z_1| = r1(\theta) = 2$ and **(iv)** $|z_2| = r2(\theta) = 4$.

Interpreting multiplication in polar form geometrically

Question

(a) Illustrate $z_1 = 2+2\sqrt{3}i = 4\,\text{cis}\left(\frac{\pi}{3}\right)$ and $z_2 = -\sqrt{3}+i = 2\,\text{cis}\left(\frac{5\pi}{6}\right)$ on the complex plane.

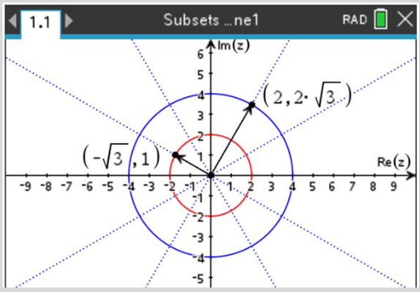
(b) Evaluate the product $z_1 z_2$ in polar form.

(c) Illustrate $z_1 z_2$ on the complex plane and interpret the result geometrically.

Solution

(a) To illustrate z_1 and z_2 as polar coordinates, on the radial grid from the previous problem:

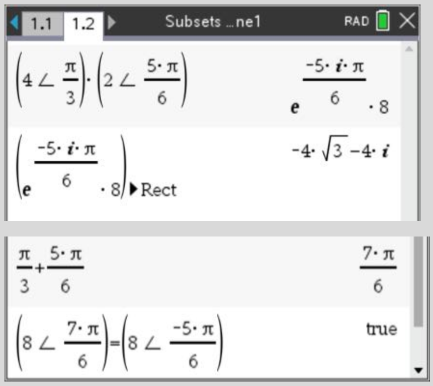
- Press menu > **Geometry** > **Points & Lines** > **Vector**.
- Click on the origin and then click **Point On** the intersection of $\text{Arg}(z_1) = \pi/3$, $|z_1| = 4$.
- Repeat for $\text{Arg}(z_2) = 5\pi/6$, $|z_1| = 2$.



- To add labels, hover over the tip of the first vector, press ctrl menu > **Label** then enter $\boldsymbol{z1}$. Repeat for $\boldsymbol{z2}$.
- Hover over the point at the tip of the $\boldsymbol{z1}$ vector, press ctrl menu > **Coordinates and Equations**.
- To force exact arithmetic, edit the y-coordinate to be $2\sqrt{3}$. Repeat for $\boldsymbol{z2}$, except edit the x-coordinate to be $-\sqrt{3}$.

(b) To evaluate $z_1 z_2$ using $(\boldsymbol{r\angle\theta})$ for $r\text{cis}(\theta)$ and **Document Settings** > **Complex** > **Polar,** on a **Calculator** page:

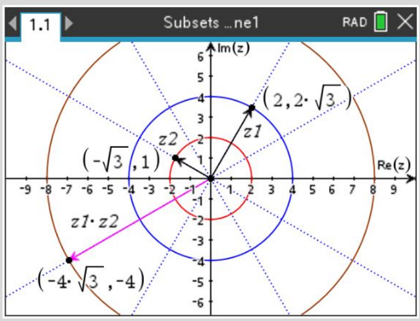
- Enter $(\boldsymbol{4\angle\pi/3})\times(\boldsymbol{2\angle5\pi/6})$.
- Press menu > **Number** > **Complex Number Tools** > **Convert to Rectangular**, then press enter.



Answer: $z_1 z_2 = 8\text{cis}(7\pi/6) = 8\text{cis}(-5\pi/6) = -4\sqrt{3} - 4i$

(c) To plot $z_1 z_2$ as a polar coordinate, on the **Graphs** page:

- Enter $\boldsymbol{r3(\theta) = 8}$.
- Press menu > **Geometry** > **Points & Lines** > **Vector**.
- Click on the origin and then click **Point On** the intersection of $r3(\theta) = 8$ and the radial line for $\text{Arg}(z_1 \cdot z_2) = -5\pi/6$.
- Hover over the point at the tip of the vector, press ctrl menu > **Coordinates and Equations**. To force exact arithmetic, edit the x-coordinate to be $-4\sqrt{3}$.



Answer: Geometric interpretation: multiplying z_1 by z_2 in polar form represents scaling z_1 by $|z_2|$ and rotating by $\text{Arg}(z_2)$.

Representing other subsets of the complex plane

Question

Illustrate on the complex plane the following subsets of C and interpret the results geometrically.

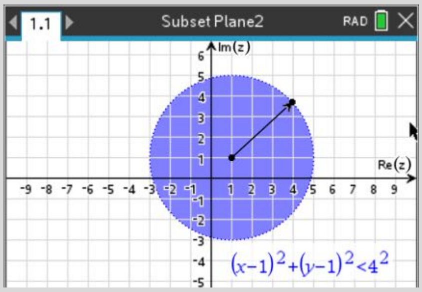
(a) $\left|z-(1+i)\right|<4$ **(b)** $|z|=\left|z-(-3+6i)\right|$

Solution

(a) To illustrate $\left|z-(1+i)\right|<4$, on a **Graphs** page:

- Press [menu] > **Graph Entry/Edit** > **Relation**.
- Enter $(\boldsymbol{x}-\mathbf{1})^2+(\boldsymbol{y}-\mathbf{1})^2<\mathbf{4}^2$.
- Press [menu] > **View** > **Grid**. Select **Lined Grid**.

Answer: The set of points less than 4 units from (1, 1).



(b) $|z|=\left|z-(-3+6i)\right|$ can be represented by the perpendicular bisector of the line segment joining the origin and $(-3,6)$.

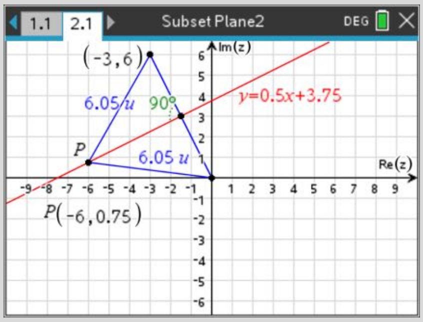
To illustrate $|z|=\left|z-(-3+6i)\right|$, on a **Graphs** page:

- Press [menu] > **Geometry** > **Points & Lines** > **Segment**.
- Click on the origin, then click on the workspace in the second quadrant. Hover over the point, press [ctrl] [menu] > **Coordinates and Equations**. Edit coordinates to $(\mathbf{-3,6})$.

Note: To make the document fully editable for future use, avoid locking onto an integer grid point in the first instance.

- Press [menu] > **Geometry** > **Construction** > **Perpendicular Bisector**, click the **segment**, then press [esc] to exit.
- Hover over the perpendicular bisector, press [ctrl] [menu] > **Coordinates and Equations**. Equation: $y=0.5x+3.75$.
- Use **Geometry** tools for additional embellishments.

Answer: $|z|=\left|z-(-3+6i)\right|$: the set of points equidistant from the origin and $(-3,6)$.



2.4 Functions, relations and graphs

2.4.1 Functions, relations and graphs

Graphing reciprocal trigonometric functions

Question

Use a pointwise approach to identify key features of the graphs of

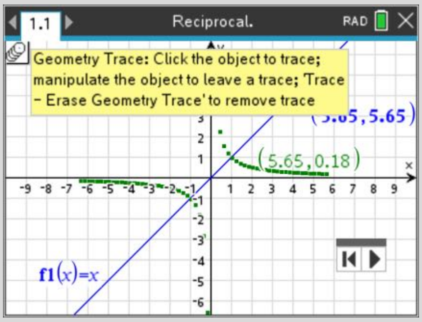
(a) $y=\dfrac{1}{x}$ **(b)** $y=\dfrac{1}{\sin(x)}$ **(c)** $y=\dfrac{1}{\tan(x)}$.

Solution

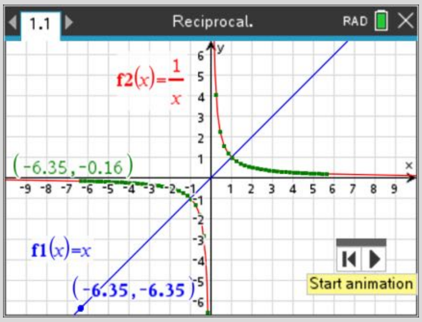
(a) To obtain a plot of $y=\dfrac{1}{x}$ using a pointwise approach, on a **Graphs** page:

- Enter $f1(x)=x$.
- Press [P] > **Point**. Click on the graph $f1$ to place a point on the graph and edit the x-coordinate to obtain $(6,6)$.

Note: *Press* [esc] *to exit any Geometry tool before proceeding.*



- Hover over the x-coordinate value, press [var] and assign the variable $x1$. Repeat for the y-coordinate with $y1$.
- Hover over the point, press [ctrl][menu] > **Attributes**. Press ▼ to **Unidirectional speed**, press [1] followed by [enter][enter].
- Press [P] > **Point by Coordinates**. Edit the coordinates to be $(x1, 1/y1)$.
- Hover over the point, press [ctrl][menu] > **Geometry Trace**.
- Use the control buttons to start, pause or reset the animation of point $(x1, y1)$.
- Enter $f2(x)=1/x$. Compare this graph and the plot.

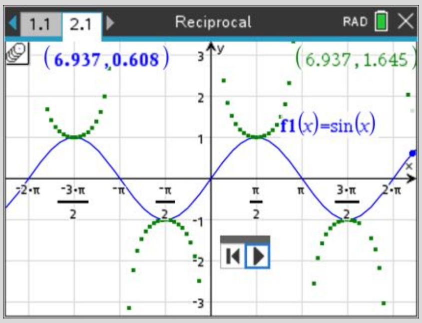


Note: *To erase a* ***Geometry Trace****, press* [ctrl][menu] *and select* ***Erase Geometry Trace****.*

Answer: As $x\to\pm\infty$, $1/x\to 0$. As $x\to 0^{+/-}$, $1/x\to +/-\infty$.

(b) To obtain a pointwise plot of $y=\dfrac{1}{\sin(x)}$:

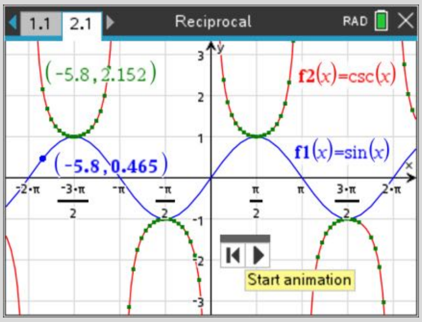
- On the document from part **(a)** above, press [ctrl] ▲. In thumbnail view, press ▲ to select **Problem 1**.
- Press [ctrl][C] then [ctrl][V] to obtain a copy of **Problem 1**.
- On **page 2.1**, hide $f2$ and edit $f1$ to $f1(x)=\sin(x)$.
- Press [menu] > **Window/Zoom** > **Window Settings**. In the dialog box that follows, enter the following values:
  Xmin: $-9\pi/4$ XMax: $9\pi/4$ XScale: $\pi/2$
  Ymin: -3.33 Ymax: 3.33 YScale: 1



... continued

Solution (continued)

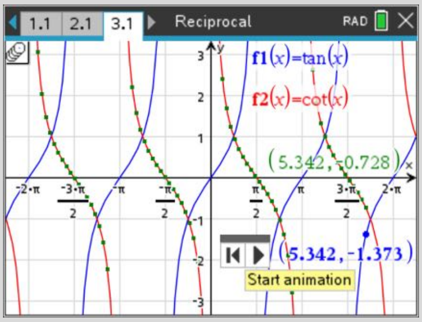
- Edit x-coordinate of $(\boldsymbol{x1}, \boldsymbol{y1})$ to **–0.58**, then hover over point $(\boldsymbol{x1}, \mathbf{1}/\boldsymbol{y1})$ and press [ctrl] [menu] **> Geometry Trace**.
- Using the control buttons, animate the point $(\boldsymbol{x1}, \boldsymbol{y1})$.
- Unhide $\boldsymbol{f2}$ and edit to $\boldsymbol{f2}(\boldsymbol{x}) = \mathbf{csc}(\boldsymbol{x})$. Observe the relationship of $\boldsymbol{f2}$ to the pointwise plot. Then edit $\boldsymbol{f2}$ to $\boldsymbol{f2}(\boldsymbol{x}) = \mathbf{1}/\mathbf{sin}(\boldsymbol{x})$ and observe the relationship.



Answer: Asymptotes: $x = n\pi, n \in Z$, Domain: $x \neq n\pi$, Range: $y \in (-\infty, -1] \cup [1, \infty)$, Period: 2π.

(c) To obtain a pointwise plot of $y = 1/\tan(x)$:

- On the document from part **(b)** above, press [ctrl] ▲. In thumbnail view, press ▲ to select **Problem 2**.
- Press [ctrl] [C] then [ctrl] [V] to obtain a copy of **Problem 2**.
- On page 3.1, hide $\boldsymbol{f2}$ and edit $\boldsymbol{f1}$ to $\boldsymbol{f1}(\boldsymbol{x}) = \mathbf{tan}(\boldsymbol{x})$.
- Edit x-coordinate of $(\boldsymbol{x1}, \boldsymbol{y1})$ to **–0.58**, then hover over point $(\boldsymbol{x1}, \mathbf{1}/\boldsymbol{y1})$ and press [ctrl] [menu] **> Geometry Trace**.
- Using the control buttons, animate the point $(\boldsymbol{x1}, \boldsymbol{y1})$.
- Unhide $\boldsymbol{f2}$ and edit to $\boldsymbol{f2}(\boldsymbol{x}) = \mathbf{cot}(\boldsymbol{x})$. Observe the relationship of $\boldsymbol{f2}$ to the pointwise plot. Then edit $\boldsymbol{f2}$ to $\boldsymbol{f2}(\boldsymbol{x}) = \mathbf{1}/\mathbf{tan}(\boldsymbol{x})$ and observe the relationship.



***Note:** To hide/unhide a graph, press [ctrl] [G] and check/uncheck the box to the left of the equation of the graph.*

Answer: Asymptotes: $x = n\pi, n \in Z$, Domain: $x \neq n\pi$, Range: R, Period: 2π, x-intercepts: $x = \frac{\pi}{2} + n\pi$.

Graphing the inverse of sine and cosine functions

Question

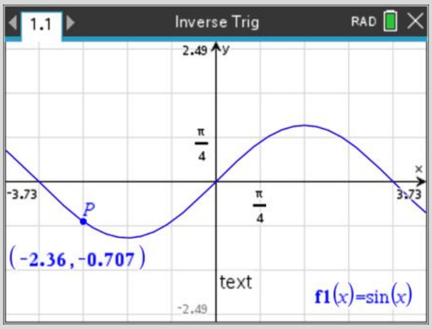
Use a pointwise approach to identify key features of the graphs of the inverses of f, g and h below, including necessary domain restrictions for the inverse functions, f^{-1}, g^{-1} and h^{-1} to exist.

(a) $f(x)=\sin(x)$ and f^{-1} **(b)** $g(x)=\cos(x)$ and g^{-1} **(c)** $h(x)=\tan(x)$ and h^{-1}

Solution

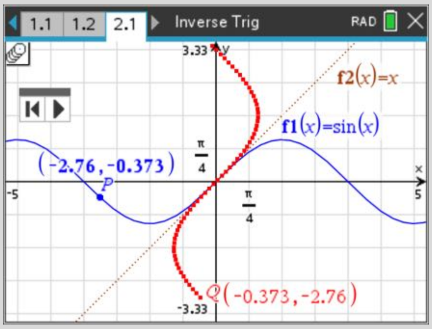
(a) To 'pointwise' plot $y=\sin^{-1}(x)$, on a **Graphs** page:

- Press [menu] > **Window/Zoom** > **Window Settings.** In the dialogue box that follows, enter the following:
  XMin: **–5** XMax: **5**, XScale: $\boldsymbol{\pi/4}$
  YMin: **–3.33** YMax: **3.33** YScale: $\boldsymbol{\pi/4}$
- Enter $\boldsymbol{f1(x)=\sin(x)}$.
- Press [P] > **Point**. Click graph $\boldsymbol{f1}$ to place a point on the graph, edit the x-coordinate to $\boldsymbol{3\pi/4}$, then [esc] to exit.
- Hover over the point, press [ctrl][menu] > **Label** and enter ***P***.



Note: *Press [esc] to exit any* ***Geometry*** *tool before proceeding.*

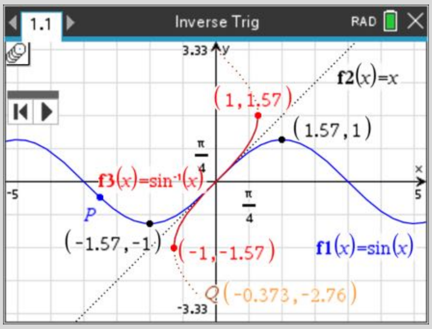
- Hover over the x-coordinate value, press [var] and assign the variable ***xc***. Repeat for the y-coordinate with ***yc***.
- Hover over the point, press [ctrl][menu] > **Attributes**. Press ▼ to **Unidirection speed**, press [1] followed by [enter][enter].
- Press [P] > **Point by Coordinates**. Edit the coordinates to be (***yc***, ***xc***). Label this point ***Q***.
- Hover over point ***Q***, press [ctrl][menu] > **Attributes**, select **Thin** point type. Select appropriate point **Colour**.
- Hover over point ***Q***, press [ctrl][menu] > **Geometry Trace**.
- Use the control buttons to start or reset the animation.
- Press [esc] to exit, then [ctrl][G] and enter $\boldsymbol{f2(x)=x}$.



Note: *To erase a* ***Geometry Trace****, press [ctrl][menu] and select* ***Erase Geometry Trace****.*

To compare the plot with the graph of f^{-1}:

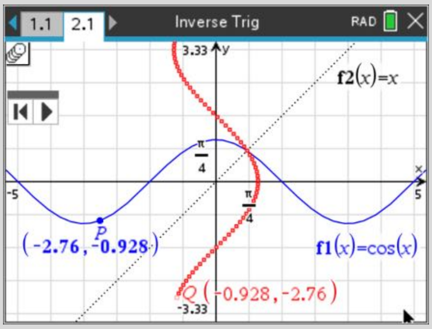
- Enter $\boldsymbol{f3(x)=\sin^{-1}(x)}$, pressing [trig] to select $\boldsymbol{\sin^{-1}}$.



Answer: The graph of the inverse is a reflection of $y=\sin(x)$ in the line $y=x$. If the domain of f is restricted to $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, then f^{-1} is defined as $f^{-1}(x)=\sin^{-1}(x), x\in[-1,1]$.

... continued

Solution (continued)

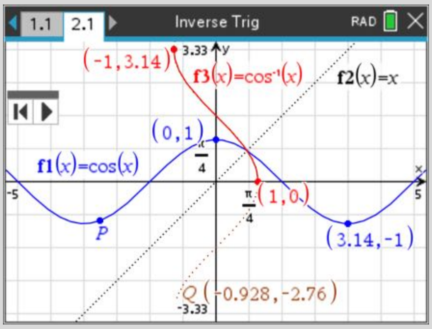


(b) To make a copy of the previous problem:

- Press [ctrl] ▲, select **Problem 1** and press [ctrl] [C] [ctrl] [V].

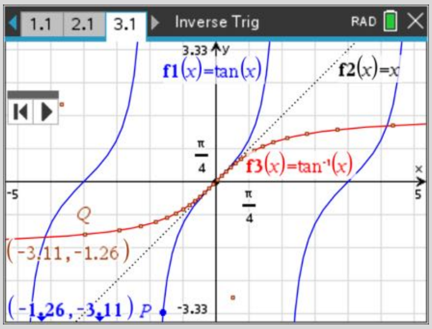
To obtain a pointwise plot of the inverse of $g(x) = \cos(x)$:

- On page 2.1, edit to $\boldsymbol{f1(x) = \cos(x)}$ and uncheck $\boldsymbol{f3}$.
- Hover over point ***Q***, press [ctrl] [menu] > **Geometry Trace**.
- Use the control buttons to start or reset the animation.



To compare the plot with the graph of f^{-1}:

- Enter $\boldsymbol{f3(x) = \cos^{-1}(x)}$, pressing [trig] to select $\boldsymbol{\cos^{-1}}$.
- **Answer:** The graph of the inverse is a reflection of $y = \cos(x)$ in the line $y = x$. If the domain of g is restricted to $x \in [0, \pi]$, then g^{-1} is defined as $g^{-1}(x) = \cos^{-1}(x), x \in [-1, 1]$.



(c) To pointwise plot the inverse of $h(x) = \tan(x)$, repeat the instructions for part **(b)** above, except on page 3.1:

- Edit to $\boldsymbol{f1(x) = \tan(x)}$ and edit the x-coordinate of point ***P*** to **–1.26**, then **Geometry Trace** point ***Q*** and start/reset the animation. Hence edit $\boldsymbol{f3(x) = \tan^{-1}(x)}$.

Answer: If the domain of h is restricted to $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then h^{-1} is defined as $h^{-1}(x) = \tan^{-1}(x), x \in R$.

Introducing the absolute value (modulus) function

Question

Plot the graphs of the following functions and determine the range of each function.

(a) $f_1(x)=|x|$ **(b)** $f_2(x)=\begin{cases} x & x\ge 0 \\ -x & x<0 \end{cases}$ **(c)** $f_3(x)=|x+1|-3$ **(d)** $f_4(x)=3-|x+1|$

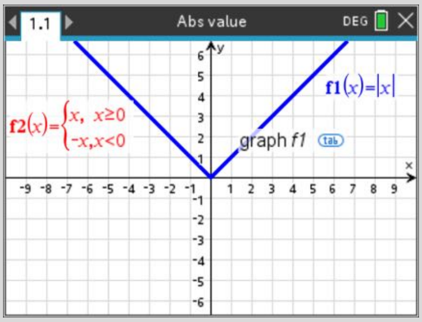
Solution

To plot the graphs for parts **(a)** and **(b)** and determine the range, on a **Graphs** page:

- Enter $\boldsymbol{f1(x)=|x|}$ by pressing to select the **absolute value** template, $|\square|$.

Note: Entering $\mathbf{abs}(\boldsymbol{x})$ *has the same effect as entering* $|\boldsymbol{x}|$.

- Press tab to open the graph entry line, then enter $\boldsymbol{f2(x)=\begin{cases} x & x\ge 0 \\ -x & x<0 \end{cases}}$ by pressing to select the **Piecewise** function template.
- Hover over a graph and press tab to toggle between the graphs, $\boldsymbol{f1}$ and $\boldsymbol{f2}$.

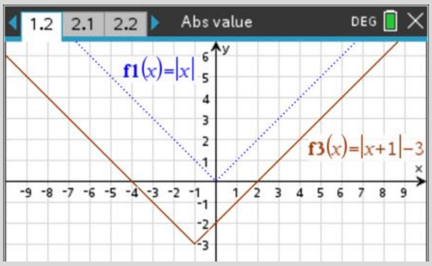


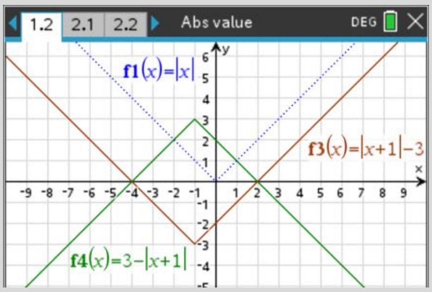
Answer: (a), (b) $f_1(x)=|x|=f_2(x)$. Range f_1, f_2 is $[0,\infty)$.

Note: *To show a lined grid, press* ctrl menu ***> Hide/Show*** *and select* ***Show Lined Grid****. To select multiple labels on the axes, hover over an axis, press* ctrl menu ***> Attributes****.*

To plot the graphs for parts **(c)** and **(d)** and determine the range, on a **Graphs** page:

- Enter $\boldsymbol{f3(x)=|x+1|-3}$ and determine the range.
- Enter $\boldsymbol{f4(x)=3-|x+1|}$ and determine the range.





Answer:

(c) $f_3(x)=f_1(x+1)-3$. Range f_3 is $[-3,\infty)$.

(d) $f_4(x)=-f_3(x)=-(f_1(x+1)-3)$. Range f_4 is $(-\infty,3]$.

Analysing relationships involving the absolute value (modulus) function

Question

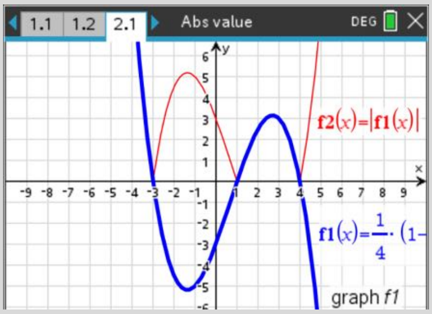
Sketch the graphs of the following functions and interpret relationships between the graphs.

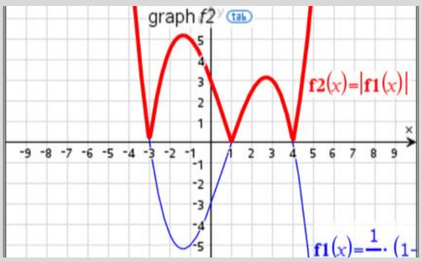
(a) $f_1(x)=\frac{1}{4}(1-x)(x+3)(x-4)$ **(b)** $f_2(x)=|f_1(x)|$ **(c)** $f_3(x)=f_1(|x|)$

Solution

To graph $y=f_1(x)$ and $y=|f_1(x)|$, on a **Graphs** page:

- Enter $\boldsymbol{f1(x)=\frac{1}{4}(1-x)(x+3)(x-4)}$, observe key features of the graph, then enter $\boldsymbol{f2(x)=|f1(x)|}$.
- Hover over the upper-left branch of the graphs and press tab to toggle between graphs and make observations.



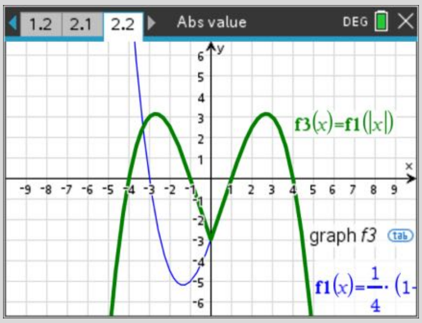


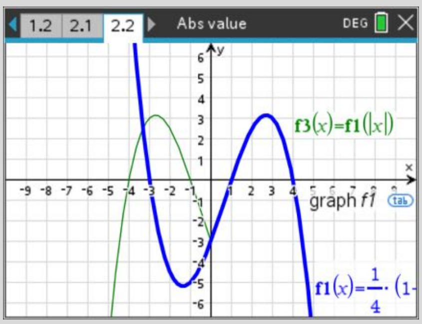
Answer: The graph of $y=|f_1(x)|$ reflects in the x-axis any points on the graph of $y=f_1(x)$ with a negative y-coordinate.

Therefore, $|f_1(x)|=\begin{cases} f_1(x), & f_1(x)\geq 0 \\ -f_1(x), & f_1(x)<0 \end{cases}$.

To plot graphs of $y=f_1(x)$ and $y=f_1(|x|)$, and determine their relationship, add a **Graphs** page to the existing problem:

- In the graph entry line, select $\boldsymbol{f1(x)}$, and press enter to plot the graph, and then enter $\boldsymbol{f3(x)=f1(|x|)}$.
- Hover over the lower-right branch of the graphs and press tab to toggle between graphs and to observe the relationship.





Answer: The graph of $y=f_1(|x|)$ is symmetric about the y-axis because $f_1(|x|)=\begin{cases} f_1(x), & x\geq 0 \\ f_1(-x), & x<0 \end{cases}$.

The graph of $y=f_1(|x|)$ for $x\in(-\infty,\infty)$ is a reflection in the y-axis of the graph of $y=f_1(x)$ for $x\in[0,\infty)$, the non-negative values of x.

Visualising the locus definition of a straight line using the Locus tool

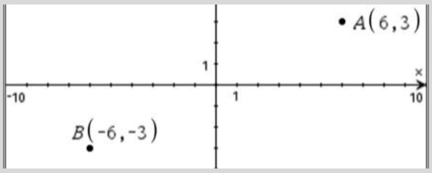
Question

Use the Locus tool to trace the geometric shape formed by the set of points that are equidistant from two distinct points A and B, then interpret the diagram produced by the Locus tool.

Solution

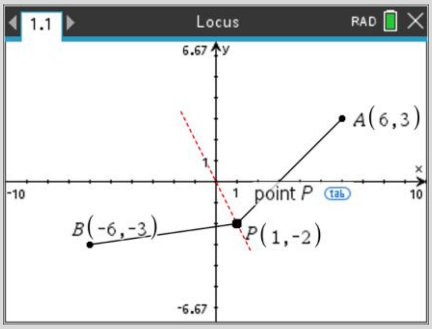
To show and label the points A and B, on a **Graphs** page:

- Press P > **Point by Coordinates**, enter **(6, 3)** then press (and enter **(–6, –3)**. Press esc to exit the tool.
- Hover over a point, press ctrl menu > **Label**. Enter ***A***, ***B***.



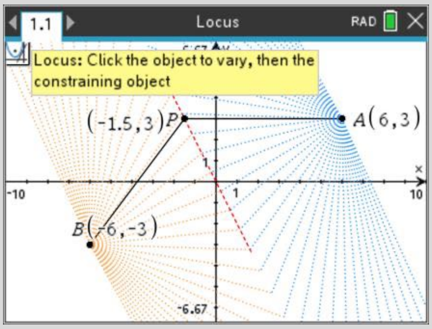
To show a point P that is equidistant from points A and B:

- Press menu > **Geometry** > **Construction** > **Perpendicular bisector** and click point ***A*** then ***B***. Press esc to exit.
- Press P > **Point,** click the line and label the point ***P***.
- Hover over point ***P***, press ctrl menu > **Coordinates …**
- Press menu > **Geometry** > **Points and Lines** > **Segment**. Click to draw segments ***AP*** and ***BP***, then press esc.



To trace the path of ***P*** showing segments ***AP*** and ***BP***:

- Press menu > **Geometry** > **Construction** > **Locus**. Click segment ***AP*** then point ***P***. Repeat for ***BP***, then press esc.
- Press ctrl menu to change colour and attributes of the loci.



Answer: The Locus tool predicts the 'future path' of segments AP and BP, which are constrained by their endpoints P being equidistant from points A and B. The locus of P is a straight line such that $y = -2x$, as illustrated by coordinates $P(-1.5, 3)$.

Leveraging the Locus tool to visualise the locus definitions of a parabola

Question

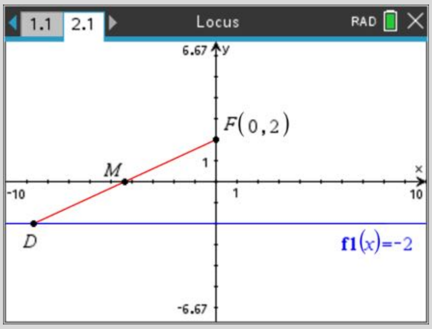
Consider a fixed point F, called the focus, and a fixed line l, called the directrix. The point P, equidistant from F and l, traces a curve. For $F(0, 2)$ and l with equation $y = -2$, use the Locus tool to display a set of tangents to this curve.

Show that the equation of the curve is $8y = x^2$ and interpret the diagram produced by the Locus tool.

Solution

To set up the diagram with F and l, on a **Graphs** page:

- Enter $\boldsymbol{f1(x) = -2}$ then press [P] > **Point by Coordinates**, enter **(0, 2)** then click graph $\boldsymbol{f1}$. Press [esc] to exit.
- Hover over a point, press [ctrl] [menu] > **Label**. Enter as shown.
- Press [menu] > **Geometry** > **Points and Lines** > **Segment**. Click point F then D to draw segment ***FD***.

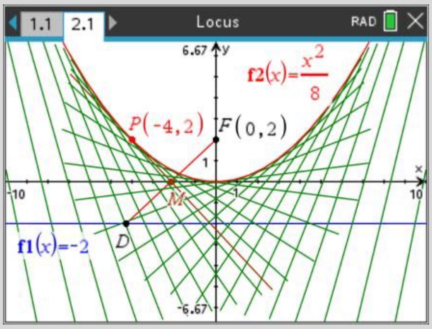


To display a set of tangents to the curve traced out by P:

- Press [menu] > **Geometry** > **Construction** > **Perpendicular Bisector.** Click **segment *FD*** and press [esc] to exit. Drag the ends to extend line.
- Repeat but select **…** > **Locus**, click $\overline{FD}$ then point ***D***.

To show that the set of tangents fit the curve $8y = x^2$:

- Enter $\boldsymbol{f2(x) = x^2 / 8}$. Add intersection point ***P*** on $\boldsymbol{f2}$.



Answer: The Locus tool displays a set of tangents constrained such that the path of P traces the parabola $8y = x^2$. Dragging point F (or changing l) changes the equation of the parabola.

Graphing circles and ellipses using equation templates

Question

(a) On the same axes, plot the graphs with equations: **(i)** $(x+3)^2+(y-1)^2=25$

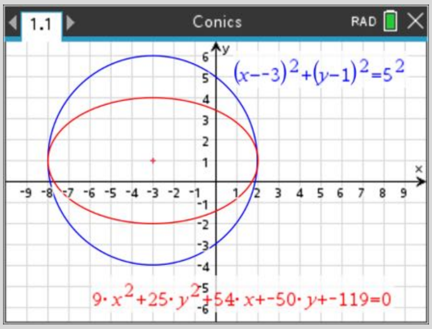
(ii) $9x^2+25y^2+54x-50y-119=0$

(b) For graph **(ii)** above, find and display on the graph the **(i)** centre, **(ii)** foci and **(iii)** directrices.

Solution

(a)(i) To graph $(x+3)^2+(y-1)^2=25$, on a **Graphs** page:

- Press [menu] > **Graph Entry/Edit** > **Equation Templates** > **Circle** > **Centre form**. Enter $(x-(\mathbf{-3}))^2+(y-\mathbf{1})^2=\mathbf{5}^2$.



(ii) To graph $9x^2+25y^2+54x-50y-119=0$:

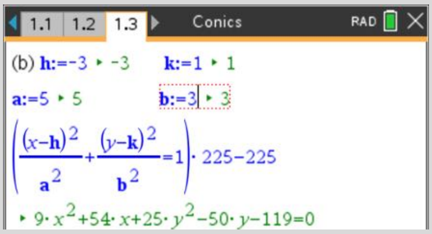
- Press [menu] > **Graph Entry/Edit** > **Equation Templates** > **Conic** > **General**.
- Enter $\mathbf{9}x^2+\mathbf{0}x\cdot y+\mathbf{25}y^2+\mathbf{54}x+(\mathbf{-50})y+(\mathbf{-119})=0$.

(b) (i) The graph is of the form $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$.

To confirm the values of h, k, a and b, on a **Notes** page:

- Press [ctrl] [M] to insert five **Maths Boxes**. Enter as shown.

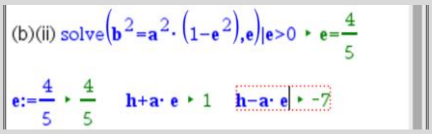
Answer: $h=-3$, $k=1$, $a=5$, $b=3$. Centre at $(-3, 1)$.



(ii) To find the foci using $F(h\pm ae,k)$ and $b^2=a^2(1-e^2)$:

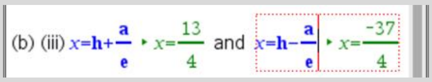
- Press [ctrl] [M] to insert four **Maths Boxes**. Enter as shown.

Answer: $e=4/5$, foci: $(h+ae,k)=(1,1)$, $(h-ae,k)=(-7,1)$



(iii) To find the equation of the directrices using $x=h\pm\frac{a}{e}$:

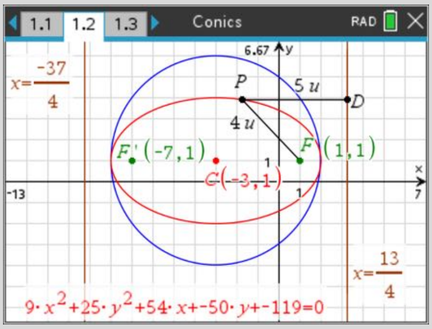
- Press [ctrl] [M] to insert two **Maths Boxes**. Enter as shown.



Answer: Equations of directrices: $x=\dfrac{13}{4}$ and $x=\dfrac{-37}{4}$.

To display the directrices, centre and foci on the graph:

- Press [menu] > **Graph Entry/Edit** > **Relation**.
- Enter $x=13/4$ and $x=-37/4$.
- Press [P] > **Point by Coordinates**, enter **(–3, 1)** then press [(] and enter **(–7, 1)** and **(1, 1)**. Press [esc] to exit the tool.



Extension: To verify geometrically that $e=4/5$, measure the distances from a point P on the curve to a focus F, and to a point D on a directrix, as shown. The ratio d(FP)/d(DP) = 4/5.

Analysing curves that are defined parametrically

Question

The curve traced out by a particle can be defined in terms of a parameter, such as time.

Plot the graphs of the following parametrically defined curves and describe their nature.

(a) $x = 4\cos(t) + 2$ and $y = 4\sin(t)$, where $t \in [0, 2\pi)$.

(b) $x = 4\cos(t) + 2$ and $y = 6\sin(t)$, where $t \in [0, 2\pi)$.

(c) $x = a\cos(2t)$ and $y = a\sin(t) + 1$, $a \in R \setminus \{0\}$. Compare with graph of $2(y-1)^2 = a(a-x)$.

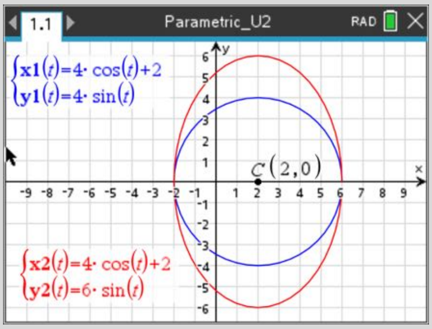
(d) $x = 3\sec(t)$ and $y = 2\tan(t)$, where $t \in [-\pi, \pi], t \neq \pm\pi/2$.

Solution

(a) To graph $x = 4\cos(t) + 2, y = 4\sin(t)$, on a **Graphs** page:

- Press menu > **Graph Entry/Edit** > **Parametric**. Enter ***x*1(*t*)=4cos(*t*) + 2** and ***y*1(*t*)=4sin(*t*)**. Accept other defaults.

(b) Similarly, enter ***x*2(*t*)=4cos(*t*) + 2** and ***y*2(*t*)=6sin(*t*)**.

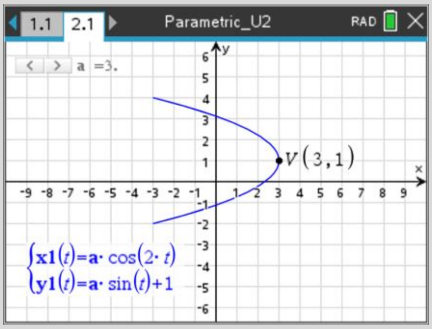


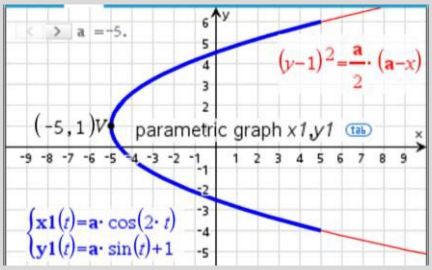
Answer: (a) Circle, $r = 4$ **(b)** ellipse, $a = 4$, $b = 6$.
Both curves centred at (2,0). Cartesian equations:

(a) $(x-2)^2 + y^2 = 4^2$, **(b)** $\dfrac{(x-2)^2}{4^2} + \dfrac{y^2}{6^2} = 1$.

(c) In a new **Problem**, enter similarly to above:

- ***x*1(*t*)=*a*×cos(2*t*)** and ***y*1(*t*)= *a*×sin(*t*) + 1**.
  When prompted to create a slider for *a*, click **OK**.
- Press menu > **Graph Entry/Edit** > **Relation** and enter $(\boldsymbol{y}-\mathbf{1})^2 = \boldsymbol{a}/\mathbf{2}\times(\boldsymbol{a}-\boldsymbol{x})$. Change graph colour to red.
- Hover over a graph and press tab to toggle the graphs.
- Use the slider to explore the effect of changing the value of *a*.





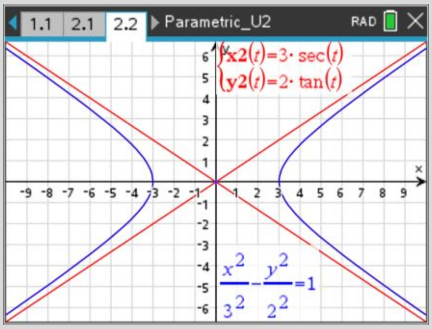
Answer: Parabola. Cartesian equation $2(y-1)^2 = a(a-x)$.

Since $\cos(2t) \in [-1,1]$, domain is $x \in [-a, a]$.

Since $\sin(t) \in [-1,1] \Rightarrow \dfrac{y-1}{a} \in [-1,1]$. Range $y \in [1-a, 1+a]$

(d) To plot, press menu > **Graph Entry/Edit** > **Parametric**.

- Enter ***x*2(*t*) = 3sec(*t*), *y*2(*t*) = 2tan(*t*)** and $-\pi \leq t \leq \pi$.



Answer: Hyperbola with domain $R\backslash(-3, 3)$ and range R.

Asymptotes: $3y = 2x$.

Plotting graphs of conics using polar coordinates

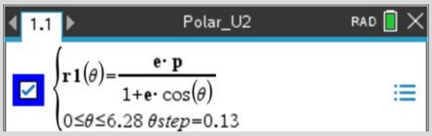
Question

(a) Conics with a focus centred at the pole satisfy the equation $r(\theta)=\dfrac{ep}{1+e\cos(\theta)}$, where e is the eccentricity and p is the focus-directrix distance. Use polar graphing to investigate and describe the nature of these curves if: **(i)** $0<e<1$, **(ii)** $e=1$, **(iii)** $e>1$.

Solution

(a)(i) To plot $r(\theta)$ with $0<e<1$ on a **Graphs** page:

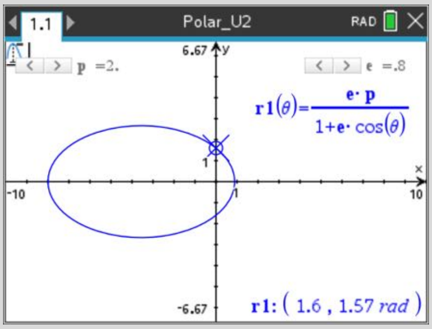
- Press [menu] > **Graph Entry/Edit** > **Polar**. Enter $\boldsymbol{r1(\theta)}$ as shown. Accept the defaults for $\boldsymbol{\theta}$ and $\boldsymbol{\theta step}$.
- When prompted to add sliders for ***e*** and ***p***, click **OK**.
- Enter the following values for the slider settings:
  Variable: ***p***, Minimum: **–5**, Maximum: **5**, Step Size: **0.5**
  Variable: ***e***, Minimum: **0**, Maximum: **5**, Step Size: **0.1**



Note: *Press* [ctrl] [📖] *to select* $\boldsymbol{\theta}$.

To display the polar coordinates for, say, $\theta=\pi/2$:

- Press [menu] > **Trace** > **Graph Trace**. Key in $\boldsymbol{\pi/2}$, [enter].
- Observe the effect of changing the values of p and e.

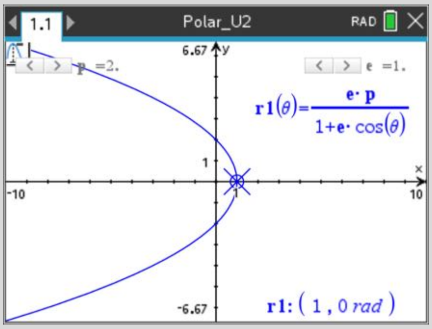


Answer: For $0<e<1$, the curve is an ellipse. The focus is at the pole (origin) and the axis of symmetry is the polar axis (the x-axis). For $p=2$ and $e=0.8$, the directrix has equation $x=p=2$. The curve passes through $(0,\pm e\cdot p)=(0,\pm1.6)$. The value $e\cdot p=1.6$ is known as the semi-latus rectum.

(ii) To plot $r(\theta)$ with $e=1$, select this value on the slider and manipulate the value of p with the other slider.

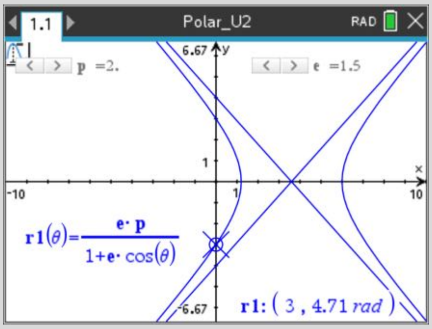
To display the polar coordinates for $\theta=0,\ \pi/2,\ 3\pi/2$:

- Press [menu] > **Trace** > **Graph Trace**. Enter each value.



Answer: For $e=1$, the curve is a parabola. At $\theta=\pi/2,\ 3\pi/2,\ r(\theta)=p$. Pole-vertex distance is $p/2$ and the directrix is $x=p$. Unlike the ellipse, the curve is unbounded because $1+\cos(\theta)=0$, hence as $\theta\to\pi$, $r\to\infty$.

(iii) To plot $r(\theta)$ with $e>1$, select values greater than 1 on the slider and manipulate the value of p.



Answer: For e > 1: hyperbola with focus at the origin and directrix at $x=p$. Asymptote angles for $p=2$ and $e=1.5$: denominator is $1+1.5\cos(\theta)=0\Rightarrow\theta=\pm\cos^{-1}\left(\frac{1}{1.5}\right)$.

Exploring spirals and their properties using polar coordinates

Question

Plot the curves defined by the following polar equations and explore the effect of changing the domain (angular interval) and the value of k. Describe key features of each type of curves.

(a) Spiral of Archimedes, $r(\theta)=k\theta$. **(b)** Fermat's spiral, $r^2(\theta)=k^2\theta$.

(c) A logarithmic spiral, $r(\theta)=k2^{\theta/5}$. Explore the angle between the tangent and the radius vector.

Solution

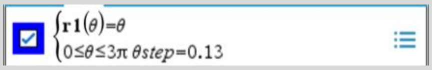
(a) To plot $r(\theta)=k\theta$, $k=\pm1$, $\theta\in[0,3\pi]$, on a **Graphs** page:

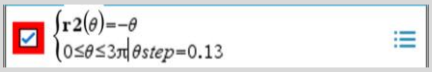
- Press [menu] > **Graph Entry/Edit** > **Polar**. Enter as shown, pressing [ctrl] [📖] (∞β°) to select $\boldsymbol{\theta}$ and [π▸] to select $\boldsymbol{\pi}$.

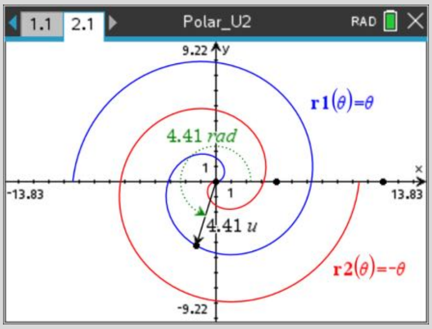
To resize the axes while maintaining a square aspect ratio:

- Grab a tick on an axis and move it towards the centre.
- Edit $\boldsymbol{r1}$ to observe the effect of changing the value of k (or alternatively, add a slider to control k for $\boldsymbol{r1(\theta)=k\times\theta}$).

Answer: The parameter k controls the spacing of the spiral. Length of the radius vector is proportional to θ.



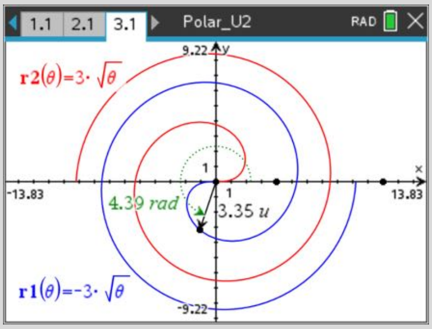




(b) To plot $r^2(\theta)=k^2\theta$, $k^2=\pm3, \theta\in[0,3\pi]$:

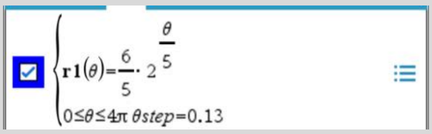
- Copy the previous problem by pressing [ctrl] ▲ followed by ▲ then [ctrl] [C] [ctrl] [V]. Press [enter] to open the page.
- Edit the equations to $\boldsymbol{r1(\theta)=-3\sqrt{\theta}}$, $\boldsymbol{r2(\theta)=3\sqrt{\theta}}$.
- Edit to observe the effect of changing the value of k.

Answer: Two symmetric arms are produced. The distance between successive turns decreases as θ increases. Compared to Archimedes spiral, it is tighter and spreads more slowly.



(c) To plot $r(\theta)=\frac{6}{5}\times2^{\theta/5}$, $\theta\in[0,4\pi]$, on a **Graphs** page:

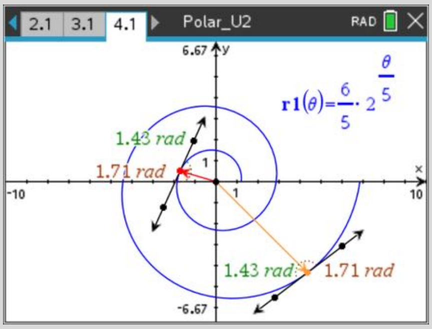
- Press [menu] > **Graph Entry/Edit** > **Polar**. Enter as shown.



To measure the angle between the tangent and radius vector:

- Press [menu] > **Geometry** > **Points & Lines** > **Tangent** and click the curve. Similarly, **… Points & Lines** > **Vector.** Click the origin then the tangent contact point, as shown.
- Press [menu] > **Geometry** > **Measurement** > **Angle**. Click the origin, contact point and tangent, then press [esc].

Answer: Moving the tangent around the spiral shows that the angle between the **r**adius vector and tangent is constant everywhere. The parameter k determines the size of the spiral, which grows exponentially with θ. Scaling the spiral (zooming in or out) produces the same shape (self-similarity).



VCE Specialist Mathematics Units 3 & 4

3.1 Discrete mathematics

3.1.1 Logic and proof

Examples in Section 1.1.1 showcase various examples of proof.

These include a proof by contradiction, use of examples and counterexamples, a proof by contrapositive and a proof by mathematical induction.

Proving by cases

Sometimes to prove a statement, dividing into cases which exhaust all possibilities and then showing the statement follows in all these cases is required.

Question

Prove that if $n \in \mathbb{Z}$, then $2n^2 + n + 1$ is not divisible by 3.

Solution

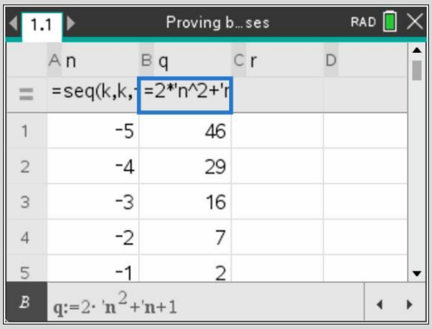
Start with an exploratory verification.

On a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the variable ***n***.
- In the column B heading cell, enter the variable ***q***.
- In the column C heading cell, enter the variable ***r***.

Generate the required sequences of values as follows:

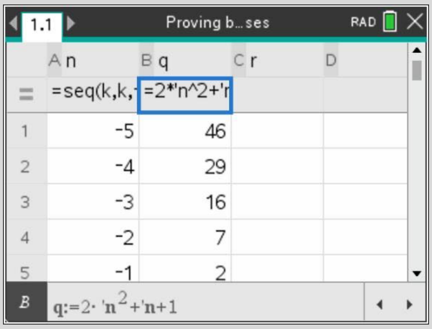
To enter $\boldsymbol{n := \text{seq}(k, k, -5, 5)}$ in the column A formula cell:



- Press 📖 1 S, scroll down and select **seq(**.
- Enter as shown.

Note: *The syntax for expressing a sequence as a list is* ***seq(Expression, Variable, Low, High[,Step])****.*
The default value for ***Step*** *is 1.*

To enter $\boldsymbol{q := 2 \cdot {'n}^2 + {'n} + 1}$ in the column B formula cell:



- Press var to select ***n***.
- Enter as shown.

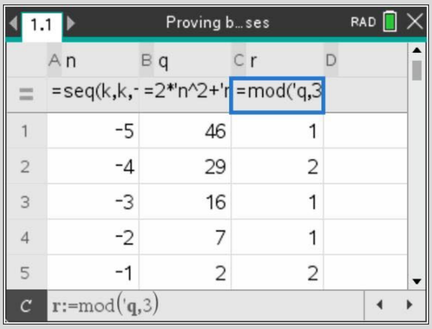
Notes: *The symbol* **'** *in* **'n** *specifies n as a variable reference. Otherwise, TI-Nspire CX II CAS will consider n as a column reference. If the* **'** *symbol is omitted, a prompt may appear asking whether the variable being used refers to a variable or to a spreadsheet column.*
Press ctrl 1 *to go to the last entry in a column.*
Press ctrl 7 *to go to the first entry in a column.*
Press ctrl 3 *to go down a page and* ctrl 9 *to go up a page.*
To go to a specific cell, press ctrl G *and enter the cell reference.*

... continued

Solution (continued)

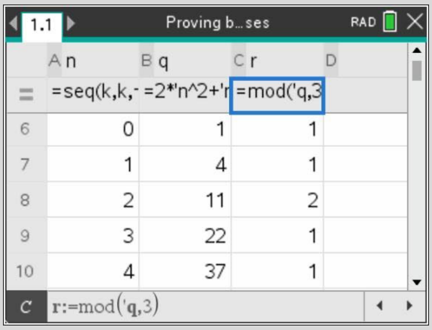
To enter $\boldsymbol{r := \text{mod}('q,3)}$ in the column C formula cell:

- Press 1 M, scroll down and select **mod(**.
- Press var to select ***q***.
- Enter as shown.



Notes: *The* ***remain(*** *command, accessed from the* ***Catalog****, can be used instead of* ***mod(****.*
To access ***mod(*** *on a* ***Calculator*** *page, press* menu *>* ***Number*** *>* ***Number Tools****.*

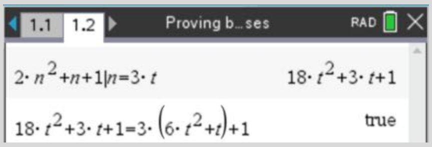
This numerical exploration (column C of the spreadsheet) suggests that if $n \in \mathbb{Z}$, then $2n^2 + n + 1$ has a remainder of 1 or 2 only and hence is not divisible by 3.



Note: *If needed, TI-Nspire CX II CAS can provide algebraic support when doing the proof. This is shown for the three cases below.*

Case (1): Consider $n = 3t$ for some $t \in \mathbb{Z}$.

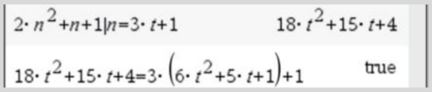
$$\begin{aligned} 2n^2 + n + 1 &= 2(3t)^2 + 3t + 1 \\ &= 18t^2 + 3t + 1 \\ &= 3(6t^2 + t) + 1 \end{aligned}$$



In this case when $2n^2 + n + 1$ is divided by 3, the remainder is 1. So $2n^2 + n + 1$ is not divisible by 3.

Case (2): Consider $n = 3t + 1$ for some $t \in \mathbb{Z}$.

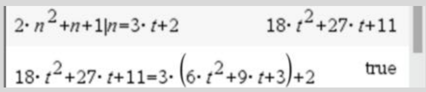
$$\begin{aligned} 2n^2 + n + 1 &= 2(3t+1)^2 + (3t+1) + 1 \\ &= 18t^2 + 15t + 4 \\ &= 3(6t^2 + 5t + 1) + 1 \end{aligned}$$



In this case when $2n^2 + n + 1$ is divided by 3, the remainder is also 1. So $2n^2 + n + 1$ is not divisible by 3.

Case (3): Consider $n = 3t + 2$ for some $t \in \mathbb{Z}$.

$$\begin{aligned} 2n^2 + n + 1 &= 2(3t+2)^2 + (3t+2) + 1 \\ &= 18t^2 + 27t + 11 \\ &= 3(6t^2 + 9t + 3) + 2 \end{aligned}$$



In this case when $2n^2 + n + 1$ is divided by 3, the remainder is 2. So $2n^2 + n + 1$ is not divisible by 3.

Since in each case $2n^2 + n + 1$ is not divisible by 3, it follows that if $n \in \mathbb{Z}$ then $2n^2 + n + 1$ is not divisible by 3.

Proving by mathematical induction – calculus

Mathematical induction is used to prove nth derivatives in calculus.

Question

Consider $y = \dfrac{1}{1+x}$, where $x \neq -1$.

(a) Conjecture an expression for the nth derivative, $\dfrac{d^n y}{dx^n}$, $\forall n \in \mathbb{Z}^+$ and where $x \neq -1$.

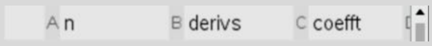
(b) Use mathematical induction to prove the conjecture made in part **(a)**.

Solution

(a) Determine the first few derivatives of y and conjecture an expression for $\dfrac{d^n y}{dx^n}$.

On a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the variable ***n***.
- In the column B heading cell, enter the variable ***derivs***.
- In the column C heading cell, enter the variable ***coefft***.



Generate the required sequences of values and derivatives as follows: To enter $\boldsymbol{n := \mathbf{seq}(k, k, 1, 6)}$ in the column A formula cell:

- Press [catalog] [1] [S], scroll down and select **seq(**.
- Enter as shown.

Note: *The syntax for expressing a sequence as a list is* ***seq(Expression, Variable, Low, High[,Step])****. The default value for* ***Step*** *is 1.*

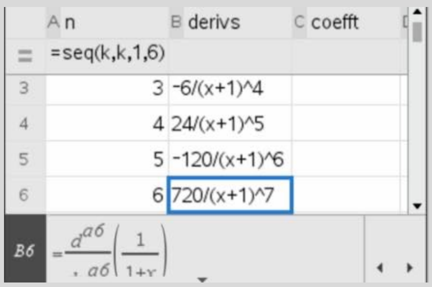
In cell B1:

- Press [catalog] [5] and select the ***n*th Derivative** template.
- Enter $\boldsymbol{= \dfrac{d^{a1}}{dx^{a1}}\left(\dfrac{1}{x+1}\right)}$ where **a1** denotes the cell reference.

To fill down to cell B6:

- Press [menu] > **Data** > **Fill**.
- Press ▼ to extend a rectangular box down to and including cell B6.
- Press [enter].

Note: *Alternatively, press* [ctrl] [menu] > ***Fill***.



The cells B1 through to B6 will be filled with the first six derivatives of y. Cell B6 shows that when $n = 6$,

$$\frac{d^6 y}{dx^6} = \frac{720}{(1+x)^7}.$$

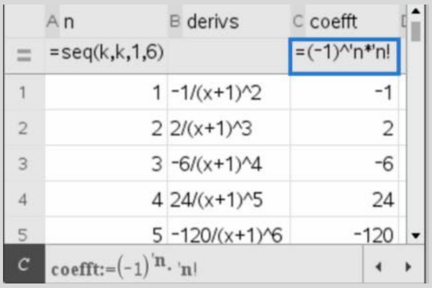
The sequence of coefficients for the first six derivatives of y are $-1, 2, -6, 24, -120, 720$.

... continued

Solution (continued)

Conjecture that the coefficient of $\frac{d^n y}{dx^n}$ is $(-1)^n\, n!$.

To enter $\boldsymbol{coefft := (-1)^{'n} \cdot 'n!}$ in the column C formula cell:



- Press [(−)] to select the negation key.
- Press [var] to select ***n***.
- Press [?!▸] to select the factorial symbol.
- Enter as shown.

Note: *The symbol* **'** *in* **'*n*** *specifies n as a variable reference. Otherwise, TI-Nspire CX II CAS will consider n as a column reference. If the* **'** *symbol is omitted, a prompt may appear asking whether the variable being used refers to a variable or to a spreadsheet column.*

Answer: $\frac{d^n y}{dx^n} = \frac{(-1)^n\, n!}{(1+x)^{n+1}},\ \forall n \in \mathbb{Z}^+$

Alternatively, use a **Notes** page to determine the first few derivatives of y and conjecture an expression for $\frac{d^n y}{dx^n}$.

Note: *Explanatory text can be added to a* ***Notes*** *page. The following shows use of this.*

Insert a **New Problem** and a **Maths Box** as follows:

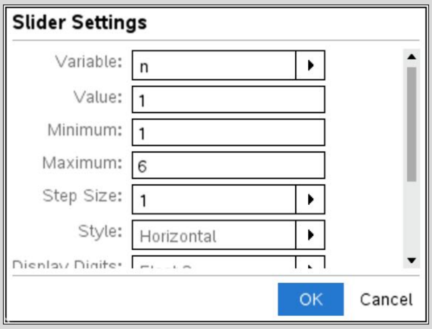
- Press [doc▾] **> Insert > Problem.**
- Press [menu] **> Insert** > **Maths Box**.

Note: *Alternatively, to insert a* ***Maths Box****, press* [ctrl] [M].

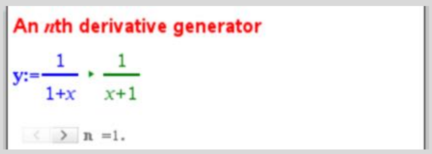
Assign y as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Enter as shown.

Insert a slider to control the value of n as follows:



- Press [menu] **> Insert > Slider**.
- Set the **Slider Settings** as shown.
- Click to check the **Minimised** box.

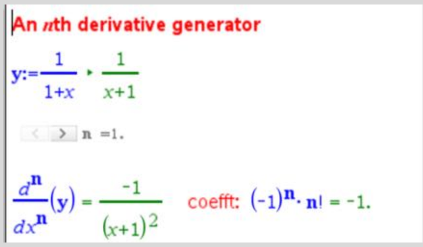


... continued

Solution (continued)

Now:

- Insert a second **Maths Box**.
- Press [📖] [5] and select the ***n*th Derivative** template.
- Enter as shown.



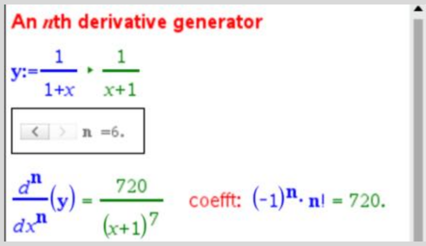
Finally:

- Insert a third **Maths Box** with accompanying text and position as shown.
- Press [?!▸] to select the factorial symbol.
- Enter as shown.

Note: *Alternatively, press* [menu] ***> Calculations >*** ***Probability*** *>* ***Factorial (!)*** *to access the factorial symbol.*

To change the display of a **Maths Box**, for example, to display an equals sign:

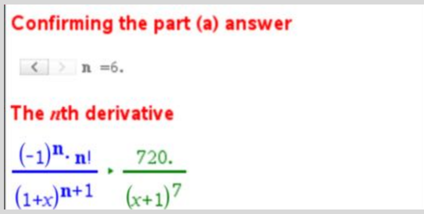
- Click on the **Maths Box**.
- Press [menu] **> Maths Box Options > Maths Box Attributes**.
- Press [tab] to highlight the **Insert Symbol** field.
- Press ▶ and select $=$.



Note: Maths Box Attributes *can also be accessed within a* ***Maths Box*** *by pressing* [ctrl] [menu].

Click on the slider to change the value of n.

To find $\frac{d^n y}{dx^n}$ for $n = 6$, manually change the value of n to 6 in the slider box.



Construct another **Notes** page as shown at right to confirm that $\frac{d^n y}{dx^n} = \frac{(-1)^n n!}{(1+x)^{n+1}}$, $\forall n \in \mathbb{Z}^+$.

(b) Let P_n be the proposition that the nth derivative of y is

$$\frac{d^n y}{dx^n} = \frac{(-1)^n n!}{(1+x)^{n+1}}, \ \forall n \in \mathbb{Z}^+.$$

Consider P_1:

LHS: $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$ and RHS: $\frac{(-1)^1 1!}{(1+x)^{1+1}} = -\frac{1}{(1+x)^2}$

So P_1 is true.

... continued

Solution (continued)

Assume P_k is true for $n = k$, i.e. $\frac{d^k y}{dx^k} = \frac{(-1)^k k!}{(1+x)^{k+1}}$ (for some $k \in \mathbb{Z}^+$). Consider P_{k+1}:

$$\frac{d^{k+1}y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$$

$$= \frac{d}{dx}\left(\frac{(-1)^k k!}{(1+x)^{k+1}}\right)$$

$$= (-1)^k k!\frac{d}{dx}\left(\frac{1}{(1+x)^{k+1}}\right)$$

$$= (-1)^k k!\left(-(k+1)(1+x)^{-(k+2)}\right)$$

$$= \frac{(-1)^{k+1}(k+1)!}{(1+x)^{k+2}}$$

Since P_1 is true and P_k true $\Rightarrow P_{k+1}$ true, P_n is proved true by mathematical induction, $\forall n \in \mathbb{Z}^+$.

Note: *The logic of the concluding statement, which is based on the structure of the induction proof, must be correct and clear.*

Proving by mathematical induction – matrices

Mathematical induction is used to prove results involving powers of matrices.

Question

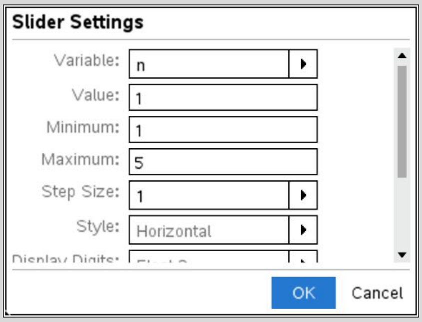
Consider $A=\begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$.

(a) Make a conjecture about A^n, $\forall n\in\mathbb{Z}^+$.

(b) Use mathematical induction to prove the conjecture made in part **(a)**.

Solution

(a) Determine the first few powers of A and conjecture an expression for A^n.



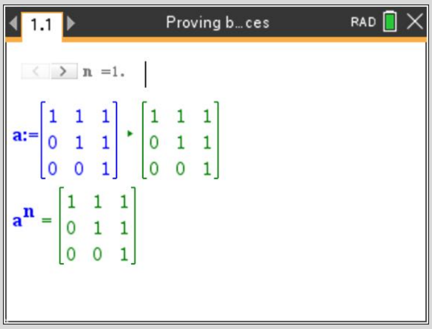
On a **Notes** page, insert a slider to control the value of n as follows:

- Press [menu] > **Insert** > **Slider**.
- Set the **Slider Settings** as shown.
- Click to check the **Minimised** box.

Insert a **Maths Box** as follows:

- Press [menu] > **Insert** > **Maths Box**.

***Note:** Alternatively, to insert a **Maths Box**, press* [ctrl] [M].



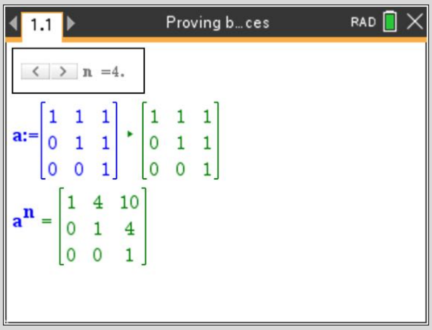
Assign A as follows:

- Press [ctrl] [var] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **3-by-3 Matrix** template and enter as shown.
- Now insert another **Maths Box** and enter a^n.

To change the display of a **Maths Box**, for example, to display an equals sign:

- Click on the **Maths Box**.
- Press [menu] > **Maths Box Options** > **Maths Box Attributes**.
- Press [tab] to highlight the **Insert Symbol** field.
- Press ▶ and select =.

***Note: Maths Box Attributes** can also be accessed within a **Maths Box** by pressing* [ctrl] [menu].



Click on the slider to change the value of n.

Answer: $A^n=\begin{bmatrix}1 & n & \dfrac{n(n+1)}{2}\\0 & 1 & n\\0 & 0 & 1\end{bmatrix}$

... continued

Solution (continued)

(b) Let P_n be the proposition that $A^n = \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}, \forall n \in \mathbb{Z}^+$.

Consider P_1:

$$A^1 = \begin{bmatrix} 1 & 1 & \frac{1(1+1)}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ (So } P_1 \text{ is true.)}$$

Assume P_k is true for $n = k$, i.e. $A^k = \begin{bmatrix} 1 & k & \frac{k(k+1)}{2} \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$, (for some $k \in \mathbb{Z}^+$).

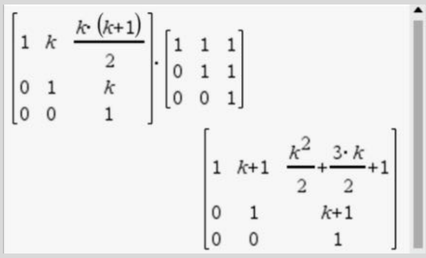
Consider P_{k+1}: $A^{k+1} = A^k A$ or $A^{k+1} = AA^k$

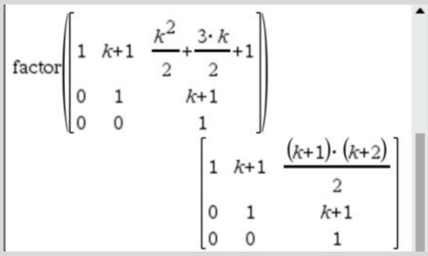
$$A^{k+1} = \begin{bmatrix} 1 & k & \frac{k(k+1)}{2} \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1+k & 1+k+\frac{k(k+1)}{2} \\ 0 & 1 & 1+k \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1+k & \frac{(k+1)(k+2)}{2} \\ 0 & 1 & 1+k \\ 0 & 0 & 1 \end{bmatrix}$$

Since P_1 is true and P_k true $\Rightarrow P_{k+1}$ true, P_n is proved true by mathematical induction, $\forall n \in \mathbb{Z}^+$.

***Notes**: The logic of the concluding statement, based on the structure of the proof, must be correct and clear in an induction proof.*

*The two screenshots right show the calculation of $A^{k+1} = A^k A$ and use of the **Factor** command on A^{k+1}.*





3.2 Functions, relations and graphs

3.2.1 Functions, relations and graphs

Determining parameter values in a rational function

A rational function has the form $f(x)=\dfrac{P(x)}{D(x)}$ where $P(x)$ and $D(x)$ are polynomials.

In quotient-remainder form, $f(x)=Q(x)+\dfrac{R(x)}{D(x)}$.

- Vertical asymptotes occur where $D(x)=0$.
- The non-vertical asymptote has equation $y=Q(x)$.
- The x-axis intercepts occur where $P(x)=0$.
- The y-axis intercept is $f(0)=\dfrac{P(0)}{D(0)}$, $D(0)\neq 0$.
- Stationary points occur where $f'(x)=0$.

Question

Consider the function $f(x)=\dfrac{ax^2+bx+c}{x-1}$, where $x\in\mathbb{R}, x\neq 1$ and a, b and c are non-zero constants.

The graph of $y=f(x)$ passes through the point $\left(3,\dfrac{23}{2}\right)$ and has a minimum point at $(2,10)$.

(a) Find the value of a, the value of b and the value of c.

(b) Plot the graph of $y=f(x)$, giving the coordinates of any turning points, points of intersection with the axes and the equations of any asymptotes.

Consider the case where $a=b=-1$ and $c\neq 2$.

(c) Find the set of values of c such that the curve has no stationary points.

Solution

(a) On a **Calculator** page, assign $f(x)$ as follows:

- Press ctrl to access the **Assign** := command.
- Press ctrl ÷ to access the **Fraction** template.
- Enter as shown.

... continued

Solution (continued)

Since the curve passes through the point $\left(3, \frac{23}{2}\right)$, then $f(3) = \frac{23}{2}$.

Since the curve passes through the point $(2,10)$, then $f(2) = 10$.

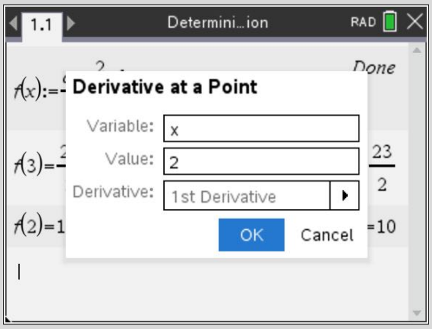
To form the first two linear equations in a, b and c:

- Enter as shown.

Since $(2,10)$ is a minimum point, then $f'(2) = 0$.

To form the third linear equation in a, b and c:

- Press menu > **Calculus** > **Derivative at a Point**.
- Complete the required fields as shown.
- Complete the **Derivative at a Point** template as shown.



*Note: Alternatively, to access the **Derivative** template, press* [templates] 5*. A more efficient alternative is to press* shift −.

By substitution and minor simplification, the following system of linear equations are formed:

$9a + 3b + c = 23$ (1)

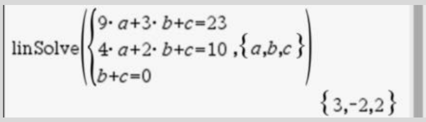
$4a + 2b + c = 10$ (2)

$b + c = 0$ (3)

Note: If required, press shift − *and complete the **Derivative** template to obtain* $f'(x) = \frac{ax^2 - 2ax - b - c}{(x-1)^2}$ *(quotient rule).*

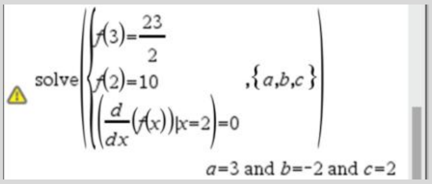
To solve the system of linear equations:

- Press menu > **Algebra** > **Solve System of Equations** > **Solve System of Linear Equations**.
- In the dialog box that follows:
  - For **Number of equations**, enter **3**.
  - For **Variables**, enter ***a*,*b*,*c***.
- Enter the equations into the template and press enter.



Answer: $a = 3, b = -2, c = 2$ and so $f(x) = \frac{3x^2 - 2x + 2}{x - 1}$

Note: Alternatively, press menu > ***Algebra** > **Solve System of Equations** > **Solve System of Equations**. Enter the three equations as shown and press* enter.

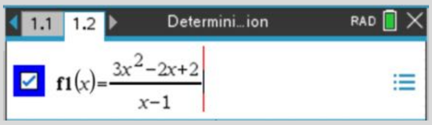


... continued

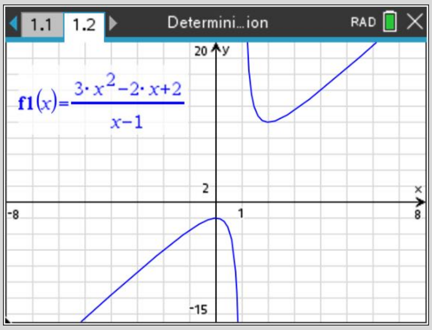
Solution (continued)

(b) On a **Graphs** page:

- Enter as shown.

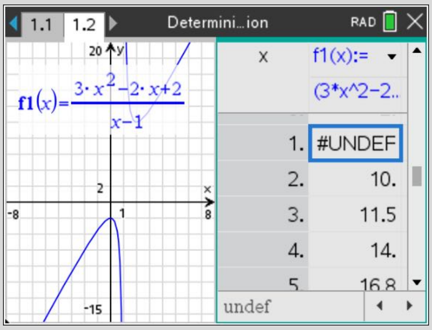


- To add a grid, press menu > **View** > **Grid** > **Lined Grid**.
- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –8 XMax = 8 XScale = 1
  YMin = –15 YMax = 20 YScale = 1



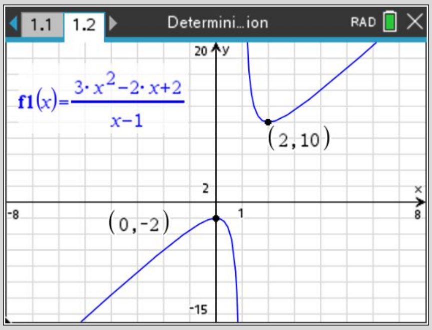
***Notes**: If* $\boldsymbol{f1(x) = f(x) \mid a = 3}$ **and** $\boldsymbol{b = -2}$ **and** $\boldsymbol{c = 2}$ *was entered, then a **Create Sliders** dialog box would appear on the screen. Sliders are not needed at this stage of the example, given that key features of the graph of* $y = f(x)$ *need to be found.*

*If needed, it can be helpful to press menu > **Window/Zoom** > **Zoom Fit** to obtain a first viewing window of the plotted graph. Otherwise, a useful tactic is to press ctrl T to show a tabular representation of the function. To edit the table settings, press menu > **Table** > **Edit Table Settings** and complete as desired. To resize the table's column widths, press menu > **Actions** > **Resize** and resize as desired.*



To find the coordinates of the local maximum and confirm the coordinates of the local minimum:

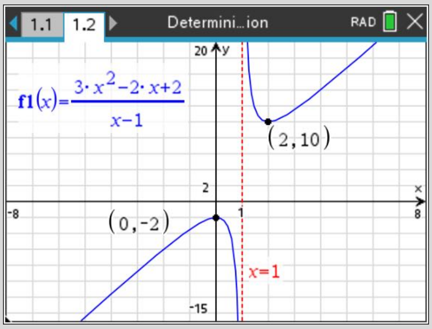
- Press menu > **Analyse Graph** > **Maximum**.
- Move the cursor to the left of the maximum for a lower bound and press enter.
- Move the cursor to the right of the maximum for an upper bound and press enter.
- Repeat the above for the minimum.



Answer: $(2,10)$ is a local minimum, $(0,-2)$ is a local maximum (it is also the y-axis intercept) and the vertical asymptote has equation $x = 1$.

To plot the vertical asymptote in relation graphing mode:

- Press menu > **Graph Entry/Edit** > **Relation**.
- Enter as shown.



To change the colour and appearance of the vertical asymptote:

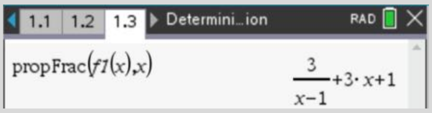
- Move the cursor over the asymptote and press ctrl menu.
- Select **Attributes**, scroll down to **Line Style** and change to a dashed line style.
- With the cursor again over the asymptote, press ctrl menu.
- Select **Colour** > **Line Colour** and change as desired.

... continued

Solution (continued)

To determine the equation of the non-vertical (oblique) asymptote on a **Calculator** page.

- Press [menu] > **Algebra** > **Fraction Tools** > **Proper Fraction**.
- Enter as shown.



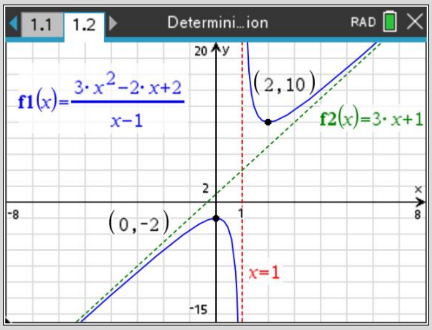
Answer: The non-vertical asymptote has equation $y=3x+1$.

To plot the non-vertical asymptote:

- Press [menu] > **Graph Entry/Edit** > **Function**.
- Enter as shown.

To change the colour and appearance of the non-vertical asymptote:

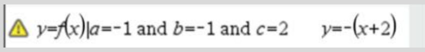
- Move the cursor over the asymptote and press [ctrl][menu].
- Select **Attributes**, scroll down to **Line Style** and change to a dashed line style.
- With the cursor again over the asymptote, press [ctrl][menu].
- Select **Colour** > **Line Colour** and change as desired.



(c) If $c=2$, then $y=\frac{-x^2-x+2}{x-1}=\frac{-(x+2)(x-1)}{x-1}$ becomes $y=-x-2$, $x\neq 1$ and has no stationary point. The graph is a straight line with a point discontinuity at $x=1$.

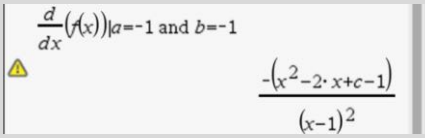
To confirm this on a **Calculator** page:

- Press [ctrl][=] to access the 'with' or 'given' symbol |.
- Enter as shown.



To find $\frac{dy}{dx}$ when $a=b=-1$:

- Press [shift][−] to access the **Derivative** template.
- Press [ctrl][=] to access the 'with' or 'given' symbol |.
- Enter as shown.



When $a=b=-1$, $\frac{dy}{dx}=\frac{-x^2+2x+1-c}{(x-1)^2}$.

Stationary points occur when $-x^2+2x+1-c=0$.

For there to be no stationary points, the equation $-x^2+2x+1-c=0$ must have no real solutions for x.

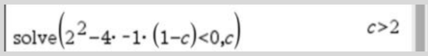
The discriminant is $2^2-4(-1)(1-c)<0$.

... continued

Solution (continued)

To solve this inequality:

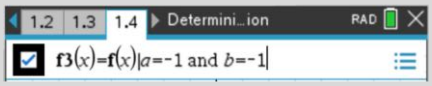
- Press menu > **Algebra** > **Solve**.
- Enter as shown.



Answer: The set of values of c is $c \in \mathbb{R}, c \geq 2$.

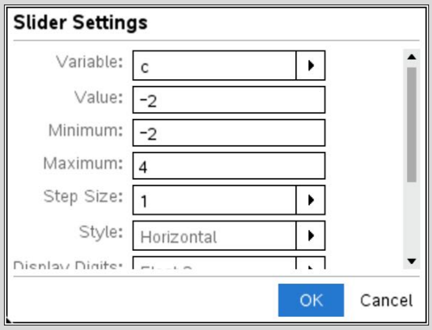
To confirm this result on a **Graphs** page:

- Enter as shown.
- Press ctrl = to access the 'with' or 'given' symbol |.
- In the **Create Sliders** dialog box, create a slider for c only (uncheck the boxes for a and b).
- Move the cursor over the label and press ctrl menu > **Hide**.



To set the **Slider Settings**:

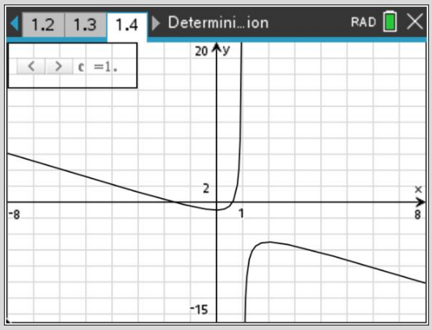
- Move the cursor over the slider.
- Press ctrl menu > **Settings**.
- Set as shown.
- Click to check the **Minimised** box.



To move the slider:

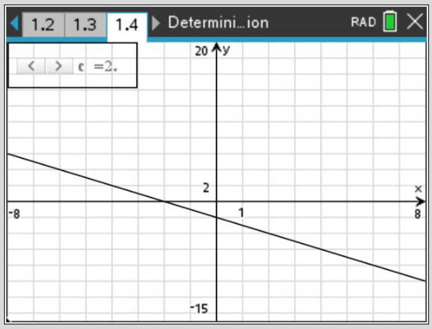
- Click on the slider, then press ctrl menu > **Move** and move it to the top left-hand corner as shown.

__Note__: The slider can be moved when it is framed by a blue border. If the blue border is not showing, click the slider, press ctrl menu *> __Move__ and move it.*



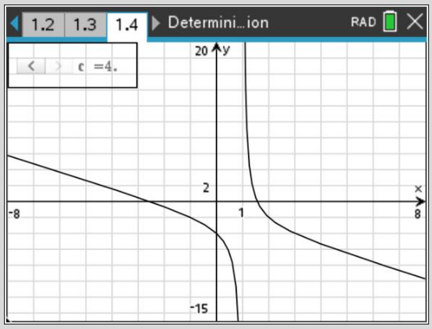
The screenshots at right show the effect of changing c.

When $c = 1$, the graph of $y = f(x)$ has stationary points at $(0, -1)$ and $(2, -5)$.



When $c = 2$, the graph of $y = f(x)$ is a straight line with a point discontinuity at $x = 1$.

For $c > 2$, the graph of $y = f(x)$ has no stationary points.



Investigating a family of rational functions

Question

Consider a family of rational functions $f(x)=\dfrac{dx+e}{ax^2+bx+c}$ where $a,d\in\mathbb{R}, a,d\neq 0$ and $b,c,e\in\mathbb{R}$.

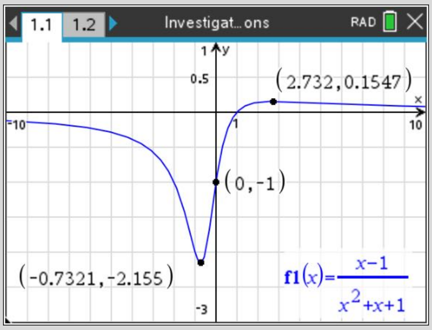
Determine a way to classify the key features of the graph of *f* and the implied domain and range of *f* for various combinations of values of *a*, *b*, *c*, *d* and *e*.

Illustrate your classifications with some well-chosen examples.

***Note**: As this is designed as an open-ended task, suggestions for how students could approach the task are outlined below. As a starting point, students could be encouraged to consider $a=c=1$ with $b^2-4ac<0$, $b^2-4ac=0$ and $b^2-4ac>0$.*

Solution

Depending on the quadratic denominator, there can be either 0, 1 or 2 vertical asymptotes.



Case 1: *a* and *c* have the same sign and $b^2-4ac<0$ i.e. $-2\sqrt{ac}<b<2\sqrt{ac}$.

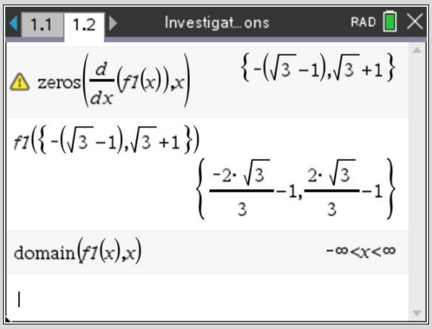
For example, $a=b=c=d=1$ and $e=-1$.

The graph of $y=\dfrac{x-1}{x^2+x+1}$ has no vertical asymptotes (as the quadratic denominator has no real solutions).

The x-axis is a horizontal asymptote.

There is an x-intercept at $(1,0)$ and a y-intercept at $(0,-1)$.

There is a local minimum at $\left(-\sqrt{3}+1,-\dfrac{2}{\sqrt{3}}-1\right)$ and a local maximum at $\left(\sqrt{3}+1,\dfrac{2}{\sqrt{3}}-1\right)$.



The implied domain is $x\in\mathbb{R}$ and the range is $y\in\left[-\dfrac{2}{\sqrt{3}}-1,\dfrac{2}{\sqrt{3}}-1\right]$.

In the general case, the graph has a similar shape to the graph of $y=\dfrac{x-1}{x^2+x+1}$.

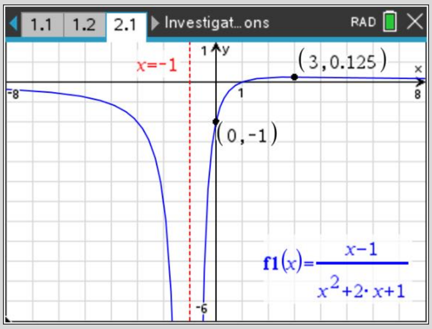
The x-axis is a horizontal asymptote. There is an x-intercept at $\left(-\dfrac{e}{d},0\right)$ and a y-intercept at $\left(0,\dfrac{e}{c}\right)$.

There are two turning points, a local minimum and a local maximum.

The implied domain is $x\in\mathbb{R}$. If the turning points have y-values, y_1 and y_2, where $y_1<y_2$, then the range is $y\in[y_1,y_2]$.

... continued

Solution (continued)



Case 2: There is a special case where $b=c=0$ and hence $x=0$ is the only vertical asymptote.

More generally, a and c have the same sign and $b^2-4ac=0 \Rightarrow b=\pm 2\sqrt{ac}$.

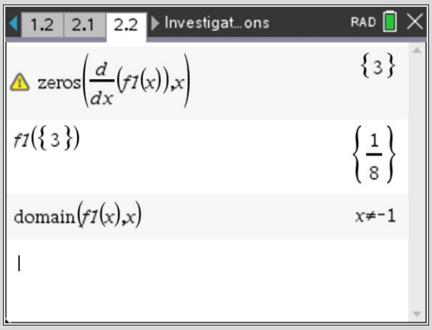
For example, $a=c=d=1$, $b=2$ and $e=-1$.

The graph of $y=\dfrac{x-1}{x^2+2x+1}$ has a single vertical asymptote with equation $x=-1$ (since the quadratic denominator is a perfect square).

The x-axis is a horizontal asymptote.

There is an x-intercept at $(1,0)$ and a y-intercept at $(0,-1)$.

There is a local maximum at $\left(3,\dfrac{1}{8}\right)$.



The implied domain is $x\in\mathbb{R}, x\neq -1$ and the range is $y\in\left(-\infty,\dfrac{1}{8}\right]$.

In the general case, the graph has a similar shape to the graph of $y=\dfrac{x-1}{x^2+2x+1}$.

There is a single vertical asymptote with equation $x=-\dfrac{b}{2a}$.

The x-axis is a horizontal asymptote.

There is an x-intercept at $\left(-\dfrac{e}{d},0\right)$ and a y-intercept at $\left(0,\dfrac{e}{c}\right)$, provided $c\neq 0$.

There is exactly one turning point.

The implied domain is $x\in\mathbb{R}, x\neq -\dfrac{b}{2a}$.

If the turning point has a y-value y_1, then if $y_1<0$, the range is $y\in[y_1,\infty)$, or if $y_1>0$, the range is $y\in(-\infty,y_1]$.

Case 3: There are three variations within this case.
(1) a and c have the same sign and $b^2-4ac>0$.
(2) a and c have different signs (so $b^2-4ac>0$ irrespective of b).
(3) $c=0$ and $b\neq 0$.

... continued

Solution (continued)

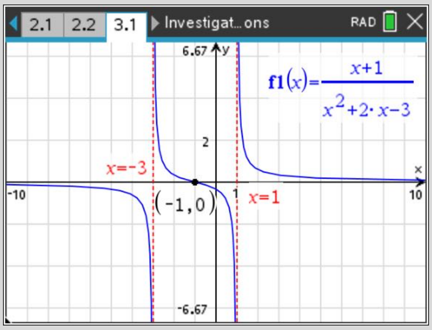
The quadratic has two solutions, say $x = x_1$ and $x = x_2$, where $x_1 < x_2$ (these could be explicitly found in terms of *a*, *b* and *c*), and the graph has vertical asymptotes through these points.
The graph is asymptotic to the x-axis, which it also crosses at $\left(-\frac{e}{d}, 0\right)$.

There is a y-intercept at $\left(0, \frac{e}{c}\right)$, provided $c \neq 0$ (if $c = 0$, the y-axis is one of the vertical asymptotes).
The graph shape depends on the location of the x-intercept.

If the x-intercept lies between the vertical asymptotes, the graph will have no turning points.
The implied domain is $x \in \mathbb{R}, x \neq x_1, x_2$ and the range is then $y \in \mathbb{R}$.
The first example of variation (2), the graph of $y = \frac{x+1}{x^2+2x-3}$, has an x-intercept at $(-1, 0)$ and vertical asymptotes with equations $x = -3$ and $x = 1$.

There are no turning points.

The implied domain is $x \in \mathbb{R}, x \neq -3, 1$ and the range is $y \in \mathbb{R}$.



If the x-intercept does not lie between the vertical asymptotes, the graph will have two turning points.

The implied domain is $x \in \mathbb{R}, x \neq x_1, x_2$.

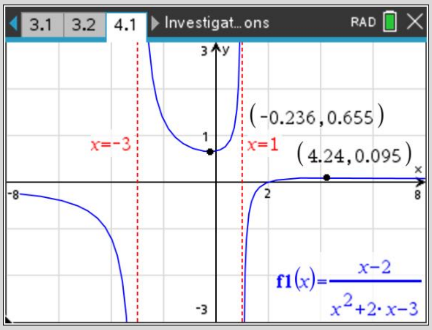
If the turning points have y-values, y_1 and y_2, where $y_1 < y_2$, then the range is $y \in (-\infty, y_1] \cup [y_2, \infty)$.

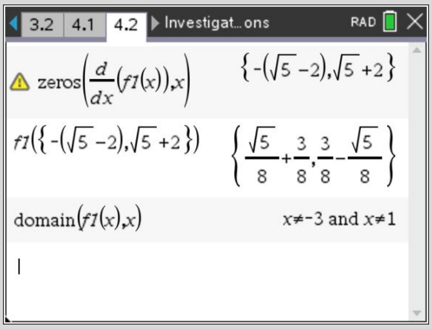
The second example of variation (2), the graph of $y = \frac{x-2}{x^2+2x-3}$, has an x-intercept at (2, 0) and vertical asymptotes with equations $x = -3$ and $x = 1$.

The coordinates of the turning points are $\left(2-\sqrt{5}, \frac{3+\sqrt{5}}{8}\right)$ and $\left(2+\sqrt{5}, \frac{3-\sqrt{5}}{8}\right)$.

The implied domain is $x \in \mathbb{R}, x \neq -3, 1$.

The range is $y \in \left(-\infty, \frac{3-\sqrt{5}}{8}\right] \cup \left[\frac{3+\sqrt{5}}{8}, \infty\right)$.





3.3 Algebra, number and structure

3.3.1 Complex numbers

Using De Moivre's theorem for integral powers

De Moivre's theorem allows us to simplify expressions of the form z^n when z is expressed in polar form.

- $z^n = \left(r\operatorname{cis}\theta\right)^n = r^n\operatorname{cis}\left(n\theta\right)$ where $n \in \mathbb{Z}$

Question

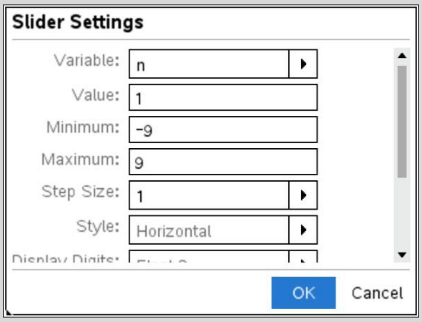
Find the values of n such that $\left(1+\sqrt{3}i\right)^n$ is a real number.

Solution

Start with an exploration.

On a **Notes** page, insert a slider to control the value of n as follows:

- Press [menu] > **Insert** > **Slider**.
- Set the **Slider Settings** as shown.
- Click to check the **Minimised** box.



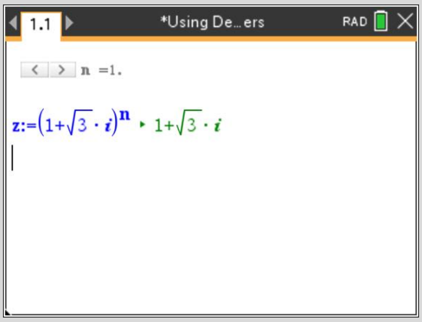
Insert a **Maths Box** as follows:

- Press [menu] > **Insert** > **Maths Box**.

*Note: Alternatively, to insert a **Maths Box**, press* [ctrl] [M].

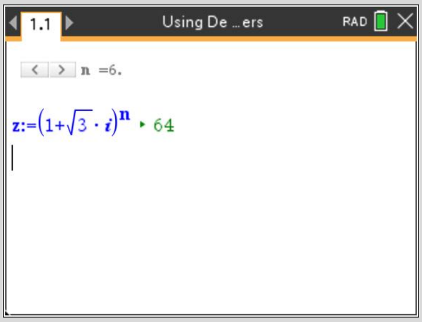
Assign z as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [ctrl] [x²] to access [√].
- Press [π▸] to access ***i***.



Click on the slider to change the value of n.

It appears that $\left(1+\sqrt{3}i\right)^n$ is a real number when $n = 0, \pm 3, \pm 6, \ldots$.



$1+\sqrt{3}i = 2\operatorname{cis}\left(\frac{\pi}{3}\right)$ in polar form.

Using De Moivre's theorem:

$$\left(1+\sqrt{3}i\right)^n = 2^n\operatorname{cis}\left(\frac{n\pi}{3}\right) = 2^n\left(\cos\left(\frac{n\pi}{3}\right) + i\sin\left(\frac{n\pi}{3}\right)\right)$$

$$\sin\left(\frac{n\pi}{3}\right) = 0 \Rightarrow n = 0, \pm 3, \pm 6, \ldots$$

Answer: $n = 3k$ where $k \in \mathbb{Z}$.

Using De Moivre's theorem to find when $z_1 = z_2$

Question

Consider $z_1 = \left(1+\sqrt{3}i\right)^m$ and $z_2 = (1-i)^n$.

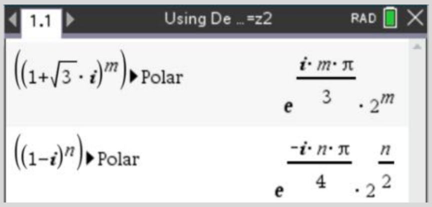
(a) Express z_1 and z_2 in polar form in terms of m and n respectively.

(b) Hence, find the smallest positive integers m and n such that $z_1 = z_2$.

Solution

(a) Convert z_1 and z_2 to polar form on a **Calculator** page as follows:

- Enter z_1 as shown.
- Press [π▸] to access ***i***.
- Press [menu] **> Number** > **Complex Number Tools >** **Convert to Polar**.
- Repeat for z_2.

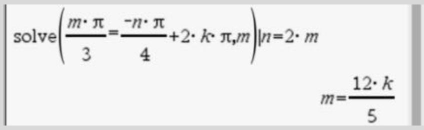


Answer: $z_1 = 2^m \operatorname{cis}\left(\dfrac{m\pi}{3}\right)$ and $z_2 = 2^{\frac{n}{2}} \operatorname{cis}\left(-\dfrac{n\pi}{4}\right)$

(b) $2^m = 2^{\frac{n}{2}} \Rightarrow n = 2m$ and $\dfrac{m\pi}{3} = -\dfrac{n\pi}{4} + 2k\pi$ where $k \in \mathbb{Z}$

To solve $\dfrac{m\pi}{3} = -\dfrac{n\pi}{4} + 2k\pi$ for m with $n = 2m$:

- Press [menu] **> Algebra** > **Solve**.
- Press [ctrl] [÷] to access the **Fraction** template.
- Press [π▸] to access π.
- Press [ctrl] [=] to access the 'with' or 'given' symbol |.



Answer: As $m = \dfrac{12k}{5}$, the smallest value of k such that $m \in \mathbb{Z}^+$ is 5 and so $m = 12$ and $n = 24$.

- To check the answer, enter as shown.

Determining the nth roots of complex numbers

Equations of the form $z^n = a$ where $a \in \mathbb{C}$ are often solved using De Moivre's theorem.

Given that $z = r\text{cis}(\theta)$ and $a = s\text{cis}(\alpha)$.

By substituting into $z^n = a$, using De Moivre's theorem and comparing both modulus and argument, it can be shown that:

- $\theta = \frac{1}{n}(\alpha + 2k\pi)$ where $k \in \mathbb{Z}$.
- The solutions of $z^n = a$ lie on a circle with centre the origin and radius $|a|^{\frac{1}{n}}$.
- There are n solutions equally spaced around the circle at intervals of $\frac{2\pi}{n}$.

Question

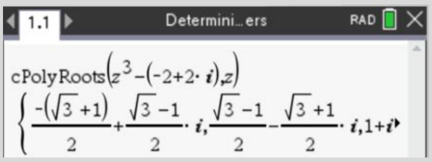
Solve the equation $z^3 = -2 + 2i$, giving your answers in

(a) Cartesian form. **(b)** polar form.

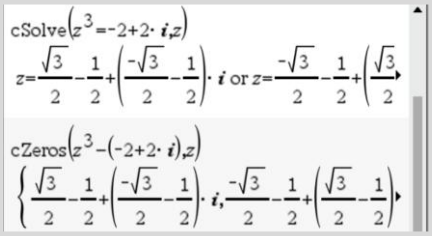
Solution

(a) On a **Calculator** page:

- Press menu > **Algebra** > **Polynomial Tools** > **Complex Roots of Polynomial**.
- Press π▸ to access ***i***.
- Enter as shown.



***Notes:** The **cPolyRoots(** command assumes an equation set to zero. Alternatively, press menu > **Algebra** > **Complex** > **Solve** to access the **cSolve(** command or press menu > **Algebra** > **Complex** > **Zeros** to access the **cZeros(** command. The **cZeros(** command also assumes an equation set to zero. The **cPolyRoots(** command gives the roots in a slightly different yet equivalent form.*

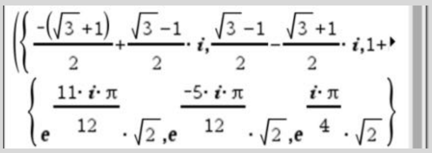


Answer: In Cartesian form, the roots are

$$\frac{-(\sqrt{3}+1)}{2} + \frac{\sqrt{3}-1}{2}i, \frac{\sqrt{3}-1}{2} - \frac{\sqrt{3}+1}{2}i, 1+i.$$

(b) To express these roots in polar form:

- Press ctrl (–) to access ans.
- Press menu > **Number** > **Complex Number Tools** > **Convert to Polar**.
- Enter as shown.



Note:** Alternatively, press* 📖 1 P*, scroll down and select* ***►Polar.

Answer: In polar form, the roots are

$$\sqrt{2}\text{cis}\left(\frac{11\pi}{12}\right), \sqrt{2}\text{cis}\left(-\frac{5\pi}{12}\right), \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right).$$

... continued

Solution (continued)

Expressing $-2+2i$ in polar form:

$-2+2i = 2\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$ and so $z^3 = 2\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$.

$z_1 = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$.

Adding $\frac{2\pi}{3}$ gives $z_2 = \sqrt{2}\text{cis}\left(\frac{\pi}{4}+\frac{2\pi}{3}\right) = \sqrt{2}\text{cis}\left(\frac{11\pi}{12}\right)$.

Subtracting $\frac{2\pi}{3}$ gives $z_3 = \sqrt{2}\text{cis}\left(\frac{\pi}{4}-\frac{2\pi}{3}\right) = \sqrt{2}\text{cis}\left(-\frac{5\pi}{12}\right)$.

Determining and examining the nth roots of unity

The solutions of the equation $z^n = 1$ are called the *n*th roots of unity.

- The solutions of $z^n = 1$ lie on the unit circle.
- There are *n* solutions equally spaced around the circle at intervals of $\frac{2\pi}{n}$.
- $z = 1$ is always a solution.

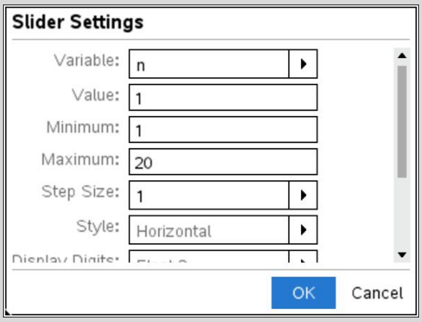
Question

Construct a dynamic demonstration of the *n*th roots of unity.

Solution

On a **Notes** page, insert a slider to control the value of *n* as follows:

- Press [menu] > **Insert** > **Slider**.
- Set the **Slider Settings** as shown.
- Click to check the **Minimised** box.

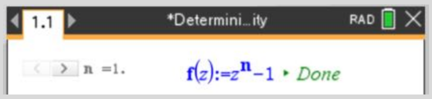


To enter $\boldsymbol{f(z) := z^n - 1}$:

- Press [menu] > **Insert** > **Maths Box**.

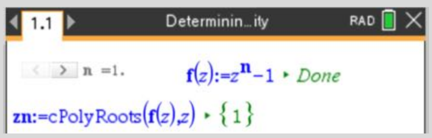
Note: *Alternatively, to insert a* ***Maths Box****, press* [ctrl] [M].

- Press [ctrl] [template] to access the **Assign** [:=] command.
- Enter as shown.



To enter $\boldsymbol{zn := \text{cPolyRoots}(f(z), z)}$:

- Press [ctrl] [M] to insert a **Maths Box**.
- Press [ctrl] [template] to access the **Assign** [:=] command.
- Press [menu] > **Calculations** > **Algebra** > **Polynomial Tools** > **Complex Roots of Polynomial**.
- Enter as shown.

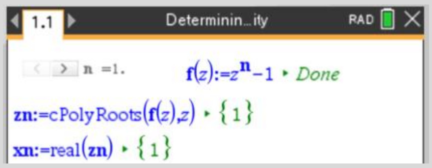


... continued

Solution (continued)

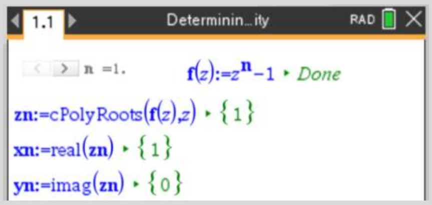
To enter $xn := \mathbf{real}(zn)$:

- Press [ctrl] [M] to insert a **Maths Box**.
- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Either type the command **real(** or press [menu] > **Calculations** > **Number** > **Complex Number Tools** > **Real Part**.
- Press [var] to access assigned/stored variables.



To enter $yn := \mathbf{imag}(zn)$:

- Press [ctrl] [M] to insert a **Maths Box**.
- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Either type the command **imag(** or press [menu] > **Calculations** > **Number** > **Complex Number Tools** > **Imaginary Part**.
- Press [var] to access assigned/stored variables.

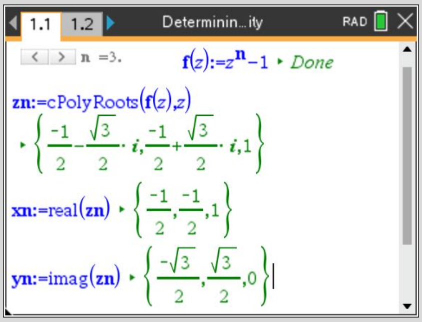


***Note:** If desired, to hide the input, press [menu] > **Maths Box Options** > **Maths Box Attributes**. Change **Show Input & Output** to **Hide Input**, press [tab] to highlight **OK** and press [enter]. Alternatively, to access **Maths Box Attributes**, press [ctrl] [menu] and select **Maths Box Attributes**.*

Click on the slider to change the value of n.

The nth roots of unity are displayed.

For example, the screenshot at right shows the roots of the equation $z^3 = 1$, namely, $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i, 1$.

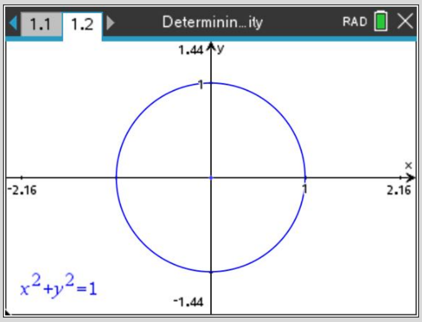


To display the roots of unity on a **Graphs** page:

- Press [menu] > **Window/Zoom** > **Window Settings**.
 In the dialog box that follows, enter the following values:
 XMin = –2.16 XMax = 2.16 XScale = 1
 YMin = –1.44 YMax = 1.44 YScale = 1

To graph the unit circle $x^2 + y^2 = 1$:

- Press [menu] > **Graph Entry/Edit** > **Relation**.
- Enter as shown.



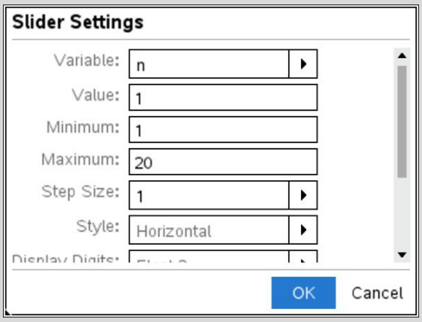
***Note:** Alternatively, to plot the unit circle, press [menu] > **Graph Entry/Edit** > **Equation Templates** > **Circle** > **Centre form** $(x-h)^2 + (y-k)^2 = r^2$.*

... continued

Solution (continued)

Insert a slider to control the value of n as follows:

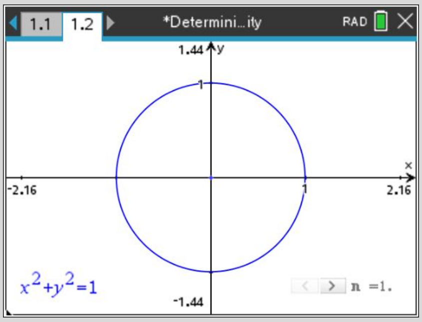
- Press menu > **Actions** > **Insert Slider**.
- Set the **Slider Settings** as shown.
- Click to check the **Minimised** box.



To move the slider:

- Click the slider, then press ctrl menu > **Move** and move it to the bottom right-hand corner as shown.

Note: *The slider is moveable when it is framed by a blue border. If the blue border is not showing, click the slider, press* ctrl menu > ***Move*** *and move it.*



To plot the nth roots of unity:

- Press menu > **Graph Entry/Edit** > **Scatter Plot**.
- Next to $\boldsymbol{x} \leftarrow$ enter $\boldsymbol{xn}$ and next to $\boldsymbol{y} \leftarrow$ enter $\boldsymbol{yn}$.

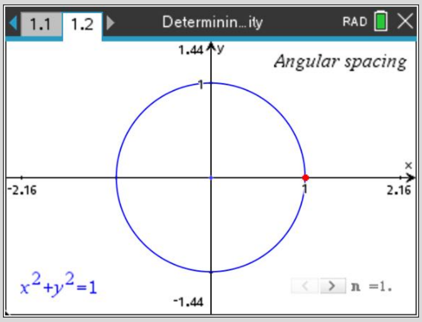
To hide the coordinates $(\boldsymbol{xn}, \boldsymbol{yn})$:

- Move the cursor over the coordinates and press ctrl menu > **Hide**.

Note: *To see hidden objects, press* menu > ***Actions*** > ***Hide/Show***. *To bring an object back onto a page, press* menu > ***Actions*** > ***Hide/Show***, *move the cursor over the object and press* enter.

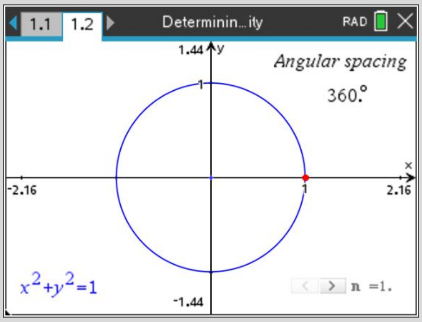
To change the colour of the point at $(1, 0)$:

- Move the cursor over the point, press ctrl menu > **Colour** > **Line Colour**.



To add the text '*Angular spacing*' to the top right-hand corner of the page:

- Press menu > **Actions** > **Text**.
- Move the cursor up towards the top right-hand corner of the page and press enter.
- Enter as shown and press enter esc.



To display the angular spacing in degrees:

- Press menu > **Actions** > **Text**.
- Move the cursor up towards the top right-hand corner of the page (underneath the text '*Angular spacing*') and press enter.
- Enter $\mathbf{360}\,/\,\boldsymbol{n}$ and press enter esc.

... continued

Solution (continued)

To display the degrees symbol:

- Press menu > **Actions** > **Text**.
- Move the cursor next to the number, press enter π▸ and select the **Degrees** symbol.
- Press enter esc.

Note: *The degrees symbol may need to be moved closer to the number, as it is a separate text object to the value of the angular spacing.*

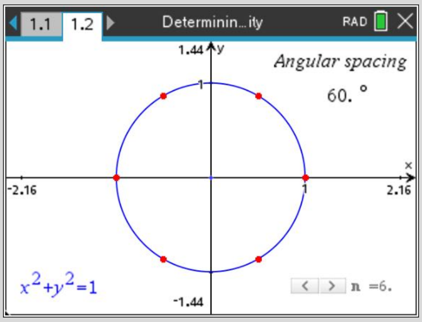
To display the angular spacing for particular values of n:

- Press menu > **Actions** > **Calculate**.
- Move the cursor over $\mathbf{\frac{360}{n}}$ and press enter L enter esc.
- Move the cursor over $\mathbf{\frac{360}{n}}$ and press ctrl menu > **Hide**.

Click on the slider to change the value of n.

The nth roots of unity appear with the angular spacing between each root displayed.

For example, the screenshot at right shows the six roots of the equation $z^6 = 1$ with these roots equally spaced around the unit circle at intervals of $\frac{2\pi}{6} = \frac{\pi}{3} = 60°$.



Applying the remainder theorem to polynomials

The remainder theorem states that when a polynomial $p(z)$ is divided by $z-\alpha$, where $\alpha \in \mathbb{C}$, the remainder is $p(\alpha)$.

Question

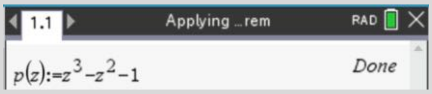
Consider $p(z)=z^3-z^2-1$.

Find the remainder in polar form when $p(z)$ is divided by $z-i$.

Solution

On a **Calculator** page, assign $p(z)$ as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Enter as shown.

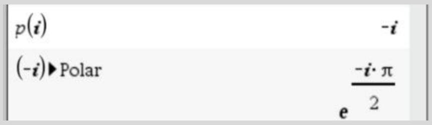


The remainder is given by $p(i)$.

- Press [π▸] to access ***i*** and enter as shown.

To express in polar form:

- Press [ctrl] [(−)] to access [ans].
- Press [menu] > **Number** > **Complex Number Tools** > **Convert to Polar**.
- Enter as shown.

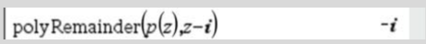


Note: *Alternatively, press [📖] [1] [P], scroll down and select* ***►Polar***.

Answer: When $p(z)$ is divided by $z-i$, the remainder is $\text{cis}\left(-\frac{\pi}{2}\right)$.

Alternatively, to find $p(i)$:

- Press [menu] > **Algebra** > **Polynomial Tools** > **Remainder of Polynomial**.
- Enter as shown.



Note: *The syntax for the* ***polyRemainder(*** *command is* ***polyRemainder(Poly1,Poly2[,Var])***.

Applying the factor theorem to polynomials

The factor theorem states that $z-\alpha$, where $\alpha \in \mathbb{C}$, is a factor of a polynomial $p(z)$ if and only if $p(\alpha)=0$.

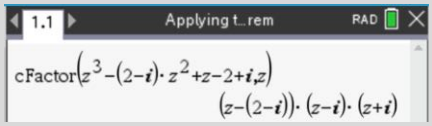
Question

Factorise $z^3-(2-i)z^2+z-2+i$ over $\mathbb{C}$.

Solution

Factorise on a **Calculator** page as follows:

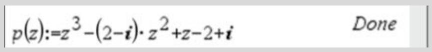
- Press [menu] > **Algebra** > **Complex** > **Factor**.
- Press [π▸] to access ***i***.
- Enter as shown.



Answer: $(z-2+i)(z-i)(z+i)$

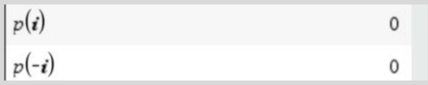
Let $p(z)=z^3-(2-i)z^2+z-2+i$.

- Press [ctrl] [⁝=] to access the **Assign** [:=] command.
- Press [π▸] to access ***i*** and enter as shown.



$p(i)=0$ and so $z-i$ is a factor.

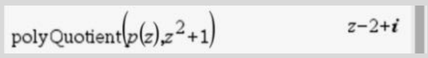
$p(-i)=0$ and so $z+i$ is a factor.



Hence $(z-i)(z+i)=z^2+1$ is a factor of $p(z)$.

To find the other (linear) factor:

- Press [menu] > **Algebra** > **Polynomial Tools** > **Quotient of Polynomial**.
- Enter as shown.



Note: *The syntax for the* ***polyQuotient(*** *command is* ***polyQuotient(Poly1,Poly2[,Var])***.

$$
\begin{aligned}
z^3-(2-i)z^2+z-2+i &= z^2(z-2+i)+1(z-2+i)\\
&=(z^2+1)(z-2+i)\\
&=(z-i)(z+i)(z-2+i)
\end{aligned}
$$

Understanding and using the complex conjugate root theorem

Let $p(z)$ be a polynomial with real coefficients. If $a+bi$, where $a,b \in \mathbb{R}$, is a solution of the equation $p(z)=0$, then the complex conjugate $a-bi$ is also a solution.

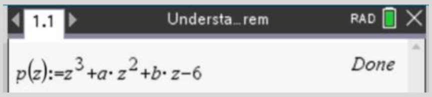
Question

Consider $p(z)=z^3+az^2+bz-6$ where $a,b \in \mathbb{R}$.

Given that $p(-1+i)=0$, find the solutions to the equation $p(z)=0$.

Solution

On a **Calculator** page, assign $p(z)$ as follows:

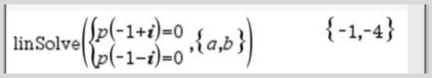


- Press ctrl [:=] to access the **Assign** [:=] command.
- Enter as shown.

From the conjugate root theorem, $p(-1-i)=0$.

To find the values of *a* and *b*, solve the system of linear equations $p(-1+i)=0$ and $p(-1-i)=0$ for *a* and *b*.

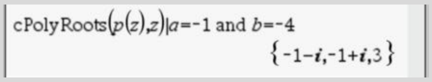
- Press menu > **Algebra** > **Solve System of Equations** > **Solve System of Linear Equations**.
- In the dialog box that follows:
  - For **Number of equations**, enter **2**.
  - For **Variables**, enter ***a*,*b***.
- Enter the equations into the template.
- Press π▸ to access ***i***.



So $a=-1, b=-4$ and the equation is $z^3-z^2-4z-6=0$.

To find the other solution to $p(z)=0$:

- Press menu > **Algebra** > **Polynomial Tools** > **Complex Roots of Polynomial**.
- Press ctrl = to access the 'with' or 'given' symbol |.
- Enter as shown.



Answer: $z=-1\pm i, 3$

Alternatively: $(z-(-1+i))(z-(-1-i))=z^2+2z+2$

Equating the coefficients of $(z^2+2z+2)(z-\alpha)$ and $p(z)$ and solving gives $\alpha=3$.

Solving over C by completing the square

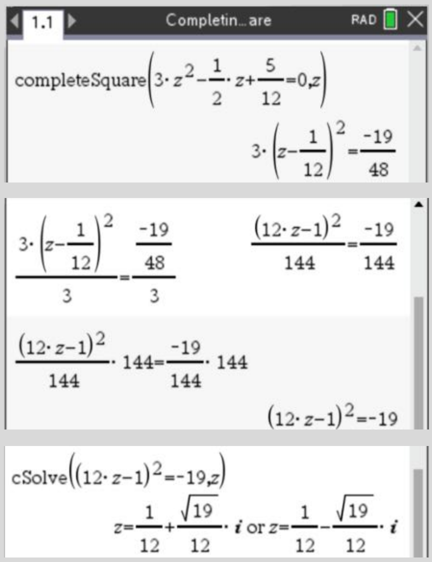
Question

By completing the square, solve $3z^2 - \frac{1}{2}z + \frac{5}{12} = 0$ where $z \in \mathbb{C}$.

Solution

Complete the square on a **Calculator** page as follows

- Press [menu] > **Algebra** > **Complete the Square**.
- Enter as shown.



Completing the square gives $3\left(z - \frac{1}{12}\right)^2 = -\frac{19}{48}$.

- Divide both sides by 3 as shown.
- Multiply both sides by 144 as shown.

This gives $(12z - 1)^2 = -19$.

To solve this equation for z:

- Press [menu] > **Algebra** > **Complex** > **Solve**.
- Enter as shown.

Answer: $z = \frac{1}{12} \pm \frac{\sqrt{19}}{12} i$

Demonstrating the fundamental theorem of algebra

Every polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ of degree n, where $n \geq 1$ and $a_i \in \mathbb{C}$ has at least one linear factor in $\mathbb{C}$.

Hence a polynomial of degree n can be factorised into n linear factors in $\mathbb{C}$:

$$P(z) = a_n (z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_n) \text{ where } \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}.$$

Every polynomial of degree n, where $n \geq 1$:

- has at least one zero, though that zero may be complex.
- with complex coefficients has precisely n zeros, as counted by their multiplicities.

Question

Construct a dynamic demonstration of the fundamental theorem of algebra for polynomials with real coefficients.

Solution

Generate random polynomials on a **Notes** page and represent the roots of the corresponding polynomial equations on a **Graphs** page.

To enter $\boldsymbol{f(x) := \mathrm{sign}(t) \cdot \mathrm{randPoly}(x, degree)}$:

- Press [menu] > **Insert** > **Maths Box**.

***Note:** Alternatively, to insert a **Maths Box**, press* [ctrl] [M].

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [menu] > **Calculations** > **Number** > **Number Tools** > **Sign**.
- Press [catalog] [1] [R], scroll down and select **randPoly(**.
- Enter as shown.

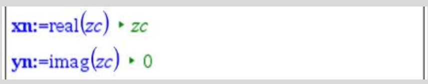
To enter $\boldsymbol{zc := \mathrm{cZeros}(f(x), x)}$:

- Press [ctrl] [M] to insert a **Maths Box**.
- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [menu] > **Calculations** > **Algebra** > **Complex** > **Zeros**.
- Enter as shown.

***Note:** For now, the output will show a **Domain error** message.*

To enter $\boldsymbol{xn := \mathrm{real}(zc)}$ and $\boldsymbol{yn := \mathrm{imag}(zc)}$:

- Press [ctrl] [M] to insert a **Maths Box**.
- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [menu] > **Calculations** > **Number** > **Complex Number Tools** > **Real Part**.
- Press [var] to access assigned/stored variables.
- Press [menu] > **Calculations** > **Number** > **Complex Number Tools** > **Imaginary Part**.
- Enter as shown.



... continued

Solution (continued)

To count the number of solutions obtained from a random polynomial equation, enter $\textit{\textbf{solutions}} := \mathbf{dim}(zi)$:

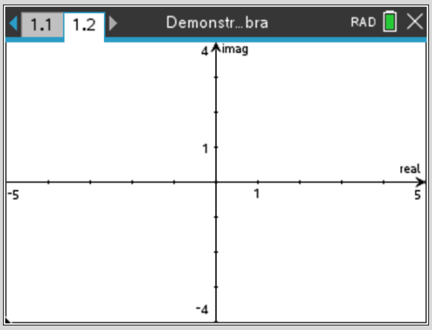
- Press ctrl M to insert a **Maths Box**.
- Press ctrl [templates] to access the **Assign** [:=] command.
- Press [catalog] 1 D, scroll down and select **dim(**.
- Press var to access assigned/stored variables.
- Enter as shown.

Notes: *For now, the output will show an* ***Invalid data type*** *error message.*

If desired, to hide the input, press menu *>* ***Maths Box Options*** *>* ***Maths Box Attributes****. Change* ***Show Input & Output*** *to* ***Hide Input****, press* tab *to highlight* ***OK*** *and press* enter*. Alternatively, to access* ***Maths Box Attributes****, press* ctrl menu *and select* ***Maths Box Attributes****.*

To set up a display of the roots of random polynomial equations on a **Graphs** page:

- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –5 XMax = 5 XScale = 1
  YMin = –4 YMax = 4 YScale = 1

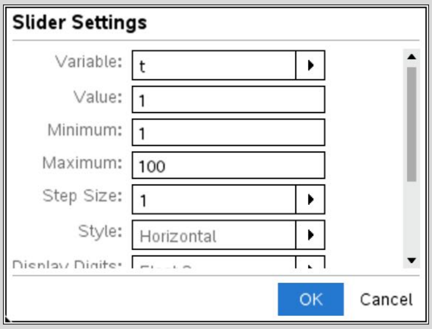


To change the axes labels:

- Move the cursor over the x-axis label.
- Press enter enter and change to **real**.
- Move the cursor over the y-axis label.
- Press enter enter and change to **imag**.

Insert a slider to control the value of t as follows:

- Press menu > **Actions** > **Insert Slider**.
- Set the **Slider Settings** as shown.
- Click to check the **Minimised** box.
- Ensure to not check the **Show Variable** box.



To move the slider:

- Click on the slider, then press ctrl menu > **Move** and move it to the top left-hand corner as shown below right.

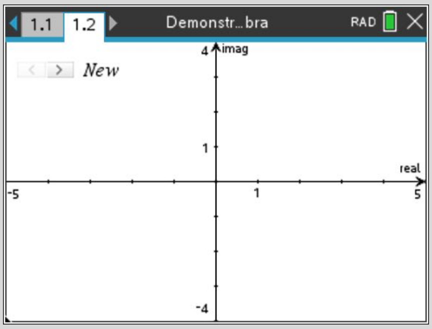
Note*: The slider can be moved when it is framed by a blue border. If the blue border is not showing, click the slider, press* ctrl menu *>* ***Move*** *and move it.*

... continued

Solution (continued)

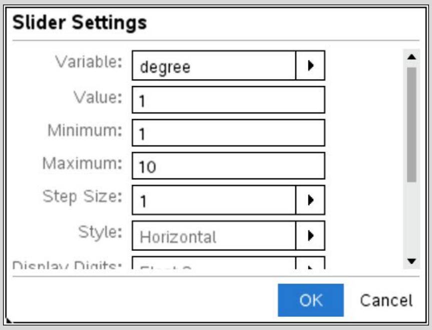
To add text next to this slider:

- Press menu > **Actions** > **Text**.
- Press enter to position and create the text box.
- Type the word '*New*'.
- Press enter esc.



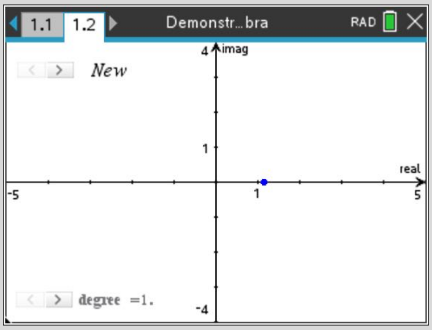
Insert a slider to control the degree of the polynomial as follows:

- Press menu > **Actions** > **Insert Slider**.
- Set the **Slider Settings** as shown.
- Click to check the **Minimised** box.
- Move the slider to the bottom left-hand corner as shown below right.



To plot the roots:

- Press menu > **Graph Entry/Edit** > **Scatter Plot**.
- Next to $x\leftarrow$ enter $\boldsymbol{xn}$ and next to $y\leftarrow$ enter $\boldsymbol{yn}$.



To hide the coordinates $(\boldsymbol{xn},\boldsymbol{yn})$:

- Move the cursor over the coordinates and press ctrl menu > **Hide**.

Notes: *To see hidden objects, press* menu *>* ***Actions*** *>* ***Hide/Show****. To bring an object back onto a page, press* menu *>* ***Actions*** *>* ***Hide/Show****, move the cursor over the object and press* enter.

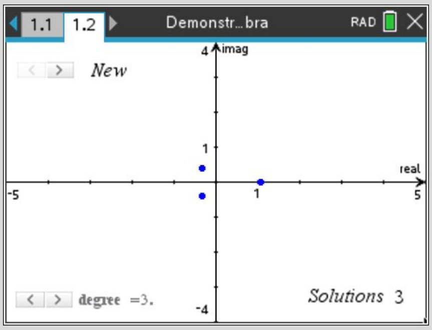
Once the sliders are constructed, the error messages on the ***Notes*** *page are resolved.*

To change the colour of the point:

- Move the cursor over the point, press ctrl menu > **Colour** > **Line Colour**.

To add the text '*Solutions*' to the bottom right-hand corner of the page:

- Press menu > **Actions** > **Text**.
- Move the cursor towards the bottom right-hand corner of the page and press enter to open a text box.
- Enter as shown and press enter esc.



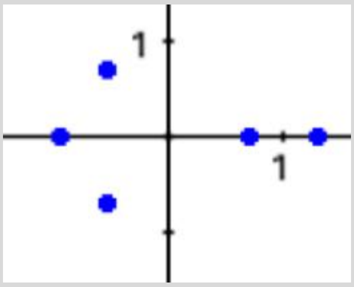
To display the number of solutions:

- Press menu > **Actions** > **Calculate**.
- Move the cursor over '*Solutions*' and press enter L enter esc.

... continued

Solution (continued)

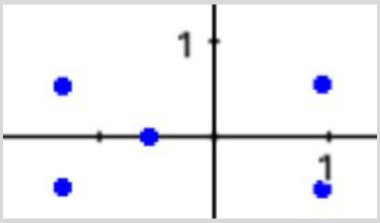
To change the random polynomial and keep the degree of the polynomial the same, click on the top-left slider called '*New*'.



To change the degree of the random polynomial, click on the bottom-left slider.

Ask students to consider and document the possible number, formation and type of roots for polynomial equations with real coefficients of varying degree (odd and even).

For example, for polynomial equations with real coefficients of degree 5, ask students to predict the possible plot shapes and hence the number and nature of roots (repeated roots are a possibility).



Geometrically, this demonstration shows that complex conjugate roots are reflections in the real axis.

Two possible examples are shown at right. What possibility is missing?

3.4 Calculus

3.4.1 Differential calculus and integral calculus

Graphing anti-derivatives of a function

Given the graph $y = f(x)$, the following deductions can be made about the graph of $y = F(x)$ where $F(x)$ is any anti-derivative of $f(x)$.

Graph of $y = f(x)$	Graph of $y = F(x)$
Negative	Negative gradient
Positive	Positive gradient
Cuts the x-axis from negative to positive	Local minimum
Cuts the x-axis from positive to negative	Local maximum
Touches the x-axis	Stationary point of inflection
Turning point	Point of inflection

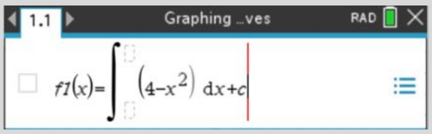
Question

Plot the graphs of $\int f(x)dx + c$ for $c = -2, -1, 0, 1, 2$ where $f(x) = 4 - x^2$.

Solution

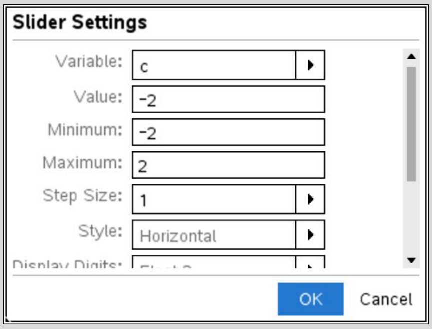
On a **Graphs** page, plot these anti-derivative graphs as follows:

- Press 📖 5 to access the **Integral** template.
- Complete the **Integral** template as shown.



***Note**: Alternatively, to access the **Integral** template press* ⇧shift +.

- In the **Create Sliders** dialog box, create a slider for *c*.
- To add a grid, press menu > **View** > **Grid** > **Lined Grid**.
- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –10 XMax = 10 XScale = 1
  YMin = –10 YMax = 10 YScale = 1



To set the **Slider Settings**:

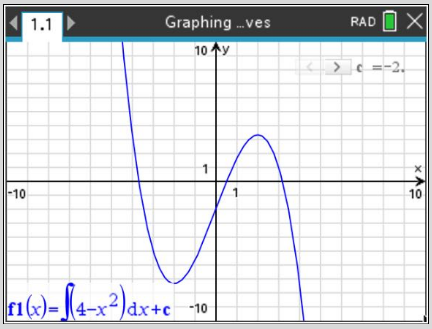
- Move the cursor over the slider.
- Press ctrl menu > **Settings**.
- Set as shown.
- Click to check the **Minimised** box.

... continued

Solution (continued)

To move the slider:

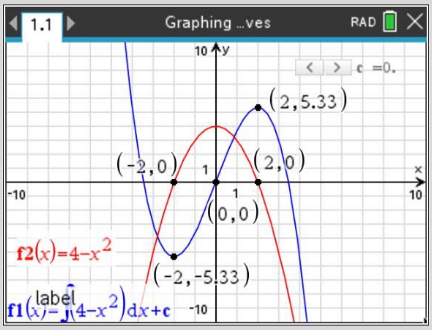
- Click on the slider, then press ctrl menu > **Move** and move it to the top right-hand corner as shown.



Note: *The slider can be moved when it is framed by a blue border. If the blue border is not showing, click the slider, press* ctrl menu *>* ***Move*** *and move it.*

- Graph $y=f(x)$ alongside the graph of $y=F(x)$, by entering $f2(x)=4-x^2$.

Answer: Comparing the graphs of $y=f(x)$ and the graph of $y=F(x)$ for $c=0$:



The graph of $y=f(x)$ is negative for $x<-2$ or $x>2$.

The graph of $y=F(x)$ has negative gradient for $x<-2$ or $x>2$.

The graph of $y=f(x)$ is positive for $-2<x<2$.

The graph of $y=F(x)$ has positive gradient for $-2<x<2$.

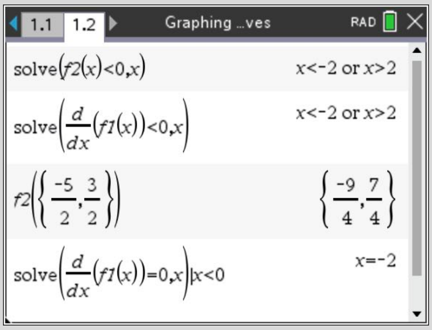
The graph of $y=f(x)$ cuts the x-axis from negative to positive at $x=-2$.

The graph of $y=F(x)$ has a local minimum at $x=-2$.

The graph of $y=f(x)$ cuts the x-axis from positive to negative at $x=2$.

The graph of $y=F(x)$ has a local maximum at $x=2$.

The graph of $y=f(x)$ has a turning point at $x=0$.



Note: *To confirm these features, press* menu *>* ***Analyse Graph*** *and select features such as* ***Zero***, ***Minimum***, ***Maximum*** *and* ***Inflection***. *Move the cursor to the left of the key feature for a lower bound and press* enter. *Move the cursor to the right of the key feature for an upper bound and press* enter.
These results can also be confirmed on a ***Calculator*** *page. A sample of this analysis is shown at right.*

Finding derivatives of inverse circular functions

Question

Find $\frac{d}{dx}\left(\tan^{-1}\left(f(x)\right)\right)$ where f is a differentiable function.

Solution

On a **Calculator** page, find as follows:

- Press [menu] > **Calculus** > **Derivative**.
- Press [trig] and select $\mathbf{tan^{-1}}$.
- Complete the **Derivative** template as shown.

***Note**: Alternatively, to access the **Derivative** template, press [📖] [5]. A more efficient alternative is to press [⇧shift] [−].*

Answer: $\frac{d}{dx}\left(\tan^{-1}\left(f(x)\right)\right)=\frac{f'(x)}{1+\left(f(x)\right)^2}$

Finding points of inflection and determining concavity

Consider a curve $y=f(x)$ where f is differentiable, $\forall x\in(a,b)$.

If $f''(x)>0$ on (a,b), then the gradient of the curve is strictly increasing over (a,b). The curve is said to be concave up.

If $f''(x)<0$ on (a,b), then the gradient of the curve is strictly decreasing over (a,b). The curve is said to be concave down.

A point of inflection is where a curve changes concavity (from concave up to concave down or from concave down to concave up). In other words, a point of inflection is where $f''(x)$ changes sign.

Question

Let X be a normally distributed random variable with mean μ and standard deviation σ.

The probability density function is given by

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \text{ where } x\in\mathbb{R}.$$

The curve $y=f(x)$ has two points of inflection.

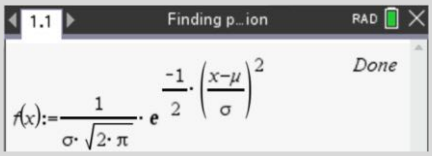
(a) Find the x-coordinates of these two points of inflection.

(b) Find the area under the curve bounded by the vertical lines containing these two points of inflection. Give your answer correct to two decimal places.

Solution

(a) On a **Calculator** page, assign $f(x)$ as follows:

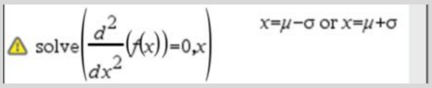
- Press ctrl [:=] to access the **Assign** := command.
- Press ctrl ÷ to access the **Fraction** template.
- Press ctrl [catalog] to access σ and μ.
- Press π to access π.
- Press e^x to access e.
- Enter as shown.



The points of inflection are where the curve changes concavity i.e. where $f''(x)=0$.

To find the x-coordinate of the two points of inflection:

- Press menu > **Algebra** > **Solve**.
- Press [catalog] 5 to access the **2nd Derivative** template.
- Complete the **2nd Derivative** template as shown.



Answer: $x=\mu\pm\sigma$ i.e. the points of inflection occur at one standard deviation below the mean ($\mu-\sigma$) and one standard deviation above the mean ($x=\mu+\sigma$).

(b) The required area is $\Pr(\mu-\sigma\le X\le\mu+\sigma)$.

To find this area, convert the x-values to z-values using $z=\dfrac{x-\mu}{\sigma}$.

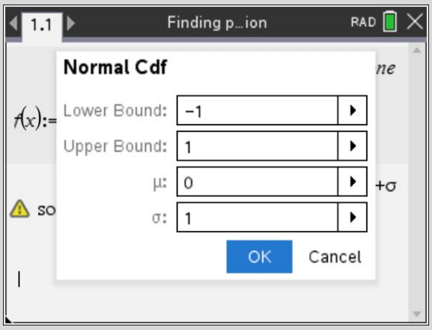
When $x=\mu-\sigma$, $z=\dfrac{(\mu-\sigma)-\mu}{\sigma}=-1$.

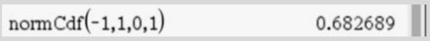
When $x=\mu+\sigma$, $z=\dfrac{(\mu+\sigma)-\mu}{\sigma}=1$.

Hence the required area is $\Pr(-1\le Z\le 1)$.

To find this:

- Press menu > **Probability** > **Distributions** > **Normal Cdf**.
- Complete the **Normal Cdf** template as shown.

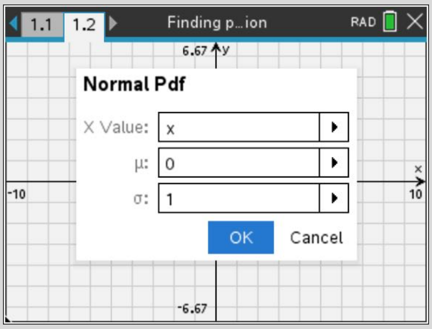




Answer: The area under the normal distribution curve between the two points of inflection is 0.68, correct to two decimal places.

Alternatively, on a **Graphs** page:

- To add a grid, press menu > **View** > **Grid** > **Lined Grid**.
- Press [catalog] 1 N, scroll down and select **normPdf(**.
- Complete the **Normal Pdf** template as shown.
- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –3 XMax = 3 XScale = 1
  YMin = –0.2 YMax = 1 YScale = 1



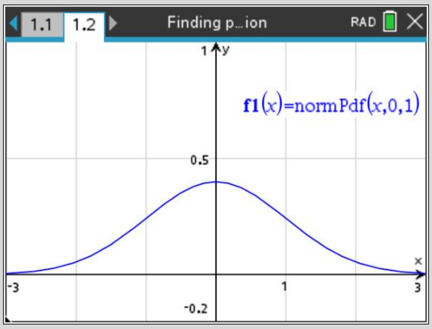
... continued

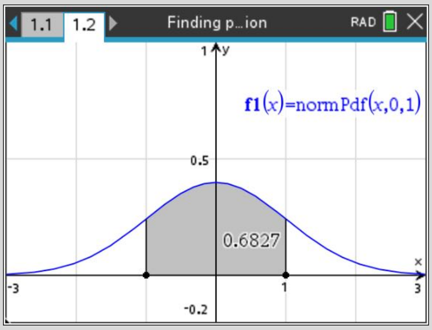
Solution (continued)

To find the required area given by $\Pr(-1 \le Z \le 1)$:

- Press menu **> Analyse Graph > Integral.**
- For the lower bound enter -1 and press enter.
- Move the vertical dashed line just to the right of the lower bound, enter 1 and press enter.

This confirms that $\Pr(-1 \le Z \le 1) = 0.68$, correct to two decimal places.





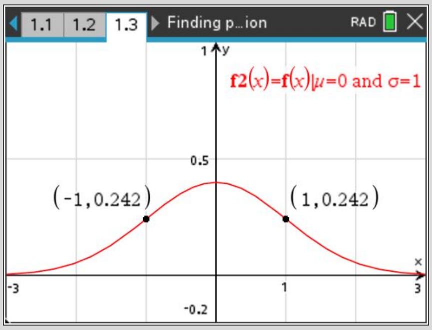
***Note**: To confirm there are points of inflection at $x = \pm 1$, add a new **Graphs** page, and then enter*
$\boldsymbol{f2(x) = f(x) \mid \mu = 0 \text{ and } \sigma = 1}$.

Decline the invitation to create sliders for μ and σ.

*Press menu **> Analyse Graph > Inflection**.*

Move the cursor to the left of the point of inflection for a lower bound and press enter.

Move the cursor to the right of the point of inflection and press enter.



Analysing implicit differentiation graphically

Question

Consider the circle with equation $x^2 + y^2 = 16$.

(a) Use implicit differentiation to show that $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

(b) Hence find the gradients of the tangents to the curve with equation $x^2 + y^2 = 16$ at the points where it intersects the graph of $y = e^{\left(\frac{x}{2}\right)} - 1$. Give your answers correct to three decimal places.

(c) Use a graphical method to verify the answer found in part **(b)** above.

(d) Generalise part **(c)** above to show graphically that the gradient of the tangent at any point $P(x, y)$ on the curve with equation $x^2 + y^2 = 16$ is given by $-\dfrac{x}{y}$ (where defined).

Solution

(a) Using implicit differentiation:

$$\frac{d(x^2)}{dx} + \frac{d(y^2)}{dy} \times \frac{dy}{dx} = \frac{d(16)}{dx}$$

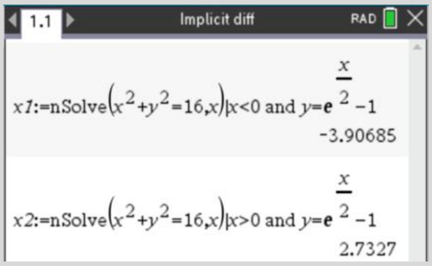
$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

(b) Find the points of intersection of the two curves, and the gradient of the tangents at these points.

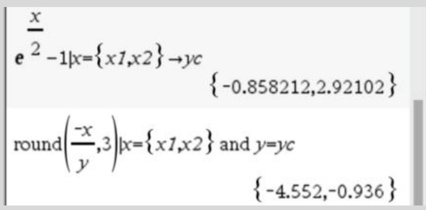
To find the points of intersection, on a **Calculator** page, assign $\boldsymbol{x1}$ and $\boldsymbol{x2}$ as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [menu] > **Algebra** > **Numerical Solve**.
- Press [ctrl] [=] to access the 'with' or 'given' symbol |.
- Press [ctrl] [÷] to access the **Fraction** template.
- Enter as shown.



To find the gradient of the tangents at these points of intersection:

- Press [var] to access assigned/stored variables.
- Enter $\boldsymbol{e^{\frac{x}{2}} - 1 \mid x = \{x1, x2\}}$, press [ctrl] [sto→], enter $\boldsymbol{yc}$, and press [enter].
- Press [menu] > **Number** > **Number Tools** > **Round**.
- Enter $\textbf{round}\left(\dfrac{-\boldsymbol{x}}{\boldsymbol{y}}, \boldsymbol{3}\right) \mid \boldsymbol{x = \{x1, x2\}}$ **and** $\boldsymbol{y = yc}$.



Answer: At $(-3.907, -0.858)$, $m = -4.552$ and at $(2.733, 2.921)$, $m = -0.936$.

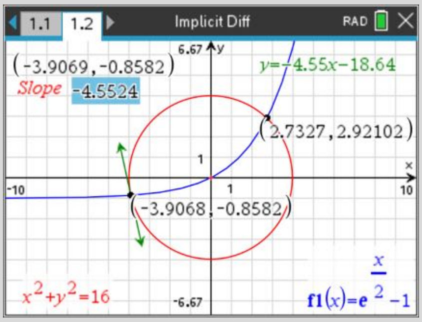
... continued

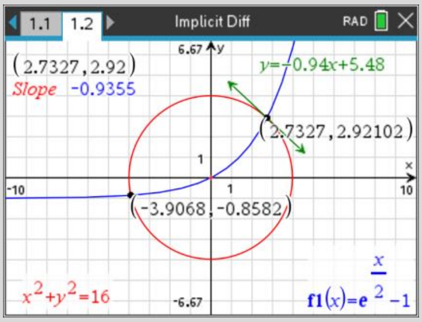
Solution (continued)

(c) To graphically verify the previous answer, proceed as follows on a **Graphs** page.

Note: *The standard viewing window will be used in this example.*

- Enter $f1(x)=e^{\frac{x}{2}}-1$.
- Press [menu] > **Graph Entry/Edit** > **Relation**.
- Enter the relation $x^2+y^2=16$.
- To add a grid, press [menu] > **View** > **Grid** > **Lined Grid**.
- Press [menu] > **Geometry** > **Points & Lines** > **Intersection Points**. Click on the first graph and then on the second graph. Press [esc].
- Press [menu] > **Geometry** > **Points & Lines** > **Tangent**. Click on the circle then press [enter]. Press [esc] to exit this tool.
- Press [menu] > **Geometry** > **Measurement** > **Slope**. Hover the cursor over the tangent line, press [enter] to select the line, then press [enter] to place the value of the slope on the page. Press [esc] to exit this tool.
- Press [menu] > **Actions** > **Text** and enter the word 'slope'.
- Grab the tangent intersection point ([ctrl] [grab]). Move the point near an intersection point, and press [esc]. Hover over the tangent intersection point and press [ctrl] [menu] > **Coordinates and Equations**. Click on the y-coordinates of this point until editable. Enter $(-3.9068,-0.8582)$ and note the slope measurement. Repeat for $(2.7327,2.9210)$.

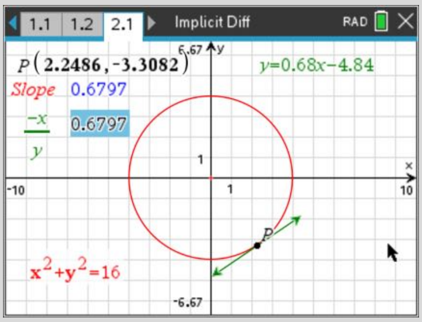




Note: *To increase the precision displayed for a measurement or value, hover over the text and press* [+] *(or* [−] *to decrease precision).*

(d) To generalise the graphical solution, copy Page 1.2 from part **(c)** above to a new problem as follows:

- Press [ctrl] ▲ and then press [menu] > **Insert Problem**.
- Select the Page 1.2 thumbnail.
- Press [ctrl] [C] and then click ▼ to move to Problem 2 and press [ctrl] [V] [enter].
- Hover over graph $f1$, and press [ctrl] [menu] > **Delete**.
- Hover over the x-coordinate of the point of intersection of the tangent and the circle.
- Press [ctrl] [menu] > **Store.**
- Enter ***var* := *x*** and press [enter]. Repeat for the y-coordinate, except enter ***var* := *y*** .
- Label the point, P: Press [ctrl] [menu] > **Label** and enter as shown.



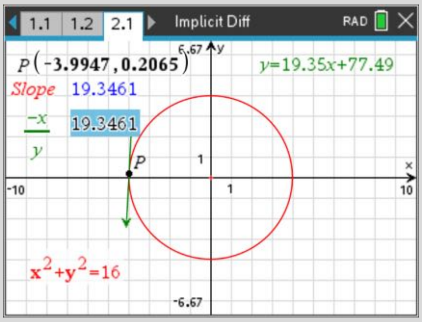
... continued

Solution (continued)

- Press ctrl menu > **Text**, then enter $\dfrac{-x}{y}$.
- Press menu > **Actions** > **Calculate**. Click on the text $\dfrac{-x}{y}$, then click on the x-coordinate and then click on the y-coordinate.
- Grab (ctrl [grab]) and move the point P around the curve.

What do you notice about the value of the measured slope with the calculated value of $\dfrac{-x}{y}$?

Note: *To release/'ungrab' P, either press [grab] or esc.*



Showing that a curve has no stationary points

Question

Consider the curve $\left(x^2+y^2\right)y^2=4x^2$, where $x>0$ and $-2<y<2$.

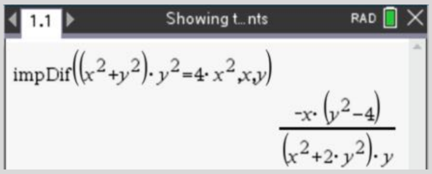
Show that the curve has no local minimum or maximum points for $x>0$.

Solution

On a **Calculator** page, find $\dfrac{dy}{dx}$ as follows:

- Press menu > **Calculus** > **Implicit Differentiation**.
- Enter as shown.

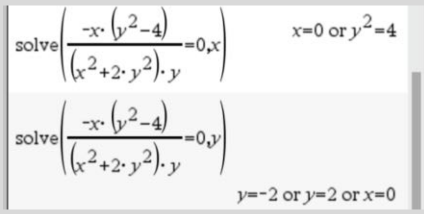
Note*: The syntax for the* ***impDif(*** *command is* ***ImpDif(Equation,Var,dependVar[,Ord])****.*



$$\frac{dy}{dx}=\frac{-x\left(y^2-4\right)}{y\left(x^2+2y^2\right)}$$

At a local maximum or minimum point, $\dfrac{dy}{dx}=0$.

- Press menu > **Algebra** > **Solve**.
- Enter as shown.



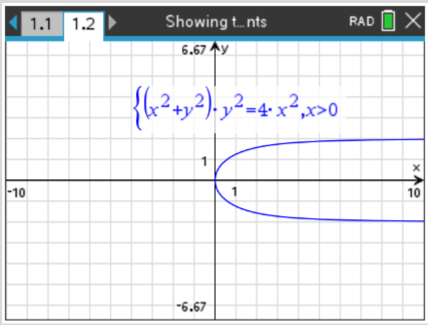
So $x=0$ or $y=\pm2$.

Considering $x>0$ and $-2<y<2$, there are no solutions and hence there are no local maximum or minimum points.

To plot the curve on a **Graphs** page:

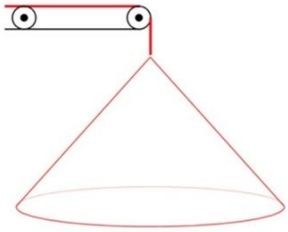
- Press menu > **Graph Entry/Edit** > **Relation**.
- Enter $\left(x^2+y^2\right)\cdot y^2=4x^2 \mid x>0$.
- To add a grid, press menu > **View** > **Grid** > **Lined Grid**.

Note: *The standard viewing window is used in this example.*



Applying related rates to the sand pile problem

Question



Sand is dropped from a conveyor belt onto a pile at a rate of 2 m^3/min.
The pile of sand is modelled as a right circular cone such that its height is always equal to the radius of its base.
At time $t = 0$, where t is measured in minutes, the volume of the pile is zero. Give the answer to parts **(a)** and **(b)** below in m/min, correct to two decimal places.

(a) Find the rate of change of the height of the cone when the volume of sand in the pile is 15 m^3.

(b) Find the rate of change of the height of the cone at $t = 55$.

Solution

(a) To find $\frac{dh}{dt}$ when $V = 15$ on a **Calculator** page, assign $\frac{dV}{dt}$ and $V(h)$ as follows:

- Press [ctrl] [␣] to insert an underscore.
- Press [ctrl] [⊞] to access the **Assign** [:=] command.
- Press [π▸] to access π.

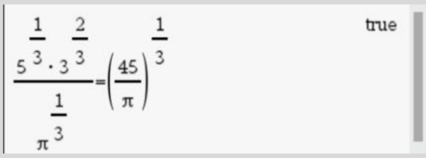
To find the value of h when $V = 15$:

- Press [menu] > **Algebra** > **Numerical Solve**.
- Enter as shown.

Solving $\frac{1}{3}\pi h^3 = 15$ for h gives $h = 2.42859...$(m).

Note: *Using the* ***solve(*** *command in* ***Auto*** *mode,* $h = \dfrac{5^{\frac{1}{3}} \times 3^{\frac{2}{3}}}{\pi^{\frac{1}{3}}}$*,*

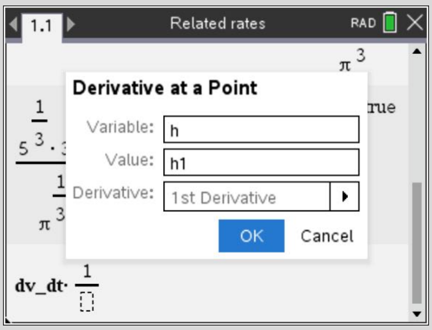
which is equivalent to $h = \left(\dfrac{45}{\pi}\right)^{\frac{1}{3}}$.



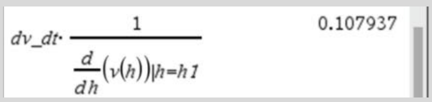
$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ where $\frac{dV}{dt} = 2$ and $\frac{dh}{dV} = \frac{1}{\pi h^2}$

So $\frac{dh}{dt} = \frac{2}{\pi h^2}$. To find $\frac{dh}{dt}$ when $V = 15$:

- Press [var] to access assigned/defined variables.
- Press [ctrl] [÷] to access the **Fraction** template.
- Press [menu] > **Calculus** > **Derivative at a Point**.
- Complete the required fields as shown.
- Complete the **Derivative at a Point** template as shown.



Note*: Alternatively, to access the* ***Derivative*** *template, press* [📖] [5]*. A more efficient alternative is to press* [⇧shift] [−].



Answer: The height is increasing at a rate of 0.11 m/min, correct to two decimal places.

... continued

Solution (continued)

(b) To find $\frac{dh}{dt}$ at $t = 55$, note that $V = 2 \times 55 = 110$ (m^3).

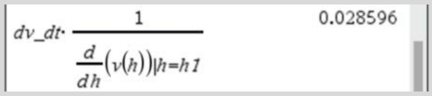
- Press ctrl [ctrl] to access the **Assign** [:=] command.
- Press menu > **Algebra** > **Numerical Solve**.
- Enter as shown.

Solving $\frac{1}{3}\pi h^3 = 2 \times 55$ for h gives $h = 4.71832...$(m).

To find $\frac{dh}{dt}$ when $h = 4.71832...$:

- Press var to access assigned/defined variables.
- Press ctrl ÷ to access the **Fraction** template.
- Press menu > **Calculus** > **Derivative at a Point**.
- Complete the **Derivative at a Point** template as shown.

Answer: The height is increasing at a rate of 0.03 m/min, correct to two decimal places.



Determining partial fractions

A rational function can be resolved into partial fractions.

- Distinct linear factors, for example, $\dfrac{5x-2}{(x-3)(2x+1)} \equiv \dfrac{A}{x-3} + \dfrac{B}{2x+1}$.
- A repeated linear factor, for example, $\dfrac{5x-2}{(x-3)(2x+1)^2} \equiv \dfrac{A}{x-3} + \dfrac{B}{2x+1} + \dfrac{C}{(2x+1)^2}$.
- An irreducible quadratic factor, for example, $\dfrac{5x-2}{(x-3)(x^2+1)} \equiv \dfrac{A}{x-3} + \dfrac{Bx+C}{x^2+1}$.

If $f(x) = \dfrac{P(x)}{D(x)}$ is an improper fraction (degree of $P(x) \geq D(x)$), then division is performed first and $f(x) = Q(x) + \dfrac{R(x)}{D(x)}$.

Question

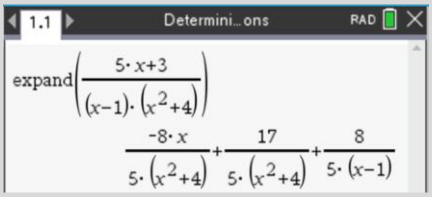
(a) Express $\dfrac{5x+3}{(x-1)(x^2+4)}$ in partial fractions.

(b) Find $\displaystyle\int_2^3 \frac{5x+3}{(x-1)(x^2+4)}\,dx$. Give your answer correct to two decimal places.

Solution

(a) On a **Calculator** page:

- Press [menu] > **Algebra** > **Expand**.
- Enter as shown.



Answer: $\dfrac{8}{5(x-1)} + \dfrac{17-8x}{5(x^2+4)}$

Alternatively, scaffold the by-hand approach as follows:

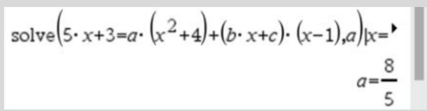
$\dfrac{5x+3}{(x-1)(x^2+4)} \equiv \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+4}$ (irreducible quadratic factor)

Multiply both sides by $(x-1)(x^2+4)$ so

$5x+3 \equiv A(x^2+4) + (Bx+C)(x-1)$.

To find the value of A:

- Press [menu] > **Algebra** > **Solve**.
- Press [ctrl] [=] to access the 'with' or 'given' symbol |.
- Enter
  $\text{solve}\left(5x+3 = a\cdot(x^2+4)+(b\cdot x+c)\cdot(x-1), a\right) \mid x=1$.



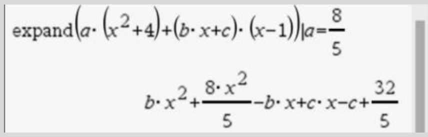
Answer: $A = \dfrac{8}{5}$.

... continued

Solution (continued)

To find the value of B and the value of C, expand the right-hand side, equate coefficients and solve as follows:

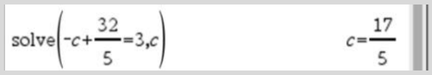
- Press menu > **Algebra** > **Expand**.
- Press ctrl = to access the 'with' or 'given' symbol |.
- Enter as shown.



For the x^2 term: $B+\frac{8}{5}=0 \Rightarrow B=-\frac{8}{5}$

For the constant term:

- Press menu > **Algebra** > **Solve**.
- Enter as shown.

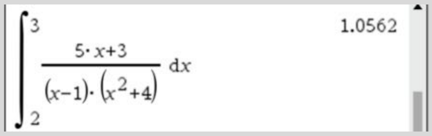


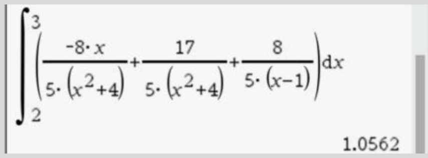
$-C+\frac{32}{5}=3 \Rightarrow C=\frac{17}{5}$

So $\frac{8}{5(x-1)}+\frac{17-8x}{5(x^2+4)}$.

(b) To find $\int_2^3 \frac{5x+3}{(x-1)(x^2+4)}dx$, either integrand can be used.

- Press menu > **Calculus** > **Integral**.
- Complete the **Integral** template as shown.





*Note: Alternatively, to access the **Integral** template, press* 5*. A more efficient alternative is to access the **Integral** template is to press* ⇧shift +.

Answer: $\int_2^3 \frac{5x+3}{(x-1)(x^2+4)}dx=1.06$, correct to two decimal places.

Note: To give the answer correct to two decimal places, press menu *> **Number** > **Number Tools** > **Round**. Enter as shown.*

Determining a sequence of definite integrals

Question

Consider the family of functions $f_n(x) = x^n e^{-x}$, where $x \geq 0$ and $n \in \mathbb{Z}^+$. The total area, A_n, of the region bounded by the graph $y = f_n(x)$ and the x-axis is given by $A_n = \int_0^\infty f_n(x)\,dx$.

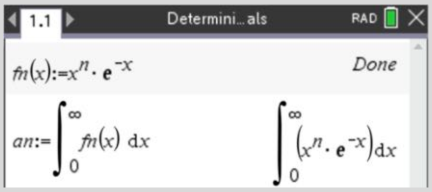
(a) Suggest an expression for A_n in terms of *n*, where $n \in \mathbb{Z}^+$.

(b) Use mathematical induction to prove your conjecture. You may assume that, for any value of *m*, $\lim_{x \to \infty} x^m e^{-x} = 0$.

Solution

(a) On a **Calculator** page, assign $f_n(x)$ and A_n as follows:

- Press [ctrl] [⊞] to access the **Assign** [:=] command.
- Press [menu] > **Calculus** > **Integral**.
- Press [π▸] to access ∞.
- Complete the **Integral** template as shown.

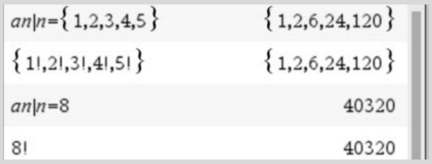


***Note**: Alternatively, to access the **Integral** template, press [📖] [5]. A more efficient alternative is to access the **Integral** template is to press [⇧shift] [+].*

- Press [var] to access assigned/defined variables.
- Press [ctrl] [=] to access the 'with' or 'given' symbol |.

To explore this:

- Press [menu] > **Probability** > **Factorial (!)**.
- Enter as shown.



For $n = 8$, for example, $A_8 = 40320 = 8!$.

For $n = 1, 2, 3, 4, 5$, the conjecture is that $A_n = n!$.

Answer: Conjecture is that $A_n = n!$ for $n \in \mathbb{Z}^+$.

(b) When $n = 1$, $A_1 = 1$ and $1! = 1$.

Hence true for $n = 1$.

Assume true for $n = k$, $A_k = \int_0^\infty x^k e^{-x}\,dx = k!$.

Consider $n = k + 1$:

Integrate by parts: $\int_0^\infty x^{k+1} e^{-x}\,dx = \left[-x^{k+1} e^{-x}\right]_0^\infty + (k+1)\int_0^\infty x^k e^{-x}\,dx$

$= 0 + (k+1)k!$

$= (k+1)!$

Hence if true for $n = k$ then also true for $n = k + 1$. As true for $n = 1$ so true for $n \in \mathbb{Z}^+$.

Determining the arc length for a parametrically determined curve

Consider a curve defined parametrically by $x = f(t)$ and $y = g(t)$ for $a \le t \le b$.

If a point $P(f(t), g(t))$ traces the curve exactly once from $t = a$ to $t = b$, then the length of the curve is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt .$$

Question

Find the exact length of the curve defined by the parametric equations

$x = e^t \cos(t)$ and $y = e^t \sin(t)$ over the interval $0 \le t \le \pi$.

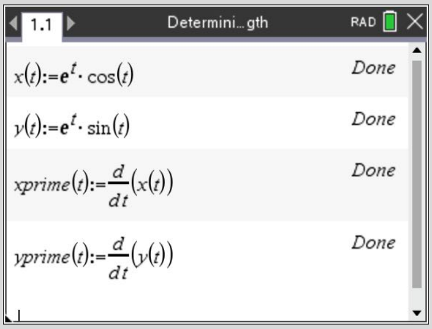
Solution

On a **Calculator** page, assign $x(t)$, $y(t)$, $x'(t)$ and $y'(t)$ as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [trig] to access **sin** and **cos**.
- Press [menu] > **Calculus** > **Derivative**.
- Complete the **Derivative** template as shown.

Note: *Alternatively, to access the* ***Derivative*** *template, press* [catalog] [5]. *A more efficient alternative is to press* [shift] [−].

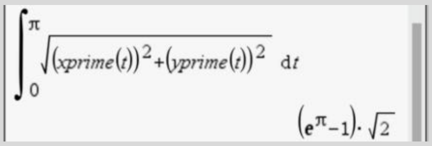
- Enter as shown.



To find the required length:

- Press [menu] > **Calculus** > **Integral**.
- Complete the **Integral** template as shown.
- Press [π▸] to access π.
- Press [var] to access assigned/stored variables.

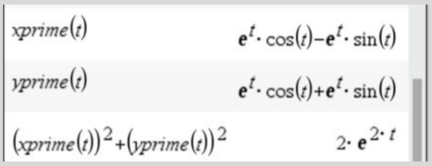
Note: *Alternatively, to access the* ***Integral*** *template, press* [catalog] [5]. *A more efficient alternative is to access the* ***Integral*** *template is to press* [shift] [+].



Answer: The exact length of the curve is $\sqrt{2}\left(e^{\pi} - 1\right)$.

... continued

Solution (continued)



The following unpacks the result in summary form.

Using the product rule:

$\frac{dx}{dt} = e^t\left(\cos(t) - \sin(t)\right)$ and $\frac{dy}{dt} = e^t\left(\sin(t) + \cos(t)\right)$

Adding the squares of the derivatives and making use of $\sin^2(t) + \cos^2(t) = 1$ gives:

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2e^{2t}$$

$$L = \int_0^{\pi} \sqrt{2e^{2t}}\, dt = \sqrt{2}\int_0^{\pi} e^t\, dt$$

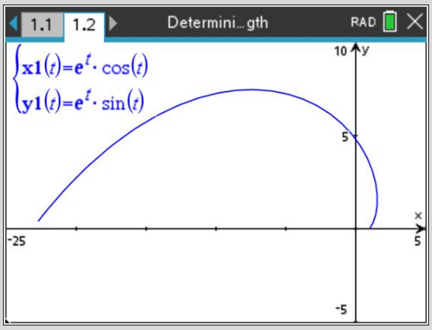
So $L = \sqrt{2}\left(e^{\pi} - 1\right)$.

This is known as the logarithmic spiral arc length.

To plot this curve in parametric graphing mode:

On a **Graphs** page:

- Press [menu] > **Graph Entry/Edit** > **Parametric**.
- Enter $\mathbf{x1}(t) = e^t \cdot \mathbf{cos}(t)$ and $\mathbf{y1}(t) = e^t \cdot \mathbf{sin}(t)$.
- Enter $\mathbf{0} \le t \le \pi$ $tstep = \mathbf{0.016}$.
- To add a grid, press [menu] > **View** > **Grid** > **Lined Grid**.
- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –25 XMax = 5 XScale = 5
  YMin = –5 YMax = 10 YScale = 5



Calculating volumes of solids of revolution about the x-axis

If the region bounded by the curve $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis is rotated $360°$ about the x-axis then $V = \pi\int_a^b (f(x))^2\,dx$.

Question

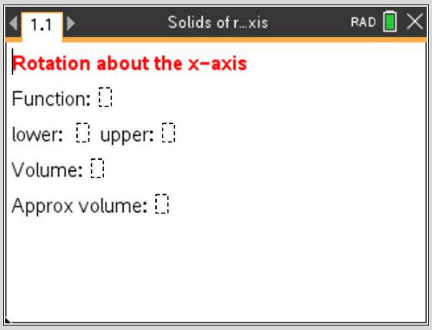
The region bounded by the curve $y = \cos(x)$, and the lines $x = -\dfrac{7\pi}{16}$ and $x = \pi$ is rotated 360° about the *x*-axis. Find the volume of this solid of revolution, correct to three decimal places.

Solution

To find the volume of solid of revolution on a **Notes** page:

- Enter the text shown in the screenshot.
- Move the cursor to the right of the word 'Function' and press [menu] > **Insert** > **Maths Box** (or press [ctrl] [M]).

Repeat to insert **Maths Boxes** next to each of the other template headings.

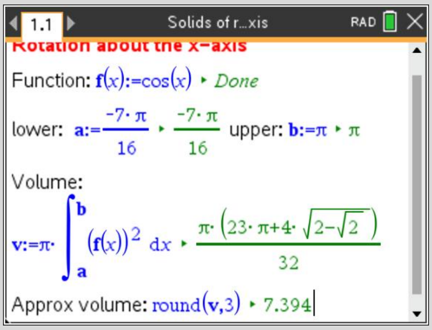


Note: *To edit the text colour, select the text by holding* [⇧shift] *and 'arrow' across the text. Then press* [menu] *>* ***Format*** *>* ***Text colour****.*

- Click on the **Maths Box** next to the word 'Function'.
- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Inside the **Maths Box**, enter $\boldsymbol{f(x) := \cos(x)}$ and press [enter].

Similarly, in the other **Maths Boxes**:

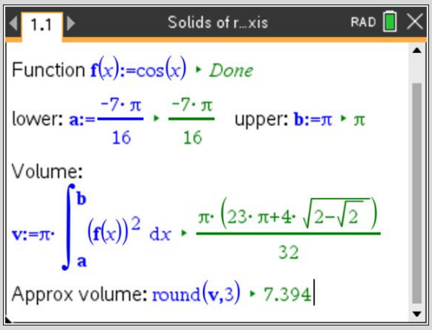
- For 'lower', enter $\boldsymbol{a := -\dfrac{7\pi}{16}}$.
- For 'upper', enter $\boldsymbol{b := \pi}$.
- For 'Volume', enter $\boldsymbol{v := \int_a^b \left(\pi \cdot (f(x))^2\right) dx}$.



Note: *To access the* ***Integral*** *template, either press* [ctrl] [5] *and select the template, or press* [⇧shift] [+].

- For 'Approx. volume', enter **round(*v*,3)**.
- Press [menu] > **Calculations** > **Number** > **Number Tools** > **Round**.

Note: *Alternatively, press* [book] [1] [R] *and scroll down to access the* ***round(*** *command.*



Answer: The volume is 7.394, correct to three decimal places.

- To ensure continuity with the next example, press [ctrl] [S] to save the document.

Displaying a solid of revolution

Question

(a) Plot a graph that shows the region bounded by the curve $y = \cos(x)$, and the lines with equations $x = -\frac{7\pi}{16}$ and $x = \pi$.

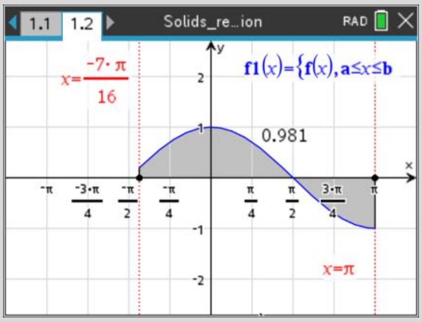
(b) Create a graph that shows the solid formed when the region bounded by the curve $y = \cos(x)$ and the lines $x = -\frac{7\pi}{16}$ and $x = \pi$ is rotated 360° about the x-axis.

Solution

(a) Open the saved document from the previous example.

To plot the bounded region on a **Graphs** page:

- Enter $\boldsymbol{f1(x) = f(x) | a \le x \le b}$.
- Press [ctrl][=] to access the 'with' or 'given' symbol | and the ≤ symbol.
- Press [menu] > **Graph Entry/Edit** > **Relation** and enter $\boldsymbol{x = -\frac{7\pi}{16}}$ and $\boldsymbol{x = \pi}$.
- Press [ctrl][÷] to access the **Fraction** template.
- Press [π▸] to access π.
- To add a grid, press [menu] > **View** > **Grid** > **Lined Grid**.
- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = −5π/4 XMax = 5π/4 XScale = π/4
  YMin = −2.7 YMax = 2.7 YScale = 1
- Press [menu] > **Analyse Graph** > **Integral**.
- Click on the lower bound point at (−7π/16, 0).
- Click on the upper bound point at (π, 0).
- Press [esc] to exit.



Answer: The above screenshot at right shows the region formed.

(b) If a surface is obtained by rotating a curve $y = f(x)$, $x \in [a,b]$, about the x-axis, then this surface of revolution has parametric equations:

$x = t, \quad y = f(t)\cdot\cos(u), \quad z = f(t)\cdot\sin(u)$, where $t \in [a,b]$ and $u \in [0, 2\pi]$.

... continued

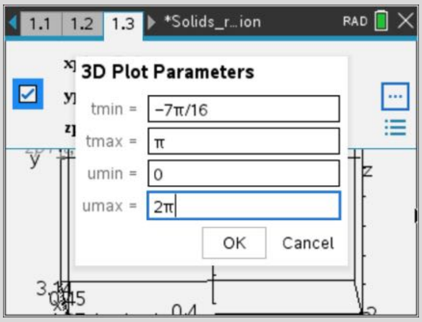
Solution (continued)

To create a graph that shows the surface of revolution on a **Graphs** page.

- Press menu > **View** > **3D Graphing**.
- Press menu > **3D Graph Entry/Edit** > **Parametric**.
- In the dialog box that follows, click the ⋯ icon.

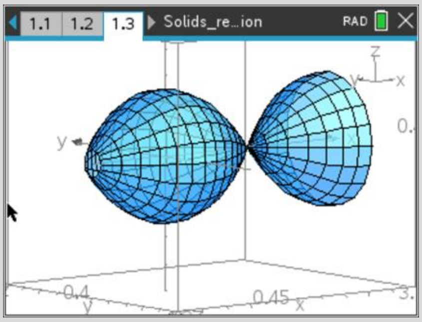
In the **3D Plot Parameters** dialog box that follows, input the following values, then press **OK**.

tmin = −7π/16 tmax = π umin = 0 umax = 2π

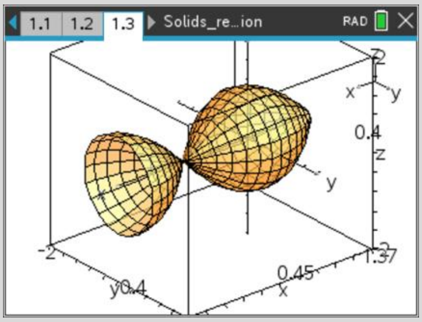


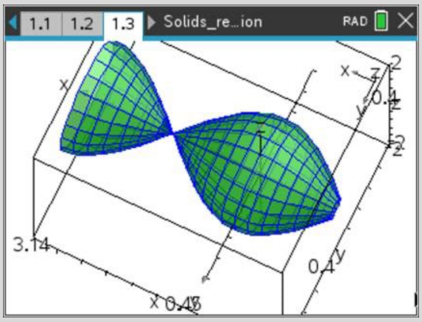
Note*: *Clicking* OK *returns to the* 3D Graph Entry/Edit *dialog box.

- Enter the parametric equation $\boldsymbol{xp1}(t,u)=t$, followed by $\boldsymbol{yp1}(t,u)=f(t)\cdot\cos(u)$ and $\boldsymbol{zp1}(t,u)=f(t)\cdot\sin(u)$.
- Press menu > **Range/Zoom** > **Range Settings**. In the dialog box that follows, enter the following values:
  XMin = −π/2 XMax = π XScale = Auto
  YMin = −1.1 YMax = 1.1 YScale = Auto
  ZMin = −1.1 ZMax = 1.1 ZScale = Auto
- Press × repeatedly to magnify the graph size. (Press the division key ÷ to shrink the graph size.)
- Move the cursor to the graph and press ctrl menu. From the **Context** menu, select **Colour** > **Fill Colour** to change colour. Select **Attributes** to change other characteristics.
- Press the arrow keys to rotate the graph in 3D.



Answer: The screenshots at right show the surface of revolution formed.

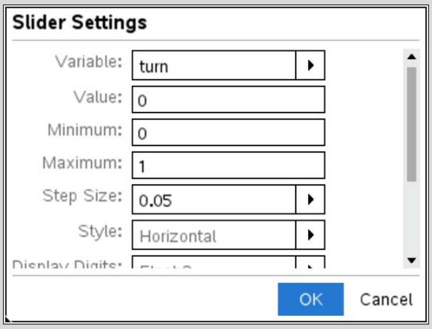




... continued

Solution (continued)

A slider can be introduced to show an animated generation of the solid.



To do this:

- Press menu > **Actions** > **Insert Slider**.
- Set the slider settings as shown.
- Do not check the **Minimised** box.

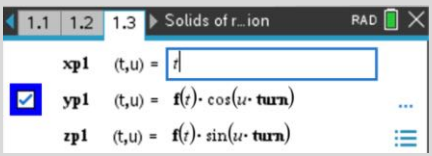
To move the slider:

- Click on the slider, then press ctrl menu > **Move** and move it to the bottom left-hand corner as shown.

Note: *The slider can be moved when it is framed by a blue border. If the blue border is not showing, click the slider, press* ctrl menu > ***Move*** *and move it.*

To update the parametric equations:

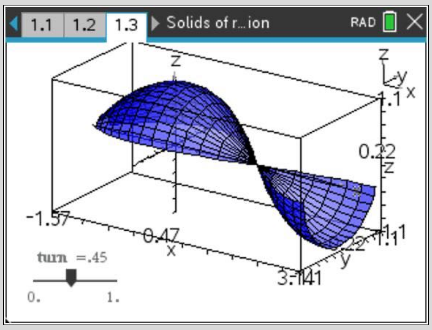
- Press menu > **3D Graph Entry Edit** > **Parametric**.
- Enter as shown.



To animate the generation of the solid:

- Move the cursor over the slider and press ctrl menu > **Animate**.

Note: To pause the animation, move the cursor over the slider and press ctrl menu > **Stop Animate**.



Calculating volumes of solids of revolution about the y-axis

If the region bounded by the curve $x = f(y)$, the lines $y = a$ and $y = b$ and the y- axis is rotated $360°$ about the y- axis, then $V = \pi \int_a^b x^2 dy = \pi \int_a^b (f(y))^2 dy$.

Question

The region bounded by the curve $y = x^2 - 4$, the line $y = 5$ and the y-axis is rotated by 360° about the y-axis to form a solid of revolution.

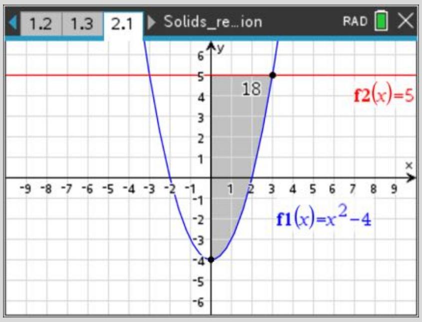
(a) Plot a graph of this region.

(b) Determine the volume of the solid of revolution, correct to two decimal places.

(c) Create a graph that shows the solid formed when the region bounded by the curve $y = x^2 - 4$, the line $y = 5$ and the y-axis is rotated by 360° about the y-axis.

Note: *The solution method can be set up similarly to the previous problem, either in a* ***New Document*** *or as a* ***New Problem*** *in the existing document. To add a new problem to the existing 'Solids of Revolution' document, press* doc▾ > ***Insert*** > ***Problem***.

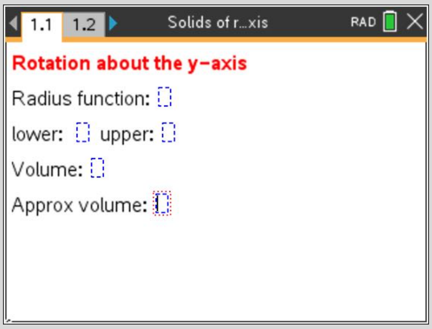
Solution



(a) To plot the bounded region on a **Graphs** page:

- Enter $f1(x)=x^2-4$ and $f2(x)=5$.
- To add a grid, press [menu] > **View** > **Grid** > **Lined Grid**.
- Press [menu] > **Analyse Graph** > **Bounded Area**.
    - For the lower bound, enter **0**.
    - For the upper bound, enter **3**.
- Press [esc] to exit this tool.

***Note:** This region is displayed in the standard window.*

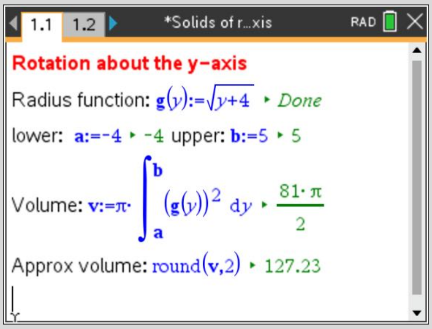


(b) To find the volume of solid of revolution on a **Notes** page:

- Enter the text shown in the screenshot.
- Move the cursor to the right of the text 'Radius function' and press [menu] > **Insert** > **Maths Box** (or press [ctrl] [M]).
- Repeat to insert **Maths Boxes** next to each of the other template headings.

If $y=x^2-4, 0\le x\le 3$, then $x=\sqrt{y+4}, -4\le y\le 5$.

- Click on the **Maths Box** next to the text 'Radius function'.
- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Inside the **Maths Box**, input $\boldsymbol{g(y):=\sqrt{y+4}}$ then press [enter].



Similarly, in the other **Maths Boxes** next to the word:

- For 'lower', enter $\boldsymbol{a:=-4}$.
- For 'upper', enter $b:=\boldsymbol{5}$.
- For 'Volume', enter $\boldsymbol{v:=\int_a^b\left(\pi\cdot(g(y))^2\right)dy}$.

Answer: The volume is 127.23, correct to two decimal places.

(c) If a surface is obtained by rotating a curve $x=g(y)$, $y\in[a,b]$, about the y-axis, then this surface of revolution has parametric equations:

$x=g(t)\cdot\sin(u),\quad y=t,\quad z=g(t)\cdot\cos(u)$, where $t\in[a,b]$ and $u\in[0,2\pi]$.

To create a graph of the surface of revolution on a **Graphs** page:

- Press [menu] > **View** > **3D Graphing**.
- Press [menu] > **3D Graph Entry/Edit** > **Parametric**.
- In the dialog box that follows, click the [...] icon.

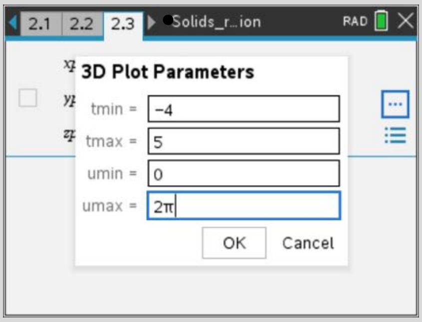
... continued

Solution (continued)

In the **3D Plot Parameters** dialog box that follows, input the following values, then press **OK**.

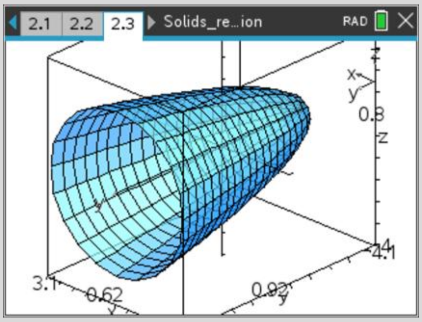
tmin = −4 tmax = 5 umin = 0 umax = 2π

Note: *Clicking* ***OK*** *returns to the* ***3D Graph Entry/Edit*** *dialog box.*

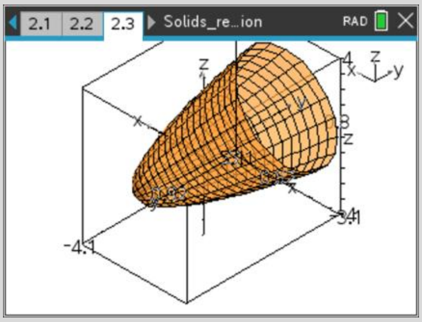


- Enter the parametric equation $xp1(t,u)=g(t)\cdot\sin(u)$ followed by $yp1(t,u)=t$ and $zp1(t,u)=g(t)\cdot\cos(u)$.
- Press [menu] > **Range/Zoom** > **Range Settings**.

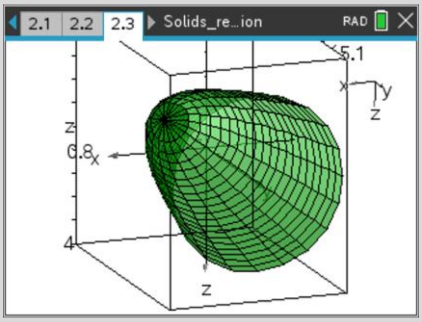
  In the dialog box that follows, enter the following values:
  XMin = −0.1 XMax = 3.1 XScale = Auto
  YMin = −4.1 YMax = 5.1 YScale = Auto
  ZMin = −4 ZMax = 4 ZScale = Auto



- Press [×] repeatedly to magnify the graph size. (Press the division key [÷] to shrink the graph size)
- Move the cursor to the graph and press [ctrl] [menu]. From the **Context** menu, select **Colour > Fill Colour** to change colour. Select **Attributes** to change other characteristics.
- Press the arrow keys to rotate the graph in 3D.



Note: *A slider can be introduced to show an animation of how the solid is generated. For instructions on how to do this, please refer to the previous problem titled 'Displaying a solid of revolution'.*



3.4.2 Differential equations

Formulating and verifying the solution of a differential equation

Question

Assume that light travels through a medium whose refractive index varies with position x, such that $\frac{dy}{dx}=\frac{1}{10-x}$, where $x\in[0,10)$ is the depth into the medium and y is the optical path length.

(a) If $y(9)=12$, determine the values of y at **(i)** $x = 2$, **(ii)** $x = 7.5$. Give the answers correct to two decimal places.

(b) Confirm that the answers above are consistent with the general solution $y=-\ln(|10-x|)+c$.

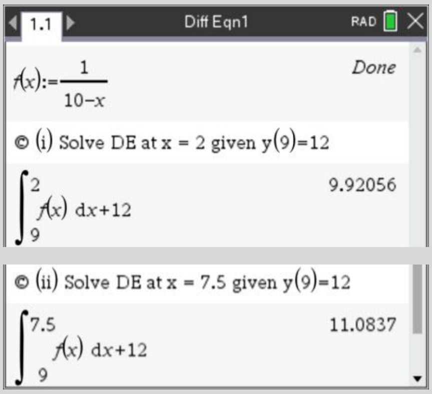
(c) Use the Differential Equation Solver to confirm the answer from parts **(a)** and **(b)** above.

Solution

(a) If $\frac{dy}{dx}=f(x)$ and $y=y_1$ at $x=x_1$, then the solution to the differential equation at $x=t$ is given by $\int_{x_1}^{t} f(x)\,dx+y_1$.

To solve for y, on a **Calculator** page enter $\boldsymbol{f(x):=\frac{1}{10-x}}$:

- **(i)** Press either [⇧shift][+] or [menu] > **Calculus** > **Integral** and key in $\int_9^2 \boldsymbol{f(x)dx+12}$, then press [ctrl][enter].
- **(ii)** Similarly, enter $\int_9^{7.5} \boldsymbol{f(x)dx+12}$.



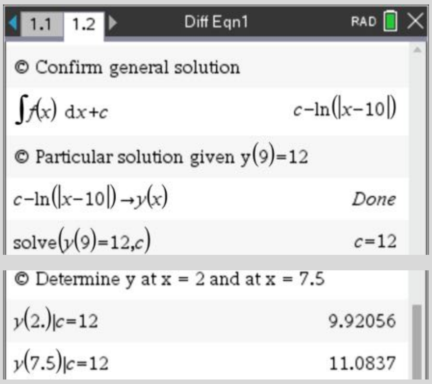
Answer: (i) y = 9.92 at x = 2. (ii) y = 11.08 at x = 7.5.

(b) To confirm the general and particular solution:

- Press [⇧shift][+] and enter $\int \boldsymbol{f(x)dx+c}$.
- Enter **Ans → *y(x)*** by pressing [ctrl][(−)] for [ans] and [ctrl][var] ([sto→]) for the **store** symbol, **→**.
- Press [menu] > **Algebra** > **Solve**. Enter **solve(*y*(9) = 12, *c*)**.

To determine the values of y at **(i)** x = 2 and **(ii)** x = 7.5:

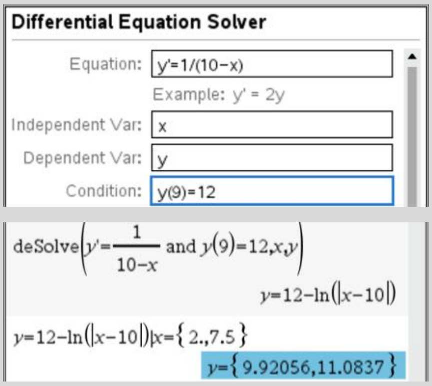
- Enter (i) ***y*(2)|*c* = 12** and (ii) ***y*(7.5)|*c* = 12**, pressing [ctrl][=] to select the **given** symbol, |.



Answer: Confirmed y = 9.92 at x = 2 and y = 11.08 at x = 7.5.

(c) To confirm results using DE Solver in a new **Problem**:

- Press [doc▾] > **Insert** > **Problem**. Select **Add Calculator**.
- Press [menu] > **Calculus** > **Differential Equation Solver**. Enter as shown, pressing [?!▸] or [π▸] for the prime ′ symbol.
- Enter $\boldsymbol{Ans\,|\,x=\{2.0,7.5\}}$, pressing [ctrl][(−)] for **Ans**.



Answer: Confirmed with DE Solver that $y=12-\ln(|10-x|)$ and y = 9.92 at x = 2 and y = 11.08 at x = 7.5.

Solving a second-order differential equations of form $y''(x) = f(x)$

Second-order differential equations arise in many practical contexts. For example, an idealised rocket model where the thrust increases steadily with time, so that the acceleration increases linearly. This situation can be represented by the equation $y''(t) = kt$, $t \ge 0$, where y denotes the position at time t, and k is a constant. Solving this equation by integrating twice yields the velocity and position functions, with two constants of integration determined by the initial conditions.

Question

At every point on a certain curve, $\frac{d^2y}{dx^2} - 8x = 0$. Assume that $P(3, 8)$ is a point on the curve and that the gradient of the tangent at P is 1. Using **(i)** integration, **(ii)** Differential Equation Solver, find:

(a) An expression for the gradient function, $y'(x) = \frac{dy}{dx}$.

(b) The equation of the curve.

Solution

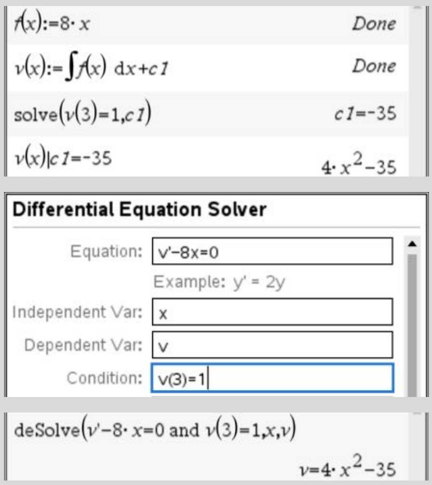
(a)(i) To find $y'(x)$ by integration, on **Calculator** page:

- Enter $\boldsymbol{f(x) := 8x}$, then $\boldsymbol{v(x) := \int f(x)\,dx + c1}$, pressing [template key] or [shift] [+] for the integral template.
- Press [menu] > **Algebra** > **Solve**. Enter **solve(*v*(3) = 1,*c*1)**, followed by ***v*(*x*)|*c*1= −35**, pressing [ctrl] [=] ([|≠≥▸]) for |.

(ii) To find $y'(x)$ using **Differential Equation Solver**:

- Press [menu] > **Calculus** > **Differential Equation Solver**. Enter as shown, pressing [?!▸] or [π▸] for the prime ′ symbol.

Answer: Both methods confirm that $y'(x) = 4x^2 - 35$.



(b)(i) To find the equation of the curve by integration, on the **Calculator** page from part **(a)(i)** above:

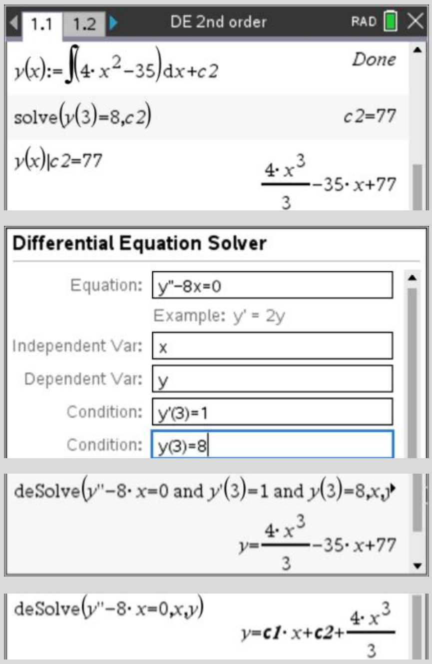
- Enter $\boldsymbol{y(x) := \int (4x^2 - 35)\,dx + c2}$, pressing [template key] or [shift] [+] for the integral template.
- Press [menu] > **Algebra** > **Solve**. Enter **solve(*y*(3) = 8, *c*2)**, followed by ***y*(*x*)|*c*2 = 77**.

(ii) To find the equation using **Differential Equation Solver**:

- Press [menu] > **Calculus** > **Differential Equation Solver**. Enter as shown, pressing [?!▸] or [π▸] for the prime ′ symbol.

Answer: Both methods confirm that the equation of the curve is $y = \frac{4}{3}x^3 - 35x + 77$.

*Note: 1. To input **y″**, key in the prime symbol, ′, twice. Do **not** use the double quotation mark symbol instead. 2. If the initial conditions are not specified, the Differential Equation Solver returns a general solution, $y = c1x + c2 + \frac{4}{3}x^3$. If the initial conditions are not specified, the Differential Equation Solver returns a general solution, $y = c1x + c2 + \frac{4}{3}x^3$.*



Modelling Newton's law of cooling with a DE of the form $dy/dx = g(y)$

In many situations, the rate of heat loss of an object is proportional to the difference between the temperature of the object and the ambient temperature. Suppose the ambient temperature remains constant at 20°C, and the object has temperature 100°C at time $t = 0$.

Let θ°C denote the temperature of the object at time t minutes. According to Newton's law of cooling, $\theta(t)$ satisfies the differential equation

$$\frac{d\theta}{dt} = -k(\theta - 20), k > 0 \text{. Therefore, } t = -\int \frac{1}{k(\theta - 20)} d\theta .$$

Question

Assume that for the above scenario, $\theta = 70$°C at $t = 10$ minutes. Answer the following, giving the answers to parts **(b)**, **(c)** and (d) in minutes or degrees Celsius, correct to one decimal place.

(a) Determine an expression for $\theta(t)$ in the form $Ae^{-kt} + 20$, $A, k \in R$. Give the values of A and k correct to two significant figures (i.e. rounded to two digits, excluding leading zeros).

(b) Determine the value of θ at $t = 20$ minutes.

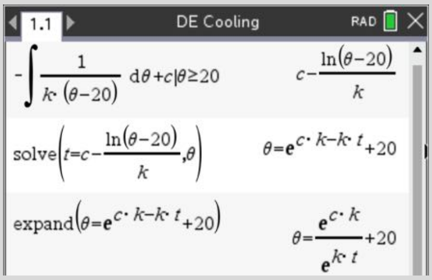
(c) Find the time, in minutes, required for the temperature of the object to decrease to 40°C.

(d) Plot a graph of the object's temperature $\theta(t)$ as a function of time. Hence, using a graphical method, determine **(i)** the times at which the excess temperature $\theta - 20$ is reduced to one-half and one-quarter of its initial value, and **(ii)** the temperature of the object after 1 hour.

Solution

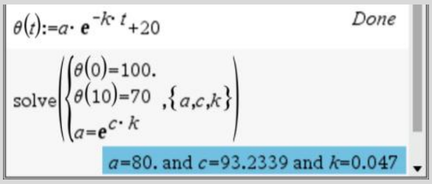
(a) To solve the DE using integration, on a **Calculator** page:

- Enter $-\int \frac{1}{k \times (\theta - 20)} d\theta + c \mid \theta \geq 20$, pressing [template key] for **integral**, [ctrl] [book] ([∞β°]) for θ, [ctrl] [=] ([≠≥▸]) for ≥ and |.
- Press [menu] > **Algebra** > **Solve**. Enter **solve(t = ans,θ)**.
- Press [menu] > **Algebra** > **Expand**. Enter **expand(ans)**.



To find the values of $A = e^{c \cdot k}$ and k:

- Enter $\boldsymbol{\theta(t) := a \times e^{-k \times t} + 20}$.
- Press [menu] > **Algebra** > **Solve System of Equations**.
- Enter: Number of Equations: **3**, Variables: ***a***, ***c***, ***k***. Enter three equations as shown, pressing [var] to select $\boldsymbol{\theta}$( ).

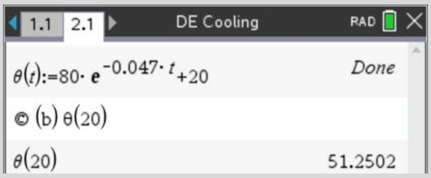


Answer: $\theta(t) = 80e^{-0.047t} + 20$.

(b) To find $\theta(20)$, on a **Calculator** page:

- Enter $\boldsymbol{\theta(t) := 80 \times e^{-0.047 \times t} + 20}$ then $\boldsymbol{\theta(20)}$.

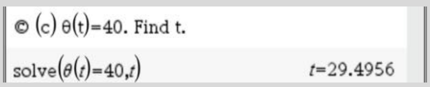
Answer: $\theta(20) = 51.3$°C.



(c) To find t such that $\theta(t) = 40$:

- Press [menu] > **Algebra** > **Solve**. Enter **solve($\theta(t)$ = 40, t)**.

Answer: $\theta(t) = 40$°C at $t = 29.5$ minutes.



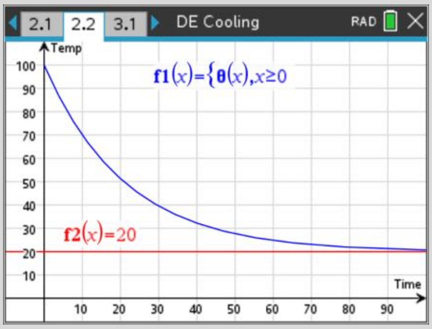
... continued

Solution (continued)

(d) To graph temperature θ as a function of time:

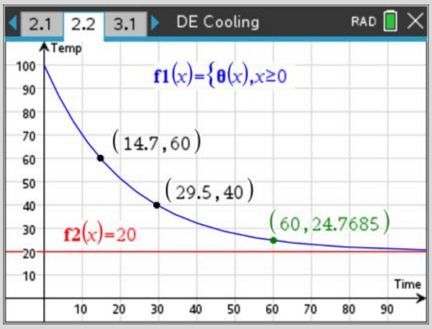
- Ensure $\boldsymbol{\theta(t)=80e^{-0.047t}+20}$ has been defined.
- Add a **Graphs** page and enter $\boldsymbol{f1(x)=\theta(x)\,|\,x\geq 0}$.
- Then enter $\boldsymbol{f2(x)=20}$
- Press [menu] > **Window/Zoom** > **Window Settings**

  In the dialog box that follows, enter the following values:
  XMin = −10 XMax = 100 XScale = 10
  YMin = −10 YMax = 110 YScale = 10



- Hover over an axis tick, press [ctrl] [menu] > **Attributes**. Select **Multiple Labels**.
- Press [ctrl] [menu] > **Hide/Show**. Select **Show Lined Grid**.
- Edit the axes labels: ***x*** to be **Time**, and ***y*** to be **Temp**.

(i) To find the times at which the excess temperature falls to one-half and one-quarter of its initial value, i.e.
$\theta = 20+\frac{1}{2}\times 80 = 60^{\circ}\text{C}$ and $\theta = 20+\frac{1}{4}\times 80 = 40^{\circ}\text{C}$:



- Press [P] > **Point**. Click graph $\boldsymbol{f1}$ twice to add two distinct points. Press [esc] to exit this tool.
- Edit the *y*-coordinate of one point to be **60** (for 60°C).
- Edit the *y*-coordinate of the other point to be **40** (for 40°C).
- If needed, to increase/decrease the precision of a coordinate, hover over the value and press [+] or [−].

Answer: The excess temperature is halved in 14.7 minutes and is one-quarter of the initial value in 29.5 minutes.

(ii) To find the temperature after 1 hour (60 minutes):

- Press [P] > **Point**. Click graph $\boldsymbol{f1}$, then press [esc] to exit.
- Edit the *x*-coordinate of the point to be **60** (for 60 minutes).

Answer: The temperature after 1 hour is 24.8°C.

Modelling logistic growth with a DE of the form $dy/dx = g(y)$

Logistic growth describes a population model that increases approximately exponentially at first, then slows as limiting factors take effect, and ultimately approaches a stable carrying capacity, M.

For example, a researcher studied the population growth of an introduced species of tropical fish.

The fish population was modelled using the logistic differential equation, $\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right)$, where P is the population at time t months, r is the intrinsic growth rate, and M is the carrying capacity.

It can be shown that the differential equation has a solution of the form $P(t) = \frac{M}{1 + Ae^{-rt}}, t \geq 0$.

Question

Assume that for the above scenario, the initial fish population in a lake was 600 and the population stabilised over time to $M = 3000$. Answer the following questions according to this model.

(a) Use the Differential Equation Solver to verify that for this model, $A = 4$.

(b) If the population was estimated to be 2000 at time $t = 18$ months, find the value of the intrinsic growth rate r, correct to three decimal places. Interpret the meaning of this value.

(c) Plot a graph of P as a function of t, $0 \leq t \leq 60$. Hence use a graphical method to determine:

(i) the time taken to double and to quadruple the initial population, in months, correct to one decimal place.

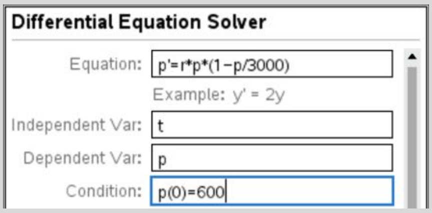
(ii) the population after 1 year and after 3 years.

(d) Find the coordinates of the point of inflection on the graph of P. Interpret this point in terms of the rate of growth of the population and its relation to the carrying capacity.

Solution

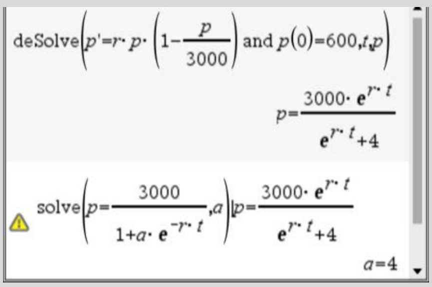
(a) To find $P(t)$ using **Differential Equation Solver**:

- Press [menu] > **Calculus** > **Differential Equation Solver**. Enter as shown, pressing [?!▸] or [π▸] for the prime ′ symbol.



To express the result in the form $P(t) = \frac{3000}{1 + Ae^{-rt}}$ and find the value of A:

- Press [menu] > **Algebra** > **Solve** and enter $\textbf{solve}\left(p = \frac{3000}{1 + a \times e^{-r \times t}}, a\right) \mid \textbf{Ans}$, pressing [ctrl] [=] ([≠≥▸]) to select the **given** symbol, |. Press [ctrl] [(−)] for [ans].



Answer: $P(t) = \frac{3000}{1 + Ae^{-rt}} = \frac{3000e^{rt}}{4 + e^{rt}}$. $A = 4 = \frac{M - P(0)}{P(0)}$, and this represents the initial ratio of unused carrying capacity.

(b) To determine the value of r, given $P(18) = 2000$:

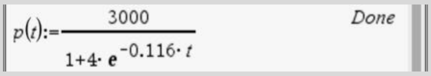
- Enter $\boldsymbol{p(t) := \frac{3000}{1 + 4e^{-r \times t}}}$.
- Enter **solve(p(18) = 2000.0, r)**.

Answer: $r = 0.116$. This means that when the population is small relative to the carrying capacity, it increases at approximately 11.6% per month.

... continued

Solution (continued)

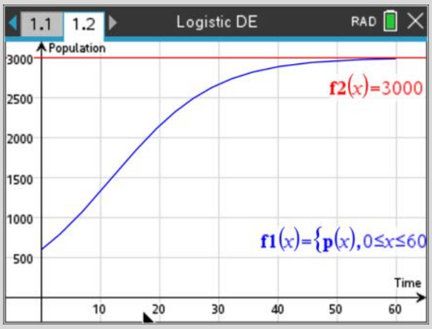
(c) To plot a graph of P as a function of time:



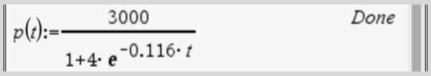
- Edit the definition of P to $\boldsymbol{p(t) := \dfrac{3000}{1+4e^{-0.116\times t}}}$
- Add a **Graphs** page. Enter $\boldsymbol{f1(x) = p(x) \mid 0 \le x \le 60}$.
- Enter $\boldsymbol{f2(x) = 3000}$.
- Press menu > **Window/Zoom > Window Settings**

  In the dialog box that follows, enter the following values:
  XMin = −5 XMax = 65 XScale = 10
  YMin = −300 YMax = 3200 YScale = 500

- Hover over an axis tick, press ctrl menu > **Attributes**. Select **Multiple Labels**.
- Press ctrl menu > **Hide/Show**. Select **Show Lined Grid**.
- Edit the axes labels: $\boldsymbol{x}$ to be **Time**, and $\boldsymbol{y}$ to be **Population**.

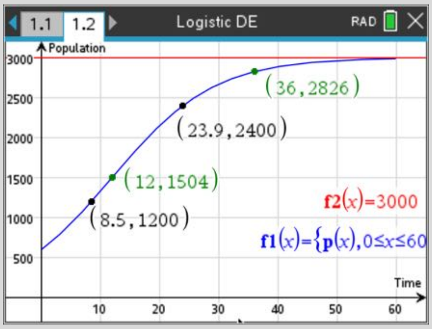


(c)(i) To find t when the population is $2P(0)$ and $4P(0)$:



- Press P > **Point**. Click graph $\boldsymbol{f1}$ twice to add two distinct points. Press esc to exit this tool.
- Edit the y-coordinate of one point to be **1200** (for $2P(0)$).
- Edit the y-coordinate of the other point to be **2400** ($4P(0)$).
- If needed, to increase/decrease the precision of a coordinate, hover over the value and press + or −.

Answer: The initial population doubles in 8.5 months and it quadruples in 23.9 months.



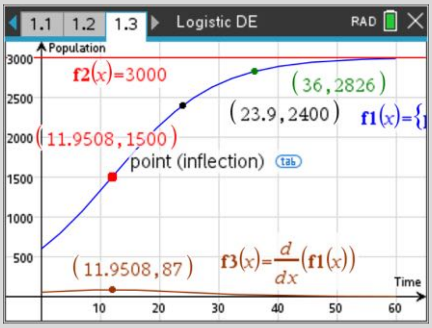
(ii) To find P when $t = 12$ (1 year) and $t = 36$ (3 years):

- Press P > **Point**. Click graph $\boldsymbol{f1}$ twice, then esc to exit.
- Edit the x-coordinate of one point to be **12**.
- Edit the x-coordinate of the other point to be **36**.

Answer: After 1 year, $P = 1504$. After 3 years, $P = 2826$.

(d) To find the coordinates of point of inflection:

- Press menu > **Analyse Graph > Inflection**. Select graph $\boldsymbol{f1}$; click on two points at approximately $t = 5$ and $t = 15$.



Answer: The inflection point occurs at $t \approx 11.95$, $P = 1500$. At the inflection point, the growth rate, dP/dt, is at its maximum value. It occurs where the population is exactly half of the carrying capacity, $M/2 = 1500$, indicating a balance between reproductive growth, that increases with P, and density-dependent constraints on growth that increase with crowding.

Exploring a differential equation of the form $dy/dx = f(x) \times g(y)$

Question

Consider the differential equation $\dfrac{dy}{dx} = xy$.

(a) Using the separation of variables method, a student obtained a general solution $y = Ae^{(x^2/2)}, A \in R$. Use the Differential Equation Solver to verify this general solution.

(b) Find the particular solution at $x = 1$, given that $y(-2) = 2$. Give the answer correct to two decimal places.

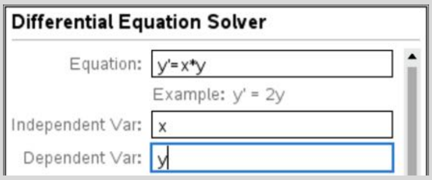
(c) Compare the answer to part **(b)** above with the answer obtained using Euler's method. Use a step size of 0.01 in the numerical calculation.

(d) Verify the result to part **(c)** above on the plotted graph on the slope field of the differential equation. Set a step size of 0.01 to plot the graph of the solution on the slope field.

Solution

(a) To solve $y' = xy$, on a **Calculator** page:

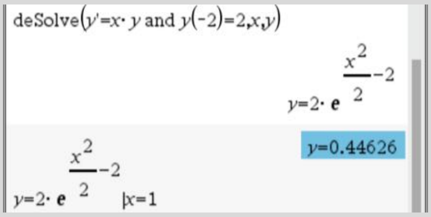
- Press [menu] > **Calculus** > **Differential Equation Solver**. Enter as shown, pressing [?!▸] or [π▸] for the prime ′ symbol.



Answer: The general solution $y = Ae^{(x^2/2)}$ is confirmed.

(b) To find y at $x = 1$, given $y(-2) = 2$ (using DE Solver):

- Edit to include the condition $y(-2) = 2$, as shown.
- Enter **Ans|x = 1**, pressing [ctrl][(−)] for [ans] and [ctrl][=]([|≠≥▸]) to select the **given** symbol, |.



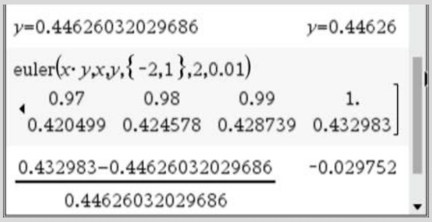
Answer: At $x = 1$, $y = 0.45$ (2 decimal places).

(c) To compare the answer to part **(b)** to Euler's method, on a **Calculator** page:

- Press [📖][1][F] and press ▲ key to select **euler(**.

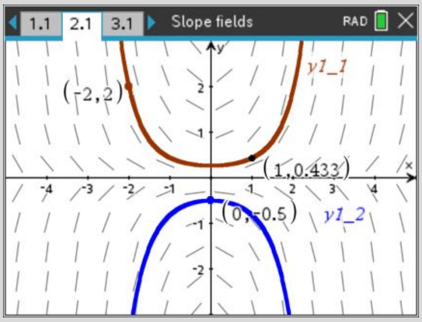
Note: The general syntax for this command is as follows: ***euler(expression,x,y,{initial x, final x},initial y, step size)***.

- Enter **euler**$(x \cdot y, x, y, \{-2, 1\}, 2, 0.01)$



Answer: Using the Euler approximation with a step size of 0.01, $y = 0.43$ (2 decimal places). This represents a percentage error of approximately 3%.

(d) To compare the result to part **(c)** above with the value on a plotted graph on the slope field, see 'Visualising solutions using a slope field for $\dfrac{dy}{dx} = xy$' below for detailed instructions.



Answer: The value on the slope field graph: at $x = 1$, $y = 0.43$.

Using a slope field to visualise solutions for a DE of the form $dy/dx = f(x)$

Question

Plot the slope field for the differential equation $\dfrac{dy}{dx} = 3 - x - x^2$.

(a) On the slope field, plot the graphs of the solutions that satisfy the following conditions.

(i) $y(0) = 3$ **(ii)** $y(0) = 0$ **(iii)** $y(0) = -2$

(b) Explore the effect of moving the initial point to dynamically display differing initial conditions.

Solution

(a) To plot the graphs of the particular solutions on the slope field, on a **Graphs** page:

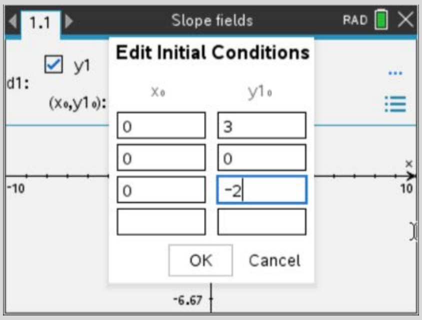
- Press [menu] > **Graph Entry/Edit** > **Diff Eq**.
- Enter the differential equation $\boldsymbol{y1' = 3 - x - x^2}$.

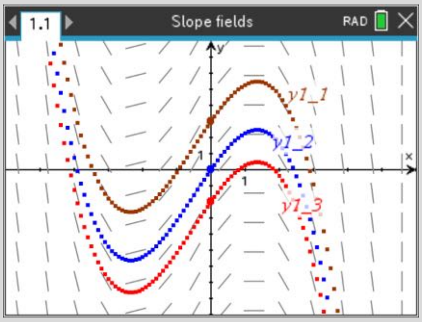
Note: *The default identifier is* ***y1*** *(not y).*

- Under the **Graph Entry** field, click the 🔍 icon.
- In the **Edit Initial Conditions** dialog box that follows, enter the initial conditions, as shown.
- Press [menu] > **Window/Zoom** > **Window Settings**

In the dialog box that follows, enter the following values:

XMin = −6	XMax = 6	XScale = 1
YMin = −9	YMax = 8	YScale = 1

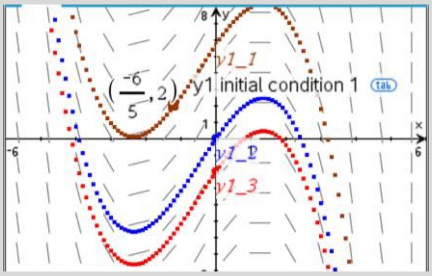


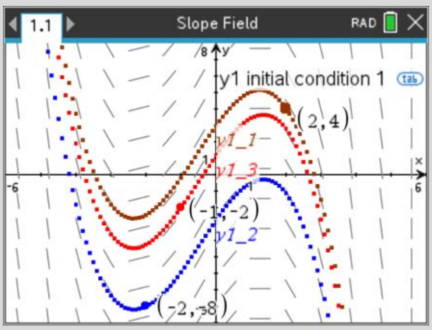


Answer: The slope field displays the gradient at each point on the plane and provides a visual guide to the shape of solution curves. When initial conditions are specified, the corresponding particular solutions can be traced through the indicated slopes, as shown by plots $y1\_1$, $y1\_2$ and $y1\_3$.

(b) To explore the effect of moving the initial point:

- Hover over the point for '**initial condition 1**' until the ☝ icon appears.
- Press [ctrl][menu] > **Coordinates and Equations**. Repeat for the points of '**initial condition**' **2** and **3**, so that the coordinates of the three points are displayed.
- Grab ([ctrl][☝]) the point for '**initial condition 1**' and move it so that the graphs for differing initial conditions are dynamically displayed. Repeat for the points for '**initial condition 2**' and '**initial condition 3**'.
- Click on the x-coordinate value of the point for **initial condition 1** and edit it to be **2**. Similarly, edit the y-coordinate to be **4** (i.e. initial condition is $y(2) = 4$).





Answer: The slope field represents every possible solution simultaneously. Moving the initial point reveals a different solution curve consistent with the same directional information.

Visualising solutions using a slope field for $dy/dx = xy$

Question

Plot the slope field for the differential equation $\dfrac{dy}{dx} = xy$.

(a) On this slope field, plot the particular solution curves that satisfy the initial conditions: **(i)** $y(0) = 1$, **(ii)** $y(0) = -0.5$.

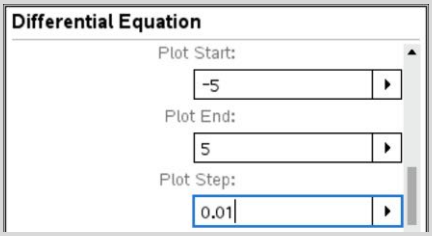
(b) On an appropriate graph, find an approximate solution to the differential equation at $x = 1$, given that $y(-2) = 2$. Give the answer correct to two decimal places.

Solution

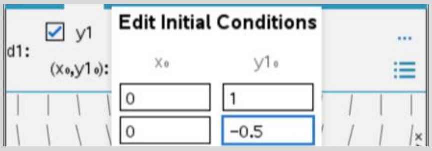
(a) To plot the graphs of the particular solutions on the slope field, on a **Graphs** page:

- Press [menu] > **Graph Entry/Edit** > **Diff Eq**.
- Enter the differential equation $\boldsymbol{y1' = x \cdot y1}$.

***Note**: The default identifier is **y1** (not y).*



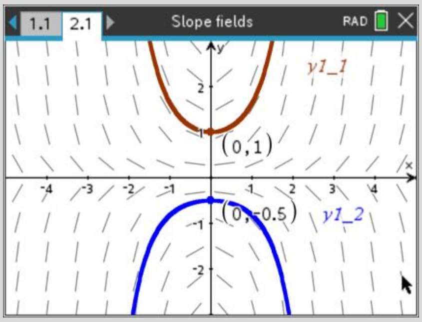
- Click the [⋯] icon on the right of the graph entry. In the dialog box that follows, enter **Plot step**: **0.01**, then [enter].
- Under the **Graph Entry** field, click the [⊕] icon. In the dialog box, enter the initial conditions, as shown.



- Press [menu] > **Window/Zoom** > **Window Settings**

  In the dialog box that follows, enter the following values:
  XMin = −5 XMax = 5 XScale = 1
  YMin = −3 YMax = 3 YScale = 1

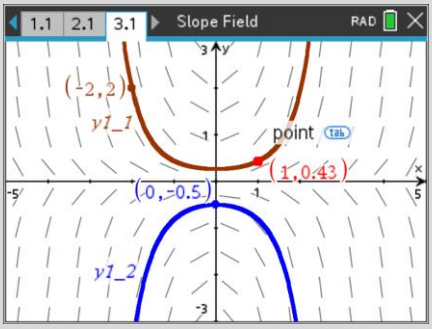
- Hover over the point for **Initial Condition 1** and press [ctrl] [menu] > **Coordinates and Equations** to display coordinates of the point. Repeat for **Initial Condition 2**.



Answer: Two of the many possible solutions are visualised.

(b) To find an approximate solution from a graph on the slope field:

- Edit '**Initial condition 1**' to $(-2, 2)$ (i.e. $y(-2) = 2$).
- Press [P] > **Point**. Click graph **y1_1**. Press [esc] to exit.
- Move the point as close as possible to the value $x = 1$.



Answer: If $\dfrac{dy}{dx} = xy$, given $(-2, 2)$ then at $x = 1$, $y \approx 0.43$.

Generating an approximate solution using Euler's Method

Let $\dfrac{dy}{dx}=f(x,y),\ y(x_0)=y_0$. Given step size h, $x_{n+1}=x_n+h$ and $y_{n+1}=y_n+h\,f(x_n,y_n)$.

Each new value is: *next value* = *current value* + (*step size* × *current gradient*).

The syntax for the *euler*() command to find an approximate value for y when $x=x_{final}$ is:

euler($f(x,y)$, x, y, { x_0, x_{final} }, y_0, *VarStep* [, *eulerStep*]), where $h=\dfrac{VarStep}{euler\ step}$.

By default, *eulerStep* = 1. The effect of varying the *eulerStep* value is demonstrated below.

Question

The differential equation $\dfrac{dy}{dx}=x+y$ models a quantity y that changes with x, and $y(0)=1$.

(a) With the aid of the Differential Equation Solver, find the value of y when $x = 0.8$, correct to two decimal places.

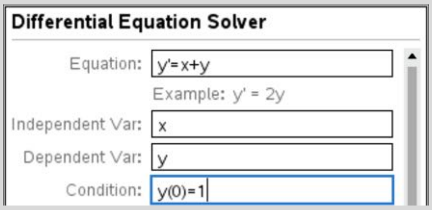
(b) Using the inbuilt command for Euler's method, estimate the value of y when $x = 0.8$ using the following step sizes: **(i)** $h = 0.2$, **(ii)** $h = 0.02$, *eulerStep* = 1, **(iii)** $h = 0.02$, *eulerStep* = 10 (**iv**) $h = 0.002$, *eulerStep* = 100. Give answers correct to two decimal places.

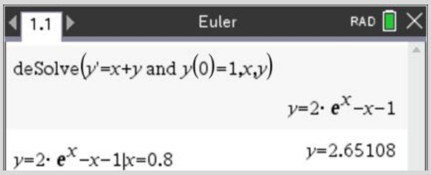
Solution

(a) To find y given $x = 0.8$, $y(0) = 1$, on a **Calculator** page:

- Press [menu] **> Calculus > Differential Equation Solver**.
- Enter as shown, pressing [?!▸] or [π▸] for the prime **′** symbol.
- Enter **ans|*x* = 0.8**, pressing [ctrl] [(–)] for [ans] and [ctrl] [=] to select the **given** symbol, |.

Answer: $y = 2.65$ (2 decimal places) when $x = 0.8$.

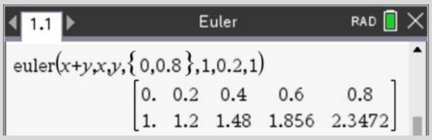




***Note**: Below, **VarStep** sets the spacing of output values (how often results are reported), while **eulerStep** sets the number of substeps per output, so that step size is calculated as: h = VarStep / eulerStep.*

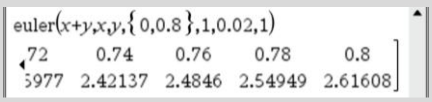
(b)(i) To estimate $y(0.8)$ with $h = 0.2$, on a **Calculator** page:

- Press [📖] [1] [E] to select euler().
- Enter **euler(*x* + *y*, *x*, *y*, {0, 0.8}, 1, 0.2, 1)**.



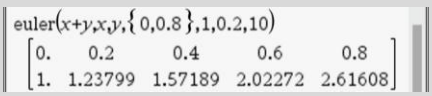
(ii) To estimate $y(0.8)$ with $h = 0.02$, *eulerStep* = 1:

- Press ▲ key to select the previous entry, then [enter].
- Edit the entry to **euler(*x* + *y*, *x*, *y*, {0, 0.8}, 1, 0.02, 1)**.



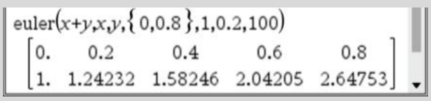
(iii) To estimate $y(0.8)$ with $h = 0.02$, *eulerStep* = 10:

- Similarly, edit to **euler(*x* + *y*, *x*, *y*, {0, 0.8}, 1, 0.2, 10)**.



(iv) To estimate $y(0.8)$ with $h = 0.002$, *eulerStep* = 100:

- Similarly, edit to **euler(*x* + *y*, *x*, *y*, {0, 0.8}, 1, 0.2, 100)**.



Answer: (i) $y \approx 2.35$ (ii), (iii) $y \approx 2.62$ (iv) $y \approx 2.65$.

Implementing pseudocode for Euler's Method in the Python application

Consider $\frac{dy}{dx} = f\left(x, y\right)$ with initial conditions $y\left(x_0\right) = y_0$. To approximate the value of y at $x = x_{final}$, the following pseudocode implementing Euler's method could be used.

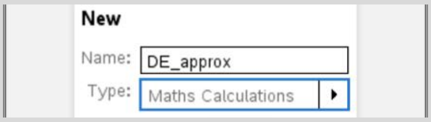
define $f\,(x, y)$	**while** $x < xfinal$
return *function rule*	$y \leftarrow y + h \times f\,(x, y)$
define *euler*$(x0, y0, h, xfinal)$	$x \leftarrow x + h$
$y \leftarrow y0$	**print** x, y (optional: intermediate approximations)
$x \leftarrow x0$	**end while**
print x, y	**return** $xfinal, y$

Question

If $\frac{dy}{dx} = x + y,\ y\left(0\right) = 1$, implement the pseudocode in the Python application to determine an approximate value of y when $x = 0.8$, using step size $h = 0.2$.

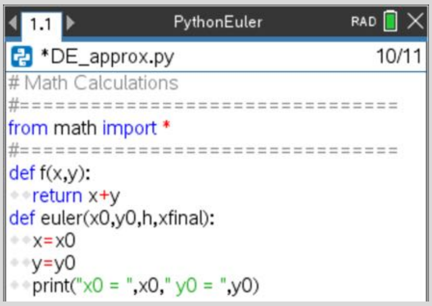
Solution

(a) To start coding, in a new **Document** or new **Problem**:
- Select **Add Python > New**.
- In the dialog box that follows, enter as shown.

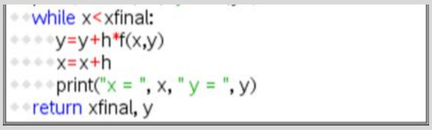


To define $f\,(x, y) = x + y$ and *euler*$(x0, y0, h,$ xfinal):
- Enter **def *f*(*x*, *y*):**, then **return *x* + *y***, as shown, by pressing [menu] > **Built-ins** > **Functions** to select **def function():** and **return**. Press [del] to remove indentation in line 7.
- Similarly, enter **def *euler*(*x*0, *y*0, *h*, *xfinal*):** .

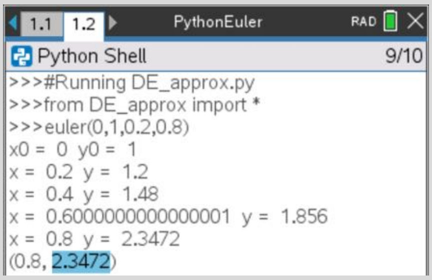
To set and display initial values, $x0$ and $y0$:
- Enter ***x* = *x*0**, then ***y* = *y*0** , with indentations as shown.
- Enter **print("*x*0 = ", *x*0, "*y*0 = ", *y*0)**, pressing [menu] > **Built-ins** > **I/O** for **print** and [?!▸] for the **"** symbol.



To set up the **while** loop until the value *xfinal* is reached:
- Enter **while *x* < *xfinal*:** , pressing [menu] > **Built-ins** > **Controls** to select **while**, and [ctrl][=] to select <.
- Enter ***y* = *y* + *h* × *f*(*x*, *y*)**, then ***x* = *x* + *h***, as shown.
- Enter **print("*x* = ", *x*, "*y* = ", *y*)** .
- Enter **return *xfinal*, *y***, with indentation as shown.



To check syntax, save and run the program:
- Press [ctrl][B] followed by [ctrl][R] (or [menu] > **Run** …).
- In the **Python Shell** page that follows, press [var], select **euler** and enter **euler(0,1,0.2,0.8)**.



Answer: By Euler's method with $h = 0.2$, at $x = 0.8$, $y \approx 2.35$.

Using the Programme Editor to implement the algorithm for Euler's method

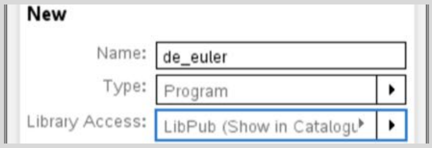
Question

Let $\frac{dy}{dx} = x + y,\ y(0) = 1$. Implement the pseudocode from the previous problem to approximate the value of y at $x = 0.8$, using step sizes: **(a)** $h = 0.2$, **(b)** $h = 0.1$, **(c)** $h = 0.02$, **(d)** $h = 0.002$.
Give answers correct to two decimal places.

Solution

To start coding, in a new **Problem** or **Document**:

- Select **Add Programme Editor > New**.
- In the dialog box that follows, enter as shown.

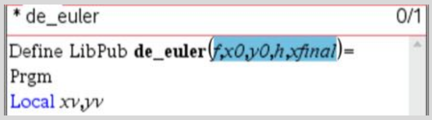


*Note: The name **euler** is a reserved word. Press* [ctrl] [␣] *for underscore in '**de_euler**'.*

The **Program Editor** will follow, ready to accept the code.

To input the function rule and the required values:

- In line 0, enter **Define LibPub de_euler(*f*,*x*0,*y*0,*h*,*xfinal*) =**
- In line 1, enter **Local *xv*, *yv***, pressing [menu] > **Define Variables** to select **Local**.



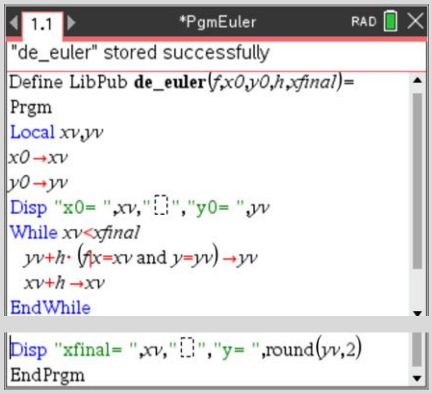
Note: xv, yv are the values that the variables x and y can take.

To set and display initial values, $x0$ and $y0$:

- Enter ***x*0 → *xv***, then ***y*0 → *yv***, pressing [ctrl] [var] for [sto→].
- Enter **Disp "*x*0 = ", *xv*, " ", "*y*0 = ", *yv***, pressing [menu] > **I/O** to select **Disp** and [?!▸] for the **"** symbol.

To set up the **while** loop until the value *xfinal* is reached:

- Press [menu] > **Control** and select **While … End While**.
- Enter **While *xv* < *xfinal*:** , pressing [ctrl] [=] to select <.
- Enter ***yv* + *h* × (*f* |*x* = *xv* and *y* = *yv*) → *yv***, pressing [ctrl] [=] ([|≠≥▸]) for **given**, |. Enter ***xv* + *h* → *xv***, as shown.
- To display the final result, after the End While command, enter **Disp "*xfinal* = ", *xv*, " ", "*y* = ", round(*yv*,2)**, pressing [📖] [1] [S] then the ▲ key to select **round**.

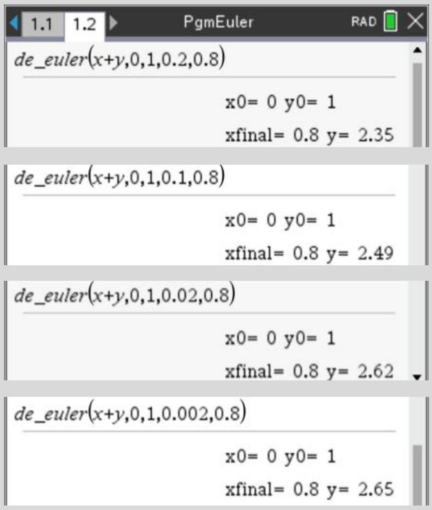


To check syntax, save and run the program:

- Press [ctrl] [B] followed by [ctrl] [R] (or [menu] > **Run** …).

To estimate $y(x_{final})$, in the **Calculator** page that follows:

- **(i)** Enter **de_euler(*x* + *y*, 0, 1, 0.2, 0.8)**, as shown.
- **(ii)** Press ▲ key to the previous entry, press [enter] and edit to **de_euler(*x* + *y*, 0, 1, 0.1, 0.8)**. Alternatively, press [var] to select **de_euler** and enter as shown.
- **(iii)** Similarly, edit to **de_euler(*x* + *y*, 0, 1, 0.02, 0.8)**.
- **(iv)** Similarly, edit to **de_euler(*x* + *y*, 0, 1, 0.002, 0.8)**.



Answer: The approximate value of y at $x = 0.8$ depends on step size: **(i)** $h = 0.2$, $y \approx 2.35$, **(ii)** $h = 0.1$, $y \approx 2.49$ **(iii)** $h = 0.02$, $y \approx 2.62$, **(iv)** $h = 0.002$, $y \approx 2.65$.

3.4.3 Kinematics: rectilinear motion

Modelling the acceleration of an object moving in a straight line

Question

A body is moving in a straight line so that at time t its displacement from a fixed origin is x, its velocity is v and its acceleration is a.

(a) Let $a = 3 - 2x$ and $v = 2$ at $x = 1$. Find the magnitude of v when $x = 2.5$, correct to two decimal places.

(b) **(i)** If $a = \frac{1+v^2}{2}$ and $v = 1$ at $x = 0$, then find x when $v = \frac{11}{2}$, correct to two decimal places.

(ii) Suppose that a student carries out the initial part of the solution to part **(i)** above by integration and finds $x = \ln\left(1+v^2\right)+c$. Use this result to complete the solution.

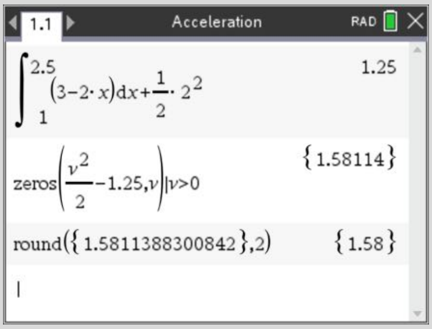
Solution

(a) Since $a = f(x)$, use $\frac{d\left(\frac{1}{2}v^2\right)}{dx}$.

If $\frac{d(y)}{dx} = f(x)$ and $y = y_1$ at $x = x_1$, then the solution to the differential equation at $x = k$ is given by $\int_{x_1}^{k} f(x)\,dx + y_1$.

To find v when $x = 2.5$, on a **Calculator** page:

- Press [⇧shift] [+] or [template key] for the integral template.
- Enter $\int_1^{2.5}(3-2x)\,dx + \left(\frac{1}{2}\cdot 2^2\right)$.
- Press [menu] > **Algebra** > **Zeros**.
- Enter $\textbf{zeros}\left(\left(v^2/2\right)-\textbf{ans},v\right)\,|\,v>0$, then **round(ans,2)**, pressing [catalog] [1] [S] to select **round**, and [ctrl] [(−)] for [ans].



Answer: v = 1.58 correct to two decimal places.

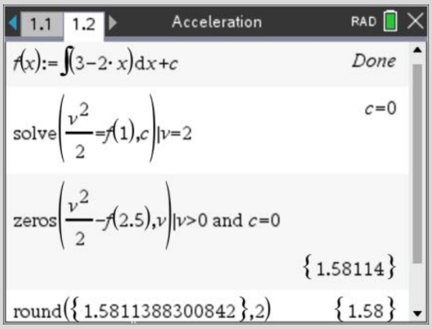
*Note: The **Solve** command could be used instead of **Zeros** as follows:*

$\textbf{solve}\left(\left(v^2/2\right)=\textbf{ans},v\right)\,|\,v>0$

Method 2. Use $\frac{1}{2}v^2 = \int(3-2x)\,dx + c$

To find v using $\frac{1}{2}v^2 = \int(3-2x)\,dx + c$, on a **Calculator** page:

- Enter $f(x) := \int(3-2x)\,dx + c$, pressing [template key] for **integral**.
- Press [menu] > **Algebra** > **Solve**.
- Enter $\textbf{solve}\left(\left(v^2/2\right)=f(1),c\right)\,|\,v=2$.
- Press [menu] > **Algebra** > **Zeros**.
- Enter $\textbf{zeros}\left(\left(v^2/2\right)-f(2.5),v\right)\,|\,v>0\ \textbf{and}\ \textbf{ans}$, then **round(ans,2)**, pressing [catalog] [1] [S] to select **round**.



Answer: v = 1.58 correct to two decimal places.

... continued

Solution (continued)

(b) (i) Since $a = g(v)$ with initial (x, v), use $v\frac{d(v)}{dx}$.

To find x when $v = 11/2$, on a **Calculator** page:

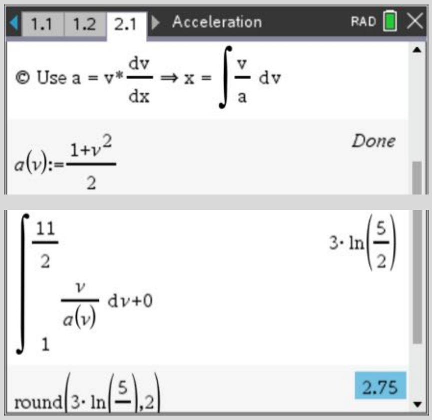
- Enter $\boldsymbol{a(v) := \frac{1+v^2}{2}}$.
- Press [⇧shift] [+] and enter $\int_1^{11/2}\left(\frac{v}{a(v)}\right)dv + 0$
- Enter **round(ans,2)**, pressing [📖] [1] [S] to select **round**.

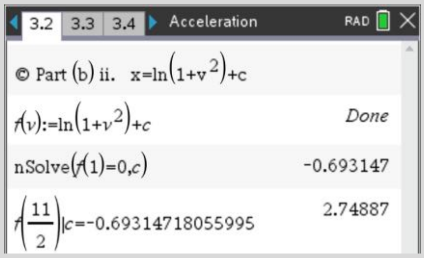
Answer: $x = 2.75$, correct to two decimal places.

(b) (ii) To find x when $v = 11/2$ using the result $x = \log_e(1+v^2)+c$, on a **Calculator** page:

- Enter $\boldsymbol{f(v) := \ln(1+v^2)+c}$.
- Press [menu] > **Algebra** > **Numerical Solve**.
- Enter $\mathbf{nSolve}(f(1) = 0, c)$.
- Enter $f\left(\frac{11}{2}\right) | c = \mathbf{ans}$.

Answer: This confirms the answer is $x = 2.75$, correct to two decimal places.





***Note:** Solve or Zeros could be used instead of nSolve.*

Solving motion problems using graphical methods

Question

A particle falls vertically in a resisting medium, from its initial rest position at the origin, O.

At time t its velocity is given by $v(t) = 40\left(1 - e^{-\frac{t}{4}}\right), t \ge 0$.

Find the following graphically, correct to two decimal places.

(a) The particle's position x relative to O at time $t = 7$.

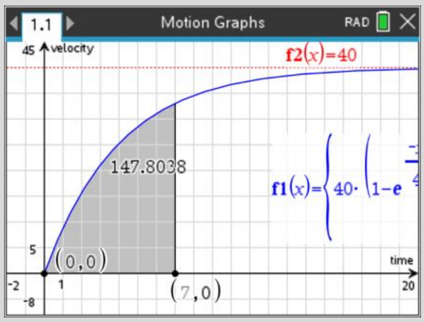
(b) The distance travelled by the particle in the interval $t = 5$ to $t = 12$.

(c) The terminal speed of the particle.

Solution

(a) To find the particle's position at $t = 7$, on a **Graphs** page:

- Enter $\boldsymbol{f1(x) = 40\left(1 - e^{-\frac{x}{4}}\right) | x \ge 0}$, then $\boldsymbol{f2(x) = 40}$.
- Press [menu] > **Window/Zoom** > **Window settings**.
  In the dialog box that follows, enter the following values:
  XMin = –2 XMax = 20 XScale = 1
  YMin = –8 YMax = 45 YScale = 1.
- Press [menu] > **Analyse Graph** > **Integral**. Click graph $f1$.
  Press [0] [enter] then [7] [enter] for **lower** and **upper bound**.



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Solution (continued)

- Hover over the lower bound point, press ctrl menu > **Coordinates & Equations**. Repeat for the upper bound.

Answer: At $t = 7$, the particle's position is $x = 147.80$.

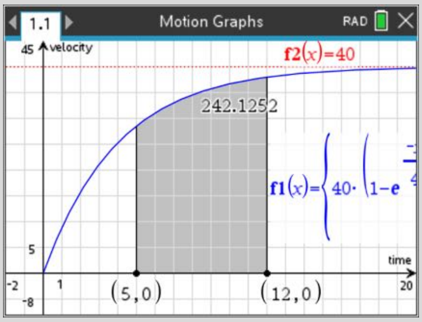
(b) To find the distance travelled in the interval $t = 5$ to $t = 12$, on the **Graphs** page from part **(a)** above:

- Edit the x-coordinate (time) of the lower bound to be **5** and the x-coordinate of the lower bound to be **12**.

Answer: distance = 242.13 units (two decimal places).

(c) To find the terminal speed, consider the velocity-time graph: as $t \to \infty$, $e^{-t/4} \to 0, v(t) \to 40$. It is apparent that $v = 40$ is a horizontal asymptote to the velocity-time graph, so that the terminal speed is $v = 40$.

Note: *To select* ***Lined Grid****, press* menu *>* ***View*** *>* ***Grid****. The axis labels* ***x*** *and* ***y*** *can be edited to* ***time*** *and* ***velocity****, as shown.*



Analysing a problem involving vertical motion under gravity

Question

An object is projected vertically upwards with a speed of 25 m/s from the top of a watchtower that is 20 m high. The watchtower is located on the edge of the top of a cliff that is 30 m high. The object ends its flight when it strikes the ground at the bottom of the cliff. Take the origin as the top of the cliff and the positive direction upwards. Assume negligible air resistance and $g = 9.8\,\text{m/s}^2$.

(a) Plot graphs of the displacement and velocity of the object as a function of time.

(b) Use graphical methods to determine the following.
  (i) The maximum height reached above the origin and the time taken to reach this height.
  (ii) The velocity with which the object strikes the ground and the time taken for the object to strike the ground.

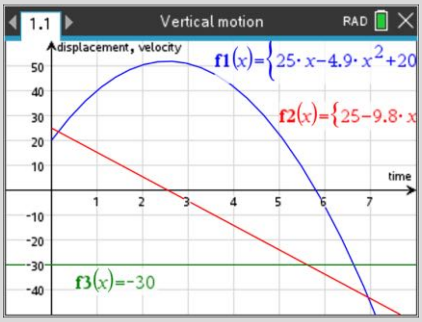
(c) Use the constant acceleration formulas to verify the graphical solutions.

Solution

(a) To obtain $s - t$ and $v - t$ graphs based on the equations $s = ut + \frac{1}{2}at^2 + s_0$ and $v = u + at$, on a **Graphs** page:

- Enter $\boldsymbol{f1(x) = 25x - 4.9x^2 \mid x \geq 0}$, $\boldsymbol{f2(x) = 25 - 9.8x \mid x \geq 0}$, and $\boldsymbol{f3(x) = -30}$.
- Press menu > **Window/Zoom** > **Window settings**.
  In the dialog box that follows, enter the following values:
  XMin = −1 XMax = 8 XScale = 1
  YMin = −50 YMax = 60 YScale = 10
- Hover over an axis tick, press ctrl menu > **Attributes** and select **Multiple Labels**.
- Press menu > **View** > **Grid** > **Lined Grid**

Answer: The $s - t$ graph is in the shape of a parabola, with a local maximum. The $v - t$ graph is linear.



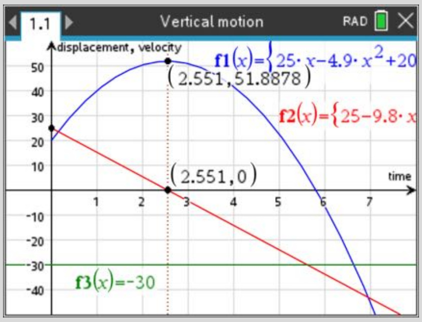
Note: *The axis labels* ***x*** *and* ***y*** *can be edited to* ***time*** *and* ***displacement****,* ***velocity****.*

... continued

Solution (continued)

(b)(i) To find the maximum height and time taken to reach maximum height using the $s-t$ graph:

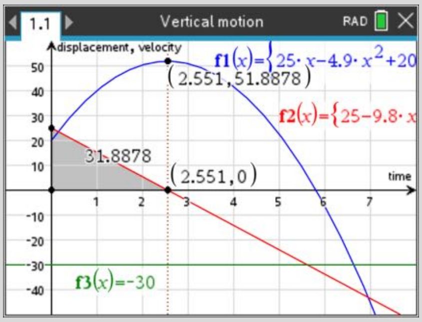
- Press [menu] > **Analyse Graph > Maximum**.
- Click graph ***f*** **1**, then click **lower** and **upper bounds** to the left and right of the local maximum.
- To increase/decrease the precision of the coordinate values, hover over the value and press [+] or [−].



Answer: Time to reach maximum height is 2.55 s.
Maximum height is 51.89 m (correct to two decimal places)

b) (i) *Alternative method*. To find the maximum height and time taken, using the area under the $v-t$ graph:

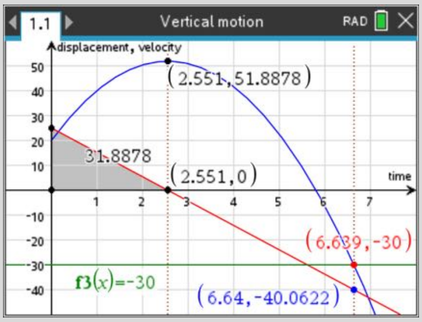
- Press [P] > **Point**. Click the **intersection point** of graph ***f*** **2** and the *x*-axis (time-axis). Hover over the intersection point and press [ctrl] [menu] > **Coordinates & Equations**.
- Press [menu] > **Analyse Graph > Integral**.
- Click graph ***f*** **2**, then click the origin (lower bound), then click the point where the graph ***f*** **2** intersects the *x*-axis (upper bound).



Answer: Maximum height above origin is 31.89 + 20 = 51.89 metres at time $t = 2.55$ s.

(b) (ii) To determine the velocity on striking the ground and the time taken to strike the ground:

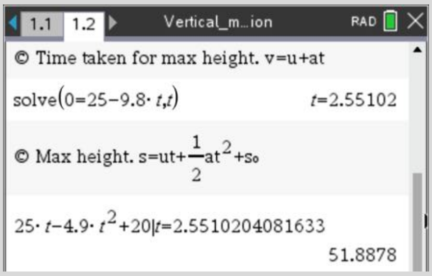
- Press [P] > **Point**. Click the **intersection point** of graphs ***f*** **1** and ***f*** **3**. The *x*-coordinate of the point is 6.639….
- Click graph ***f*** **2**, press [esc] to exit, then edit the *x*-coordinate of the point to be **6.639**.



Answer: The ground is struck with velocity –40.06 m/s at time 6.64 s, correct to two decimal places.

(c) (i) To find the maximum height and the time to reach maximum height using formulas, on a **Calculator** page:

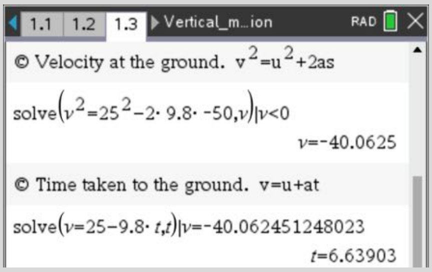
- Press [menu] > **Algebra > Solve**, then enter $\mathbf{solve}\left(\mathbf{0=25-9.8}\boldsymbol{t},\boldsymbol{t}\right)$.
- Enter $\mathbf{25}\boldsymbol{t}\mathbf{-4.9}\boldsymbol{t}^{\mathbf{2}}\mathbf{+20|ans}$, pressing [ctrl] [(−)] for [ans].



Answer: Maximum height is 51.89 m at $t = 2.55$ s.

(c) (ii) To find the velocity and time taken to hit the ground:

- Press [menu] > **Algebra > Solve**, then enter $\mathbf{solve}\left(\boldsymbol{v}^{\mathbf{2}}\mathbf{=25^2-2\times9.8\times-50,}\boldsymbol{v}\right)\mathbf{|}\boldsymbol{v}\mathbf{<0}$.
- Enter $\mathbf{solve}\left(\boldsymbol{v}\mathbf{=25-9.8}\boldsymbol{t},\boldsymbol{t}\right)\mathbf{|ans}$



Answer: Strikes ground with $v = -40.06$ m/s at $t = 6.64$ s.

Note: *To add a comment, press* [menu] ***> Actions > Insert Comment****.*

3.5 Space and measurement

3.5.1 Vectors

Showing that a set of vectors is linearly dependent

A vector $\underset{\sim}{w}$ is a linear combination of vectors, $\underset{\sim}{v}_1, \underset{\sim}{v}_2, ..., \underset{\sim}{v}_n$ if it can be expressed in the form

$$\underset{\sim}{w} = c_1\underset{\sim}{v}_1 + c_2\underset{\sim}{v}_2 + ... + c_n\underset{\sim}{v}_n \text{ for some } c_1, c_2, ..., c_n \in \mathbb{R}.$$

Two vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, are linearly dependent if and only if there exists $k, l \in \mathbb{R}$, not both zero, such that $k\underset{\sim}{a} + l\underset{\sim}{b} = \underset{\sim}{0}$. Hence two non-zero vectors are linearly dependent if and only if the two vectors are parallel.

Suppose that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are non-zero, non-parallel vectors. The vectors, $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependent if and only if there exists $m, n \in \mathbb{R}$ such that $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$.

Question

Show that the following set of vectors is linearly dependent:

$$\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$$

$$\underset{\sim}{b} = 4\underset{\sim}{i} + 5\underset{\sim}{j} + 6\underset{\sim}{k}$$

$$\underset{\sim}{c} = 7\underset{\sim}{i} + 8\underset{\sim}{j} + 9\underset{\sim}{k}$$

Solution

Vectors are linearly dependent if one can be written as a linear combination of the others.

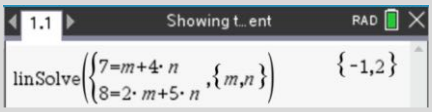
Vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are not parallel and hence suppose that $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$.

By substitution, the system of linear equations formed are:

$$7 = m + 4n \quad (1)$$
$$8 = 2m + 5n \quad (2)$$
$$9 = 3m + 6n \quad (3)$$

On a **Calculator** page:

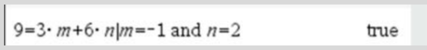
- Press menu > **Algebra** > **Solve System of Equations** > **Solve System of Linear Equations**.
- In the dialog box that follows:
  - For **Number of equations**, enter **2**.
  - For **Variables**, enter ***m*,*n***.
- For example, enter equations (1) and (2) into the template and press enter.



... continued

Solution (continued)

To show that $m = -1$ and $n = 2$ satisfy (3):



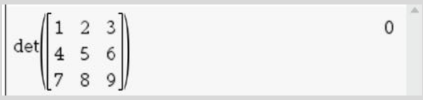
- Enter equation (3).
- Press ctrl = to access the 'with' or 'given' symbol |.

Answer: As $m = -1$ and $n = 2$ also satisfy equation (3), the vectors are linearly dependent.

Equation (3) can be expressed as $3 = m + 2n$ and hence solving (1) and (3) simultaneously gives $n = 2$ directly.
It can also be noted from the question that $\underset{\sim}{a} + \underset{\sim}{c} = 2\underset{\sim}{b}$ and hence $\underset{\sim}{a} - 2\underset{\sim}{b} + \underset{\sim}{c} = \underset{\sim}{0}$.

Alternatively, if these vectors are linearly dependent then matrix A formed by them will have a determinant of zero.

To find $\det(A)$:

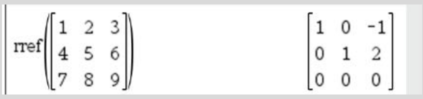


- Press menu > **Matrix & Vector** > **Determinant**.
- Press 5, select the **m-by-n Matrix** template, fix the dimensions as 3-by-3 and enter as shown.

Answer: Since $\det(A) = 0$, the vectors are linearly dependent.

Alternatively using reduced row-echelon form:

- Press menu > **Matrix & Vector** > **Reduced Row-Echelon Form**.
- If required, press 5, select the **m-by-n Matrix** template, fix the dimensions as 3-by-3 and enter as shown.



The third row has become a row of zeros.

There are two pivots (leading non-zero elements in rows 1 and 2) and there are three vectors.

Answer: Since the number of pivots is less than the number of vectors, the vectors are linearly dependent. The zero row confirms that one vector can be expressed as a linear combination of the other two.

Showing that a set of vectors is linearly independent

A set of vectors is linearly independent if no vector in the set can be expressed as a linear combination of the other vectors in the set.

Question

Show that the following set of vectors is linearly independent:

$$\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$$

$$\underset{\sim}{b} = 2\underset{\sim}{i} + 5\underset{\sim}{j} + 3\underset{\sim}{k}$$

$$\underset{\sim}{c} = 3\underset{\sim}{i} + 7\underset{\sim}{j} + 6\underset{\sim}{k}$$

Solution

Vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are non-zero and not parallel.

Hence suppose that $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$.

By substitution, the system of linear equations formed are:

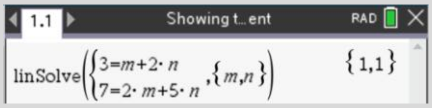
$$3 = m + 2n \quad (1)$$

$$7 = 2m + 5n \quad (2)$$

$$6 = 2m + 3n \quad (3)$$

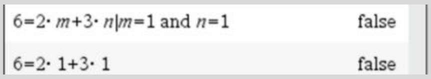
On a **Calculator** page:

- Press menu > **Algebra** > **Solve System of Equations** > **Solve System of Linear Equations**.
- In the dialog box that follows:
  - For **Number of equations**, enter **2**.
  - For **Variables**, enter ***m*,*n***.
- For example, enter equations (1) and (2) into the template and press enter.



To show that $m = 1$ and $n = 1$ do not satisfy equation (3):

- Enter equation (3).
- Press ctrl = to access the 'with' or 'given' symbol |.

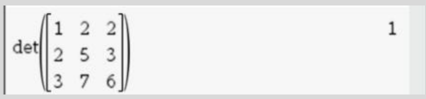


Answer: As $m = 1$ and $n = 1$ do not satisfy equation (3) ($6 \neq 5$), the vectors are linearly independent.

Alternatively, if these vectors are linearly independent then matrix A formed by them will have a non-zero determinant.

To find $\det(A)$:

- Press menu > **Matrix & Vector** > **Determinant**.
- Press 5, select the **m-by-n Matrix** template, fix the dimensions as 3-by-3 and enter as shown.



Answer: Since $\det(A) = 1(\neq 0)$, the vectors are linearly independent.

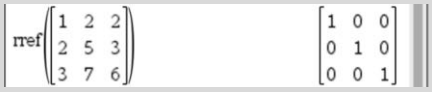
***Note**: If $\det(A) = 0$, it shows that at least one vector is a linear combination of the others.*

... continued

Solution (continued)

Alternatively using reduced row-echelon form:

- Press [menu] > **Matrix & Vector** > **Reduced Row-Echelon Form**.
- If required, press [⌸] [5], select the **m-by-n Matrix** template, fix the dimensions as 3-by-3 and enter as shown.

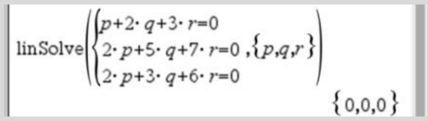


A reduces to I, the identity matrix.

Answer: As every row has a pivot (non-zero element), there are no 'free variables'. Hence the only solution to $p\underset{\sim}{a}+q\underset{\sim}{b}+r\underset{\sim}{c}=\underset{\sim}{0}$ is $p=q=r=0$. The vectors are linearly independent.

To check this on a **Calculator** page:

- Press [menu] > **Algebra** > **Solve System of Equations** > **Solve System of Linear Equations**.
- In the dialog box that follows:
  - For **Number of equations**, enter **3**.
  - For **Variables**, enter ***p,q,r***.
- Enter the three equations into the template and press [enter].



Thus confirming the three vectors are linearly independent.

Calculating the magnitude of a vector

The magnitude of $\underset{\sim}{a}=x\underset{\sim}{i}+y\underset{\sim}{j}+z\underset{\sim}{k}$ is $|\underset{\sim}{a}|=\sqrt{x^2+y^2+z^2}$.

Question

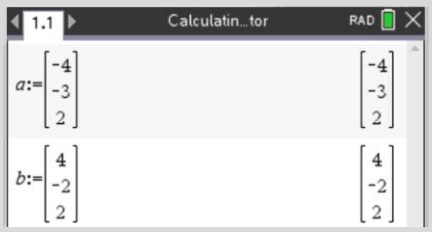
Consider the vectors $\underset{\sim}{a}=-4\underset{\sim}{i}-3\underset{\sim}{j}+2\underset{\sim}{k}$ and $\underset{\sim}{b}=4\underset{\sim}{i}-2\underset{\sim}{j}+2\underset{\sim}{k}$.

Find the exact value of $|\underset{\sim}{a}+2\underset{\sim}{b}|$.

Solution

On a **Calculator** page, assign $\underset{\sim}{a}$ and $\underset{\sim}{b}$ as follows:

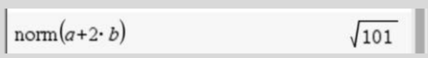
- Press [ctrl] [⊞] to access the **Assign** [:=] command.
- Press [⌸] [5], select the **m-by-n Matrix** template, fix the dimensions as 3-by-1 and enter as shown.



Note: *Column vectors are used as there is enough screen space to display all the calculations. In some subsequent examples, row vectors will be used.*

To find $|\underset{\sim}{a}+2\underset{\sim}{b}|$:

- Press [menu] > **Matrix & Vector** > **Norms** > **Norm**.
- Enter as shown.



Note: *If required, press* [ctrl] [enter] *to obtain a decimal magnitude.*

... continued

Solution (continued)

Answer: $\left|\underset{\sim}{a}+2\underset{\sim}{b}\right|=\sqrt{101}$

$$\begin{aligned}\left|\underset{\sim}{a}+2\underset{\sim}{b}\right| &= \left|-4\underset{\sim}{i}-3\underset{\sim}{j}+2\underset{\sim}{k}+2\left(4\underset{\sim}{i}-2\underset{\sim}{j}+2\underset{\sim}{k}\right)\right| \\ &= \left|4\underset{\sim}{i}-7\underset{\sim}{j}+6\underset{\sim}{k}\right| \\ &= \sqrt{4^2+(-7)^2+6^2} \\ &= \sqrt{101}\end{aligned}$$

The vector $\underset{\sim}{a}+2\underset{\sim}{b}$ can be entered and calculated before finding its magnitude.

Finding a vector parallel to another vector

A unit vector, $\hat{\underset{\sim}{a}}$, in three-dimensional space is given by $\hat{\underset{\sim}{a}}=\dfrac{\underset{\sim}{a}}{\left|\underset{\sim}{a}\right|}$.

Question

A vector parallel to $\underset{\sim}{i}-2\underset{\sim}{j}+5\underset{\sim}{k}$ and with magnitude 6 is

A. $6\left(\underset{\sim}{i}-2\underset{\sim}{j}+5\underset{\sim}{k}\right)$

B. $\dfrac{\sqrt{30}}{5}\left(\underset{\sim}{i}-2\underset{\sim}{j}+5\underset{\sim}{k}\right)$

C. $-\dfrac{6}{\sqrt{30}}\left(\underset{\sim}{i}+2\underset{\sim}{j}+5\underset{\sim}{k}\right)$

D. $\dfrac{1}{5}\left(\underset{\sim}{i}-2\underset{\sim}{j}+5\underset{\sim}{k}\right)$

Solution

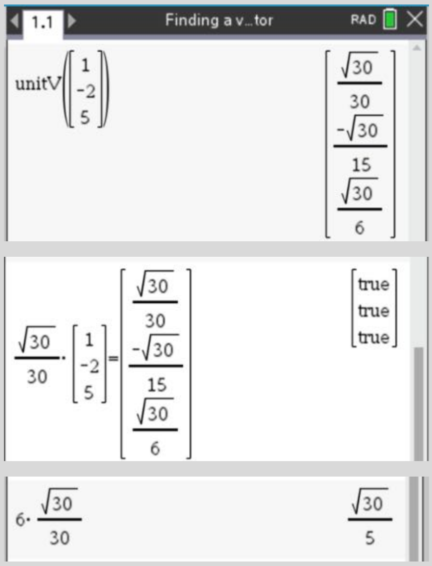
On a **Calculator** page:

- Press [menu] > **Matrix & Vector** > **Vector** > **Unit Vector**.
- Press [templates] [5], select the **m-by-n Matrix** template, fix the dimensions as 3-by-1 and enter as shown.

A unit vector in the direction of $\underset{\sim}{i}-2\underset{\sim}{j}+5\underset{\sim}{k}$ is

$$\frac{\sqrt{30}}{30}\underset{\sim}{i}-\frac{\sqrt{30}}{15}\underset{\sim}{j}+\frac{\sqrt{30}}{6}\underset{\sim}{k}=\frac{\sqrt{30}}{30}\left(\underset{\sim}{i}-2\underset{\sim}{j}+5\underset{\sim}{k}\right).$$

The screenshot at right confirms the two equivalent forms.



Hence a vector with magnitude 6 parallel to $\underset{\sim}{i}-2\underset{\sim}{j}+5\underset{\sim}{k}$ is

$$\frac{6\sqrt{30}}{30}\left(\underset{\sim}{i}-2\underset{\sim}{j}+5\underset{\sim}{k}\right)=\frac{\sqrt{30}}{5}\left(\underset{\sim}{i}-2\underset{\sim}{j}+5\underset{\sim}{k}\right)$$

Answer: B

Option **C**, $-\dfrac{6}{\sqrt{30}}\left(\underset{\sim}{i}+2\underset{\sim}{j}+5\underset{\sim}{k}\right)$ has a magnitude of 6 but is not parallel to $\underset{\sim}{i}-2\underset{\sim}{j}+5\underset{\sim}{k}$.

Using position vectors in Cartesian form

Question

The position vectors of points A and B are given by $\overrightarrow{OA} = 5\underset{\sim}{i} + 10\underset{\sim}{j} + \underset{\sim}{k}$ and $\overrightarrow{OB} = 4\underset{\sim}{i} - 2\underset{\sim}{k}$.
Find the exact distance between points A and B.

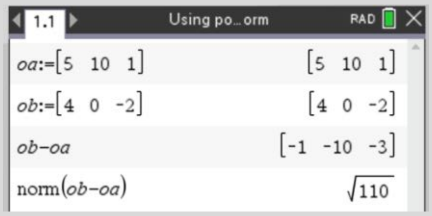
Solution

On a **Calculator** page, assign $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as row vectors as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [book] [5], select the **m-by-n Matrix** template, fix the dimensions as 1-by-3 and enter as shown.

To find the exact distance between points A and B:

- Press [menu] > **Matrix & Vector** > **Norms** > **Norm**.
- Enter as shown.



Answer: The exact distance between A and B is $\sqrt{110}$.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \left(4\underset{\sim}{i} - 2\underset{\sim}{k}\right) - \left(5\underset{\sim}{i} + 10\underset{\sim}{j} + \underset{\sim}{k}\right)$

$= -\underset{\sim}{i} - 10\underset{\sim}{j} - 3\underset{\sim}{k}$

$\left|\overrightarrow{AB}\right| = \sqrt{(-1)^2 + (-10)^2 + (-3)^2}$

$= \sqrt{110}$

Note: *Press [var] to access assigned/stored variables.*

Using the scalar (dot) product to find the angle between two vectors

Given that $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$, the scalar (dot) product is defined as:

- $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$
- $\left(a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}\right) \cdot \left(b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}\right) = a_1b_1 + a_2b_2 + a_3b_3$

Question

Points P and Q are defined by the position vectors $\underset{\sim}{p}$ and $\underset{\sim}{q}$ respectively, where

$\underset{\sim}{p} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{q} = -4\underset{\sim}{i} - 3\underset{\sim}{k}$.

Find the angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, giving your answer correct to the nearest tenth of a degree.

... continued

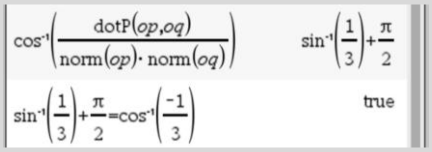
Solution

On a **Calculator** page, assign $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ as row vectors as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **m-by-n Matrix** template, fix the dimensions as 1-by-3 and enter as shown.

To determine the angle, θ, between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$:

- Press [trig] and select $\cos^{-1}$.
- Press [ctrl] [÷] to access the **Fraction** template.
- Press [menu] > **Matrix & Vector** > **Vector** > **Dot Product**.
- Enter the numerator as shown.
- Press [menu] > **Matrix & Vector** > **Norms** > **Norm**.
- Enter the denominator as shown.

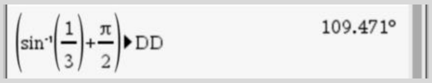


***Notes:** Press [var] to access assigned/stored variables.*

The exact output $\sin^{-1}\left(\frac{1}{3}\right)+\frac{\pi}{2}$ *is equivalent to the more familiar intermediate answer* $\cos^{-1}\left(-\frac{1}{3}\right)$ *as shown right.*

To express θ correct to the nearest tenth of a degree:

- Press ▲ [enter] to paste the exact angle (in radians) to a new entry line.
- Press [catalog] [1] [D], scroll down and select ►DD.
- Press [ctrl] [enter] to obtain a decimal angle in degrees.



Answer: The angle between the two vectors is $109.5°$, correct to the nearest tenth of a degree.

As $\overrightarrow{OP}\cdot\overrightarrow{OQ}=\left|\overrightarrow{OP}\right|\left|\overrightarrow{OQ}\right|\cos(\theta)$,

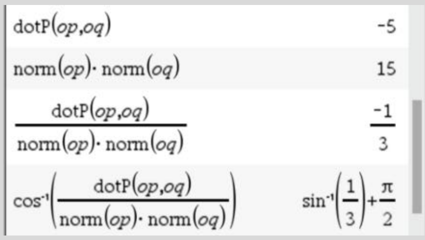
$$\theta=\cos^{-1}\left(\frac{\overrightarrow{OP}\cdot\overrightarrow{OQ}}{\left|\overrightarrow{OP}\right|\left|\overrightarrow{OQ}\right|}\right)=\cos^{-1}\left(\frac{\left(2\underset{\sim}{i}+2\underset{\sim}{j}-\underset{\sim}{k}\right)\cdot\left(-4\underset{\sim}{i}-3\underset{\sim}{k}\right)}{\left|2\underset{\sim}{i}+2\underset{\sim}{j}-\underset{\sim}{k}\right|\left|-4\underset{\sim}{i}-3\underset{\sim}{k}\right|}\right)$$

$$=\cos^{-1}\left(\frac{-8+0+3}{3\times 5}\right)$$

$$=\cos^{-1}\left(-\frac{1}{3}\right)$$

$$=109.471...°$$

$$=109.5°$$



***Notes:** Instead of performing all the steps at once on TI-Nspire CX II CAS, it is a good idea from a teaching viewpoint to show the required steps one at a time as shown at right. The approach shown in this example can be adapted to finding the angle between two lines, the angle between two planes and the angle between a line and a plane.*

Using the scalar (dot) product to determine when two vectors are perpendicular

Using the scalar product, $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$ if and only if $\underset{\sim}{a} = \underset{\sim}{0}$ or $\underset{\sim}{b} = \underset{\sim}{0}$ or $\underset{\sim}{a} \perp \underset{\sim}{b}$.

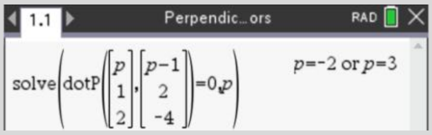
Question

Find the value(s) of p for which the vectors $\underset{\sim}{u} = p\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{v} = (p-1)\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}$ are perpendicular.

Solution

On a **Calculator** page:

- Press [menu] > **Algebra** > **Solve**.
- Press [menu] > **Matrix & Vector** > **Vector** > **Dot Product**.
- Press [📖] [5], select the **m-by-n Matrix** template, fix the dimensions as 3-by-1 and enter the vectors as shown.



Answer: $p = -2$ or $p = 3$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left(p\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}\right) \cdot \left((p-1)\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}\right)$$
$$= p(p-1) + 2 - 8$$
$$= p^2 - p - 6$$

$$p^2 - p - 6 = 0$$
$$(p-3)(p+2) = 0$$
$$p = -2, 3$$

Finding the vector projection (resolute) of one vector onto another

The scalar projection (resolute) of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ is defined as:

- $|\underset{\sim}{a}|\cos(\theta) = \underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$

The vector projection (resolute) of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ is defined as:

- $|\underset{\sim}{a}|\cos(\theta)\hat{\underset{\sim}{b}} = \left(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}} = \left(\dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\right)\underset{\sim}{b}$

Question

Find the vector resolute of $\underset{\sim}{a} = 4\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$ in the direction of $\underset{\sim}{b} = -\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$.

***Note**: The correct wording for questions of this type should be:*

Find the vector projection of $\underset{\sim}{a} = 4\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$ onto $\underset{\sim}{b} = -\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$.

The question wording used here is VCAA wording.

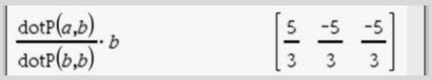
Solution

On a **Calculator** page, assign $\underset{\sim}{a}$ and $\underset{\sim}{b}$ as row vectors as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **m-by-n Matrix** template, fix the dimensions as 1-by-3 and enter the vectors as shown.

To determine the vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$:

- Press [ctrl] [÷] to access the **Fraction** template.
- Press [menu] > **Matrix & Vector** > **Vector** > **Dot Product**.
- Enter the numerator and denominator as shown.
- Press [menu] > **Matrix & Vector** > **Vector** > **Dot Product**.
- Enter as shown.



Answer: The vector resolute of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $\frac{5}{3}\underset{\sim}{i} - \frac{5}{3}\underset{\sim}{j} - \frac{5}{3}\underset{\sim}{k} = -\frac{5}{3}\left(-\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right)$.

Let $\underset{\sim}{u}$ be the vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$.

$$\underset{\sim}{u} = \left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\right)\underset{\sim}{b}$$

$$= \left(\frac{-4+2-3}{3}\right)\left(-\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right)$$

$$= -\frac{5}{3}\left(-\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right)$$

Note that the scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is $-\frac{5}{\sqrt{3}}$.

Modelling and solving problems with vectors

Question

Let $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ be unit vectors in the east, north and vertically up directions respectively.

Po leaves her base camp at point O and walks on flat terrain for 6 km in a SE direction to point A. She then walks 4.5 km east to point B.

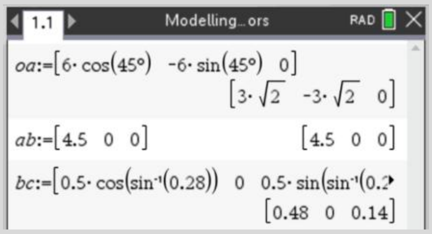
From B, Po walks 0.5 km east up a steep slope inclined at an angle of $\sin^{-1}(0.28)$ to the horizontal to point C. Find $\left|\overrightarrow{OC}\right|$, giving your answer correct to one decimal place.

Solution

***Note:** In **Document Settings > Real or Complex** (accessed by pressing [⌂ on]), there is a choice to set the TI-Nspire CX II CAS to either **Real** or **Rectangular** or **Polar** mode. In this example, TI-Nspire CX II CAS was set to **Rectangular** mode and **Radian** mode.*

On a **Calculator** page, assign $\overrightarrow{OA}$ and $\overrightarrow{AB}$ as row vectors as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [📖] [5], select the **m-by-n Matrix** template, fix the dimensions as 1-by-3.
- Press [trig] to access **cos**, **sin** and **sin⁻¹** as required.
- Press [π▸] to access the degree symbol.

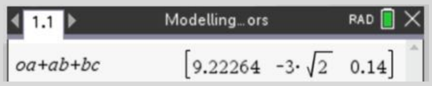


$\overrightarrow{OA} = 3\sqrt{2}\underset{\sim}{i} - 3\sqrt{2}\underset{\sim}{j}$, $\overrightarrow{AB} = 4.5\underset{\sim}{i}$ and

$\overrightarrow{BC} = 0.5\cos\left(\sin^{-1}(0.28)\right)\underset{\sim}{i} + 0.5\sin\left(\sin^{-1}(0.28)\right)\underset{\sim}{k}$

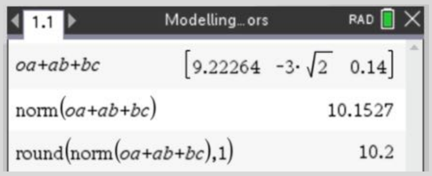
- Enter $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$ as shown.

***Note:** Press [var] to access assigned/stored variables.*



To find $\left|\overrightarrow{OC}\right|$:

- Press [menu] > **Matrix & Vector** > **Norms** > **Norm**.
- Press ▲ to select the row vector and press [enter].
- Press [menu] > **Number** > **Number Tools** > **Round** to express $\left|\overrightarrow{OC}\right|$ correct to one decimal place.



***Note:** The syntax for the **Round** command is **round(Value [,Digits])**.*

Answer: $\left|\overrightarrow{OC}\right| = 10.2$ (km), correct to one decimal place.

$\overrightarrow{OC} = 9.22264...\underset{\sim}{i} - 3\sqrt{2}\underset{\sim}{j} + 0.14\underset{\sim}{k}$

$$\begin{aligned}\left|\overrightarrow{OC}\right| &= \sqrt{(9.22264...)^2 + \left(-3\sqrt{2}\right)^2 + (0.14)^2} \\ &= 10.1526... \\ &= 10.2\end{aligned}$$

Using vectors to prove geometric results in two dimensions

The following properties of the scalar product will be useful in the following proof:

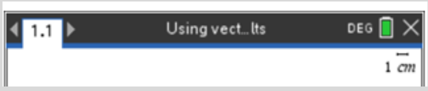
- Two non-zero vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, are perpendicular if and only if $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$.
- $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$

Question

Prove that the diagonals of a rhombus are perpendicular.

Solution

A rhombus is a quadrilateral that has two pairs of parallel, congruent sides.



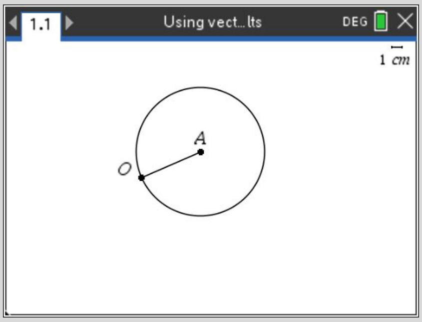
To construct a rhombus, use the centres and points on three intersecting circles to determine the vertices of the rhombus and then connect these vertices to form its sides.

Start with a geometric verification that the diagonals of a rhombus are perpendicular.

On a **Geometry** page construct the rhombus as follows:

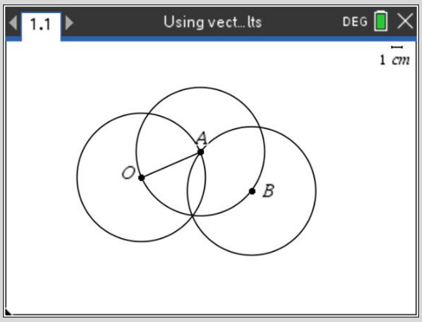
***Note:** After using each geometric tool, press* [esc] *to ensure completion of its use.*

If required, to set the **Geometry Angle** on the page to **Degree** mode:
- Activate the cursor and click (press) on **RAD**.



Construct a line segment *OA*:
- Press [menu] > **Points & Lines** > **Segment**.
- Click (press) to create the initial point of the segment and click (press) to create the end point of the segment.
- Hover the cursor over the initial point.
- Press [ctrl] [menu] > **Label** and label as *O*.
- Hover the cursor over the end point.
- Press [ctrl] [menu] > **Label** and label as *A*.

Construct a circle centred at *A* with radius *OA*:
- Press [menu] > **Construction** > **Compass**.
- Click (press) on point *O*, click (press) on point *A* and click (press) on point *A* again.



Construct a point *B* as the third vertex of the rhombus on the circle centred at *A*.
- Press [menu] > **Points & Lines** > **Point On**.
- Hover the cursor over the circle and click twice (press).
- Press [ctrl] [menu] > **Label** and label as *B*.
- Use the **Compass** tool to construct a circle centred at *B* with radius *OA*.
- Use the **Compass** tool to construct another circle centred at *O* with radius *OA*.

... continued

Solution (continued)

Generate the intersection point of the circles to locate the fourth vertex, point C, of the rhombus:

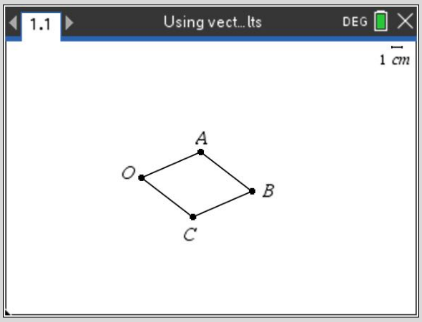
- Press [menu] > **Points & Lines** > **Intersection Point(s)**.
- Click (press [click]) on two of the circles.
- Press [ctrl] [menu] > **Label** and label as C.

Construct line segments OC, AB and BC.

To hide the three circles:

- Hover the cursor over each circle and press [ctrl] [menu] > **Hide**.

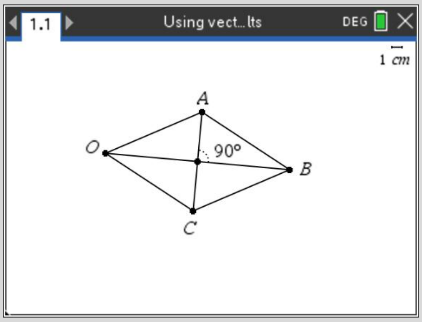
***Note:** Alternatively, press [menu] **> Actions > Hide/Show**. Use the cursor to hide or show objects as appropriate.*



- Grab and drag points O and B around the page.
- Construct diagonal line segments OB and AC as shown.

To measure the angle between the diagonals OB and AC:

- Press [menu] > **Measurement** > **Angle**.
- Click (press [click]) on point A, click (press [click]) where the diagonals meet and click (press [click]) on point B.



The diagonals OB and AC are $\perp$.

$OABC$ is a rhombus.

Let $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{c} = \overrightarrow{OC}$.

The diagonals of the rhombus are OB and AC.

$$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} = \overrightarrow{OC} + \overrightarrow{OA} = \underset{\sim}{c} + \underset{\sim}{a}$$

$$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\underset{\sim}{a} + \underset{\sim}{c}$$

$$\begin{aligned}\overrightarrow{OB} \cdot \overrightarrow{AC} &= (\underset{\sim}{c} + \underset{\sim}{a}) \cdot (\underset{\sim}{c} - \underset{\sim}{a}) \\ &= \underset{\sim}{c} \cdot \underset{\sim}{c} - \underset{\sim}{c} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{a} \\ &= \underset{\sim}{c} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{a} \\ &= |\underset{\sim}{c}|^2 - |\underset{\sim}{a}|^2\end{aligned}$$

A rhombus has four sides of equal length and hence $|\underset{\sim}{c}| = |\underset{\sim}{a}|$.

$$\overrightarrow{OB} \cdot \overrightarrow{AC} = |\underset{\sim}{c}|^2 - |\underset{\sim}{a}|^2 = 0 \Rightarrow \overrightarrow{AC} \perp \overrightarrow{OB}$$

Hence the diagonals of a rhombus are perpendicular.

3.5.2 Vector and Cartesian equations

Defining and using the vector (cross) product

If $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$, then

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\underset{\sim}{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\underset{\sim}{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\underset{\sim}{k}$$

$$= \left(a_2b_3 - a_3b_2\right)\underset{\sim}{i} - \left(a_1b_3 - a_3b_1\right)\underset{\sim}{j} + \left(a_1b_2 - a_2b_1\right)\underset{\sim}{k}.$$

In column vector form, $\underset{\sim}{a} \times \underset{\sim}{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$.

$\left|\underset{\sim}{a} \times \underset{\sim}{b}\right| = \left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|\sin\left(\theta\right)$ where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

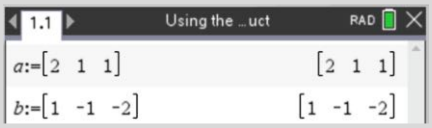
Question

Find a vector perpendicular to the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$.

Solution

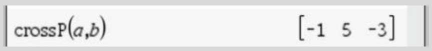
On a **Calculator** page, assign $\underset{\sim}{a}$ and $\underset{\sim}{b}$ as row vectors as follows:

- Press [ctrl] [⊞] to access the **Assign** [:=] command.
- Press [📖] [5], select the **m-by-n Matrix** template, fix the dimensions as 1-by-3 and enter as shown.



To calculate $\underset{\sim}{a} \times \underset{\sim}{b}$:

- Press [menu] > **Matrix & Vector** > **Vector** > **Cross Product**.
- Enter as shown.



Answer: $\underset{\sim}{a} \times \underset{\sim}{b} = -\underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{k}$

$$\left(2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right) \times \left(\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}\right)$$
$$= \left((1\times-2) - (1\times-1)\right)\underset{\sim}{i} - \left((2\times-2) - (1\times1)\right)\underset{\sim}{j} + \left((2\times-1) - (1\times1)\right)\underset{\sim}{k}$$
$$= -\underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{k}$$

Note: $\underset{\sim}{b} \times \underset{\sim}{a} = \underset{\sim}{i} - 5\underset{\sim}{j} + 3\underset{\sim}{k}$. *There is not a unique normal (perpendicular) vector. If the vector* $\underset{\sim}{n}$ *is normal (perpendicular) to 2 other vectors, then so are the vectors* $s\underset{\sim}{n}$ *and* $-s\underset{\sim}{n}$ *for* $s \in \mathbb{R}^+$.

Using vectors to determine the area of a triangle

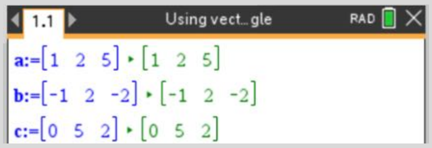
Question

Find the area of the triangle *ABC* with vertices $A(1,2,5)$, $B(-1,2,-2)$ and $C(0,5,2)$.

Solution

On a **Notes** page, assign $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ as row vectors as follows:

- Press menu > **Insert** > **Maths Box**.
- Press ctrl [:=] to access the **Assign** [:=] command.
- Press [templates] 5, select the **m-by-n Matrix** template, fix the dimensions as 1-by-3 and enter as shown.



***Note:** Alternatively, to insert a **Maths Box**, press* ctrl M.

- Press ctrl M and enter $\underset{\sim}{b}-\underset{\sim}{a}$.
- Press ctrl M and enter $\underset{\sim}{c}-\underset{\sim}{a}$.

To display an equals sign in a **Maths Box**:

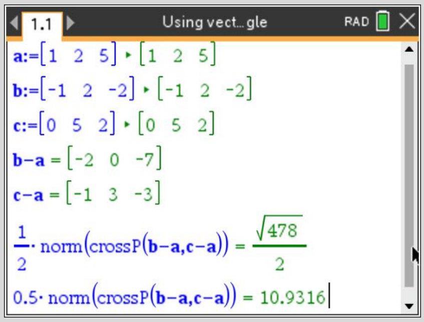
- Click on the **Maths Box**.
- Press menu > **Maths Box Options** > **Maths Box Attributes**.
- Press tab to highlight the **Insert Symbol** field.
- Press ▸ and select $=$.

***Note:** **Maths Box Attributes** can also be accessed within a **Maths Box** by pressing* ctrl menu.

Let A be the area of the triangle where $A=\frac{1}{2}\left|\left(\underset{\sim}{b}-\underset{\sim}{a}\right)\times\left(\underset{\sim}{c}-\underset{\sim}{a}\right)\right|$.

Enter this formula as follows:

- Press ctrl ÷ to access the **Fraction** template.
- Press menu > **Calculations** > **Matrix & Vector** > **Norms** > **Norm**.
- Press menu > **Calculations** > **Matrix & Vector** > **Vector** > **Cross Product**.
- Press menu > **Maths Box Options** > **Maths Box Attributes**, press tab to highlight the **Insert Symbol** field, press ▸ and select $=$.



***Note:** There are many ways to express this area in decimal form. The easiest way is to change* $\frac{1}{2}$ *to* 0.5 *as shown.*

Answer: $A=\frac{\sqrt{478}}{2}$

… continued

Solution (continued)

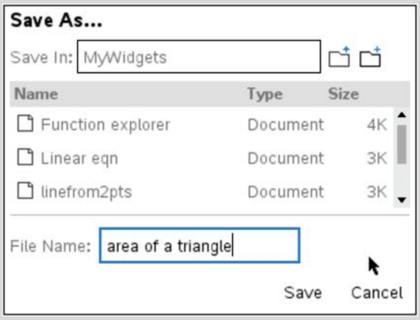
$$\begin{aligned}\overrightarrow{AB} &= \underset{\sim}{b} - \underset{\sim}{a} \\ &= -\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k} - \left(\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k}\right) \\ &= -2\underset{\sim}{i} - 7\underset{\sim}{k}\end{aligned}$$

$$\begin{aligned}\overrightarrow{AC} &= \underset{\sim}{c} - \underset{\sim}{a} \\ &= 5\underset{\sim}{j} + 2\underset{\sim}{k} - \left(\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k}\right) \\ &= -\underset{\sim}{i} + 3\underset{\sim}{j} - 3\underset{\sim}{k}\end{aligned}$$

$$\begin{aligned}A &= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| \\ &= \frac{1}{2}\left|\left(-2\underset{\sim}{i} - 7\underset{\sim}{k}\right) \times \left(-\underset{\sim}{i} + 3\underset{\sim}{j} - 3\underset{\sim}{k}\right)\right| \\ &= \frac{1}{2}\left|21\underset{\sim}{i} + \underset{\sim}{j} - 6\underset{\sim}{k}\right| \\ &= \frac{\sqrt{478}}{2}\end{aligned}$$

Note: *A **Notes** page such as this one that calculates the area of a triangle can be saved as a widget. As a widget, this page can be accessed and used to solve similar area of a triangle problems.*

To save the page as a widget, press doc▾ ***> File > Save As > MyWidgets****. Press* tab *to highlight **Save** and press* enter*. Ensure that you give the widget a name such as 'area of a triangle'. To access the widget, press* doc▾ ***> Insert > Widget****, select the widget, press* tab *to highlight **Add** and press* enter*.*



Widgets can also be constructed to determine the angle between two vectors and associated applications such as the angle between two lines, the distance between two skew lines, the angle between two planes and the angle between a line and a plane.

Determining Cartesian equations of curves from vector equations

Question

Find the Cartesian equation for the curve represented by the vector equation $\underset{\sim}{r}(t) = \cos^2(t)\underset{\sim}{i} + \sin^2(t)\underset{\sim}{j}$, where $t \in \mathbb{R}$.

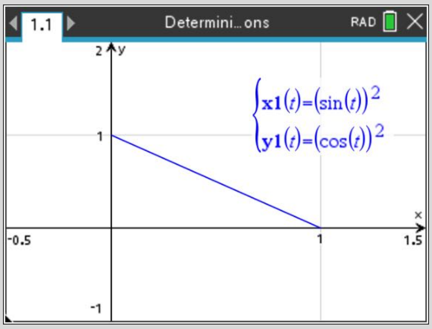
State the domain and range of the Cartesian relation.

Solution

Use parametric graphing mode to plot the curve.

On a **Graphs** page:

- Press menu > **Graph Entry/Edit** > **Parametric**.
- Enter $\boldsymbol{x1(t) = \cos(t)^2}$ and $\boldsymbol{y1(t) = \sin(t)^2}$.
- To add a grid, press menu > **View** > **Grid** > **Lined Grid**.
- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –0.5 XMax = 1.5 XScale = 1
  YMin = –1 YMax = 2 YScale = 1



***Note:** To plot this curve, there is no need to change the settings for t in the parametric graphing entry line.*

The curve appears to have Cartesian equation $y = 1 - x$ for $0 \le x \le 1$.

***Note:** To confirm this Cartesian equation, return to function graphing mode and plot* $\boldsymbol{f1(x) = 1 - x \mid 0 \le x \le 1}$.

Let (x, y) be any point on the curve defined by $\underset{\sim}{r}(t)$.

The parametric equations are:

$x = \cos^2(t)$ (1) and $y = \sin^2(t)$ (2)

$$\begin{aligned} y &= \sin^2(t) \\ &= 1 - \cos^2(t) \\ &= 1 - x \end{aligned}$$

As $0 \le \cos^2(t) \le 1$ for $t \in \mathbb{R}$, the domain is $[0,1]$.

As $0 \le \sin^2(t) \le 1$ for $t \in \mathbb{R}$, the range is $[0,1]$.

Answer: $y = 1 - x$ where $0 \le x \le 1$ and $0 \le y \le 1$.

Finding where two lines intersect

A line, l, in three-dimensional space can be described by $\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{d}$ where $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ is the position vector of a point A on l, $\underset{\sim}{d} = d_1\underset{\sim}{i} + d_2\underset{\sim}{j} + d_3\underset{\sim}{k}$ is a vector parallel to l and $t \in \mathbb{R}$.

Given position vectors, $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$, of two points on a line, l, then,

$$\underset{\sim}{r} = \underset{\sim}{a} + t\left(\underset{\sim}{b} - \underset{\sim}{a}\right) \text{ where } \overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a} \text{ and } t \in \mathbb{R}.$$

Alternatively,

$$\underset{\sim}{r} = (1-t)\underset{\sim}{a} + t\underset{\sim}{b} \text{ where } t \in \mathbb{R}.$$

The parametric equations of a line, l, are given by

$$x = a_1 + td_1,\ y = a_2 + td_2,\ z = a_3 + td_3 \text{ where } t \in \mathbb{R}.$$

The Cartesian equation of a line, l, is given by

$$\frac{x-a_1}{d_1} = \frac{y-a_2}{d_2} = \frac{z-a_3}{d_3}.$$

Two lines $l_1 : \underset{\sim}{r}_1(t) = \underset{\sim}{a}_1 + t\underset{\sim}{d}_1$ and $l_2 : \underset{\sim}{r}_2(s) = \underset{\sim}{a}_2 + s\underset{\sim}{d}_2$ intersect if there exists $t, s \in \mathbb{R}$ such that $\underset{\sim}{r}_1(t) = \underset{\sim}{r}_2(s)$.

Question

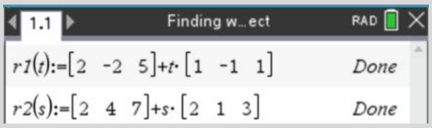
Find the position vector of the point of intersection of the lines

$\underset{\sim}{r}_1(t) = 2\underset{\sim}{i} - 2\underset{\sim}{j} + 5\underset{\sim}{k} + t\left(\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right)$ and $\underset{\sim}{r}_2(s) = 2\underset{\sim}{i} + 4\underset{\sim}{j} + 7\underset{\sim}{k} + s\left(2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right)$ where $t, s \in \mathbb{R}$.

Solution

On a **Calculator** page, assign $\underset{\sim}{r}_1(t)$ and $\underset{\sim}{r}_2(s)$ as row vectors as follows:

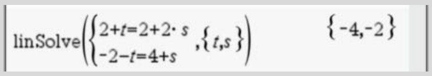
- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **m-by-n Matrix** template, fix the dimensions as 1-by-3 and enter as shown.



By substitution, the system of linear equations formed are:

$$2 + t = 2 + 2s \quad (1)$$
$$-2 - t = 4 + s \quad (2)$$
$$5 + t = 7 + 3s \quad (3)$$

- Press [menu] > **Algebra** > **Solve System of Equations** > **Solve System of Linear Equations**.
- In the dialog box that follows:
  - For **Number of equations**, enter **2**.
  - For **Variables**, enter ***t*,*s***.
- For example, enter equations (1) and (2) into the template and press [enter].



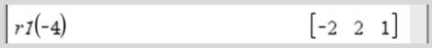
... continued

Solution (continued)

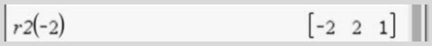
$(1)+(2)$ gives $0=6+3s \Rightarrow s=-2$.

Substituting into one of the other equations and solving for t gives $t=-4$.

Checking $t=-4$ and $s=-2$:



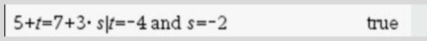
$$\underset{\sim}{r}_1(-4)=2\underset{\sim}{i}-2\underset{\sim}{j}+5\underset{\sim}{k}-4\left(\underset{\sim}{i}-\underset{\sim}{j}+\underset{\sim}{k}\right)$$
$$=-2\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k}$$



$$\underset{\sim}{r}_2(-2)=2\underset{\sim}{i}+4\underset{\sim}{j}+7\underset{\sim}{k}-2\left(2\underset{\sim}{i}+\underset{\sim}{j}+3\underset{\sim}{k}\right)$$
$$=-2\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k}$$

Answer: The point of intersection of the lines has the position vector $-2\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k}$. (The lines intersect at $(-2,2,1)$.)

To show that $t=-4$ and $s=-2$ satisfy equation (3):



- Enter equation (3).

Press [ctrl] [=] to access the 'with' or 'given' symbol |.

Analysing two skew lines

Two lines are skew lines if they do not intersect and are not parallel.

A vector $\overrightarrow{PQ}$ joins the two lines and is perpendicular to both lines.

The distance between the two lines is defined as $\left|\overrightarrow{PQ}\right|$.

Given two skew lines, $l_1 : \underset{\sim}{r}_1(s) = \underset{\sim}{a}_1 + s\underset{\sim}{d}_1$ and $l_2 : \underset{\sim}{r}_2(t) = \underset{\sim}{a}_2 + t\underset{\sim}{d}_2$, where $s, t \in \mathbb{R}$, the distance between them is given by

$$d = \left|\left(\underset{\sim}{a}_2 - \underset{\sim}{a}_1\right)\cdot \hat{\underset{\sim}{n}}\right| \text{ where } \hat{\underset{\sim}{n}} \text{ is a unit vector parallel to } \overrightarrow{PQ} \text{ and } \hat{\underset{\sim}{n}} = \frac{\underset{\sim}{d}_1 \times \underset{\sim}{d}_2}{\left|\underset{\sim}{d}_1 \times \underset{\sim}{d}_2\right|}.$$

Question

Consider the lines $l_1 : \underset{\sim}{r}_1(s) = \underset{\sim}{i} + \underset{\sim}{j} + s\left(2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right)$ and $l_2 : \underset{\sim}{r}_2(t) = 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k} + t\left(3\underset{\sim}{i} - 5\underset{\sim}{j} + 2\underset{\sim}{k}\right)$ where $s, t \in \mathbb{R}$.

(a) Show that l_1 and l_2 are skew lines.

(b) Find the distance between l_1 and l_2.

Solution

(a) The lines are not parallel since
$2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k} \neq \lambda\left(3\underset{\sim}{i} - 5\underset{\sim}{j} + 2\underset{\sim}{k}\right),\ \forall \lambda \in \mathbb{R}$.

Two skew lines do not intersect.

By substitution, the system of linear equations formed is:

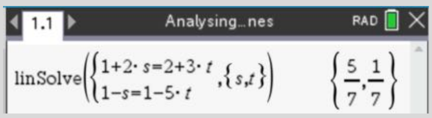
$1 + 2s = 2 + 3t \quad (1)$

$1 - s = 1 - 5t \quad (2)$

$s = -1 + 2t \quad (3)$

On a **Calculator** page:

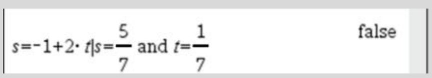
- Press [menu] > **Algebra > Solve System of Equations > Solve System of Linear Equations**.
- In the dialog box that follows:
  - For **Number of equations**, enter **2**.
  - For **Variables**, enter ***s,t***.
- For example, enter equations (1) and (2) into the template and press [enter].



From (1) and (2), $s = \frac{5}{7}, t = \frac{1}{7}$.

To show that $s = \frac{5}{7}$ and $t = \frac{1}{7}$ do not satisfy equation (3):

- Enter equation (3).



Press [ctrl] [=] to access the 'with' or 'given' symbol |.

... *continued*

Solution (continued)

Answer: As $s = \dfrac{5}{7}$ and $t = \dfrac{1}{7}$ do not satisfy equation (3), there are no $s, t \in \mathbb{R}$ such that $\underset{\sim}{r}_1(s) = \underset{\sim}{r}_2(t)$. The lines are not parallel and do not intersect. Hence l_1 and l_2 are skew lines.

(b) Comparing $\underset{\sim}{r}_1(s) = \underset{\sim}{a}_1 + s\underset{\sim}{d}_1$ and $\underset{\sim}{r}_2(t) = \underset{\sim}{a}_2 + t\underset{\sim}{d}_2$, with

$\underset{\sim}{r}_1(s) = \underset{\sim}{i} + \underset{\sim}{j} + s\left(2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right)$ and

$\underset{\sim}{r}_2(t) = 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k} + t\left(3\underset{\sim}{i} - 5\underset{\sim}{j} + 2\underset{\sim}{k}\right)$ where $s, t \in \mathbb{R}$:

$\underset{\sim}{a}_1 = \underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{d}_1 = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$,

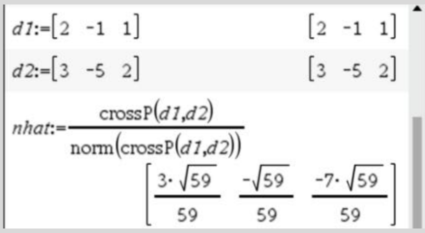
$\underset{\sim}{a}_2 = 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{d}_2 = 3\underset{\sim}{i} - 5\underset{\sim}{j} + 2\underset{\sim}{k}$.

P is a point on l_1 and Q is a point on l_2 such that $\left|\overrightarrow{PQ}\right|$ is the distance between l_1 and l_2.

As $\overrightarrow{PQ}$ is $\perp$ to l_1 and l_2, it is $\parallel$ to $\underset{\sim}{d}_1 \times \underset{\sim}{d}_2$.

On a **Calculator** page, assign $\underset{\sim}{d}_1$, $\underset{\sim}{d}_2$ and $\hat{\underset{\sim}{n}}$ as row vectors as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **m-by-n Matrix** template, fix the dimensions as 1-by-3 and enter as shown.



To determine $\hat{\underset{\sim}{n}} = \dfrac{\underset{\sim}{d}_1 \times \underset{\sim}{d}_2}{\left|\underset{\sim}{d}_1 \times \underset{\sim}{d}_2\right|}$:

- Press [ctrl] [÷] to access the **Fraction** template.
- Press [menu] **> Matrix & Vector > Vector > Cross Product**.
- Press [var] to access assigned/stored variables.
- Enter the numerator for $\hat{\underset{\sim}{n}}$ as shown.
- Press [menu] **> Matrix & Vector > Norms > Norm**.
- Enter the denominator for $\hat{\underset{\sim}{n}}$ as shown.

$$\hat{\underset{\sim}{n}} = \frac{1}{\sqrt{59}}\left(3\underset{\sim}{i} - \underset{\sim}{j} - 7\underset{\sim}{k}\right)$$

To determine $d = \left|\left(\underset{\sim}{a}_2 - \underset{\sim}{a}_1\right) \cdot \hat{\underset{\sim}{n}}\right|$ on a **Calculator** page:

- Assign $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$ as row vectors.
- Press [catalog] [1] [A] and select **abs(**.
- Press [menu] **> Matrix & Vector > Vector > Dot Product**.
- Press [var] to access assigned/stored variables.

Answer: $d = \left|\left(\underset{\sim}{i} - \underset{\sim}{k}\right) \cdot \left(\dfrac{1}{\sqrt{59}}\left(3\underset{\sim}{i} - \underset{\sim}{j} - 7\underset{\sim}{k}\right)\right)\right| = \dfrac{10}{\sqrt{59}}$

Finding and plotting the Cartesian equations of planes

The vector equation of a plane is given by

$$\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}.$$

The Cartesian equation of a plane is given by

$$ax + by + cz + d = 0.$$

Question

The plane Π contains the points $A(0,1,1)$, $B(-2,0,3)$ and $C(2,1,0)$.

(a) Find a Cartesian equation for the plane Π.

(b) Use the **3D Graphing** feature to plot the plane Π.

(c) State the axis intercepts of Π.

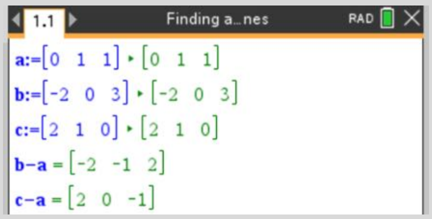
Solution

(a) On a **Notes** page, assign $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ as row vectors as follows:

- Press [menu] > **Insert** > **Maths Box**.
- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [book] [5], select the **m-by-n Matrix** template, fix the dimensions as 1-by-3 and enter as shown.

Note: *Alternatively, to insert a* ***Maths Box****, press* [ctrl] [M].

- Press [ctrl] [M] and enter $\boldsymbol{b - a}$.
- Press [ctrl] [M] and enter $\boldsymbol{c - a}$.



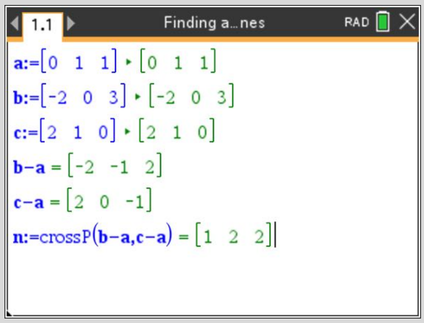
To display an equals sign in a **Maths Box**:

- Click on the **Maths Box**.
- Press [menu] > **Maths Box Options** > **Maths Box Attributes**.
- Press [tab] to highlight the **Insert Symbol** field.
- Press ▶ and select =.

Note: Maths Box Attributes *can also be accessed within a* ***Maths Box*** *by pressing* [ctrl] [menu].

Assign $\underset{\sim}{n}$, a vector normal to Π, as follows:

- Press [ctrl] [M] and enter $\boldsymbol{n}$.
- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [menu] > **Calculations** > **Matrix & Vector** > **Vector** > **Cross Product**.
- Enter as shown.
- Press [menu] > **Maths Box Options** > **Maths Box Attributes**, press [tab] to highlight the **Insert Symbol** field, press ▶ and select =.

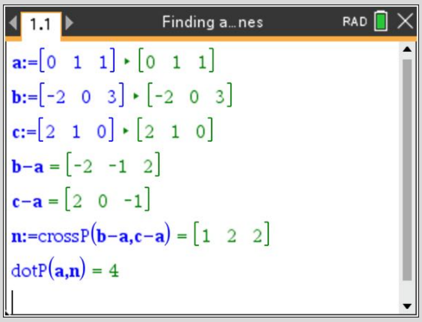


... continued

Solution (continued)

To calculate $\underset{\sim}{a} \cdot \underset{\sim}{n}$ enter as follows:

- Press ctrl M.
- Press menu > **Calculations > Matrix & Vector > Vector > Dot Product**.
- Press menu > **Maths Box Options > Maths Box Attributes**, press tab to highlight the **Insert Symbol** field, press ▶ and select =.



Answer: A Cartesian equation for Π is $x+2y+2z=4$.

$\underset{\sim}{b}-\underset{\sim}{a}=-2\underset{\sim}{i}-\underset{\sim}{j}+2\underset{\sim}{k}$ and $\underset{\sim}{c}-\underset{\sim}{a}=2\underset{\sim}{i}-\underset{\sim}{k}$

$(\underset{\sim}{b}-\underset{\sim}{a})\times(\underset{\sim}{c}-\underset{\sim}{a})=\underset{\sim}{i}+2\underset{\sim}{j}+2\underset{\sim}{k}$

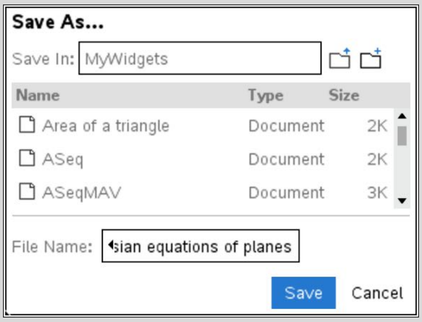
So $\underset{\sim}{n}=\underset{\sim}{i}+2\underset{\sim}{j}+2\underset{\sim}{k}$ is a vector normal to Π.

$\underset{\sim}{a}\cdot\underset{\sim}{n}=(\underset{\sim}{j}+\underset{\sim}{k})\cdot(\underset{\sim}{i}+2\underset{\sim}{j}+2\underset{\sim}{k})=4$

Using $\underset{\sim}{r}\cdot\underset{\sim}{n}=\underset{\sim}{a}\cdot\underset{\sim}{n}$, a Cartesian equation for Π is $x+2y+2z=4$.

Note: *A* ***Notes*** *page such as this one that calculates the Cartesian equation of a plane can be saved as a widget. As a widget, this page can be accessed and used to solve similar problems involving planes.*

To save the page as a widget, press doc▾ ***> File > Save As > MyWidgets****. Press tab to highlight* ***Save*** *and press enter. Ensure that you give the widget a name such as 'Cartesian equations of planes'. To access the widget, press doc▾ >* ***Insert > Widget****, select the widget, press tab to highlight* ***Add*** *and press enter.*

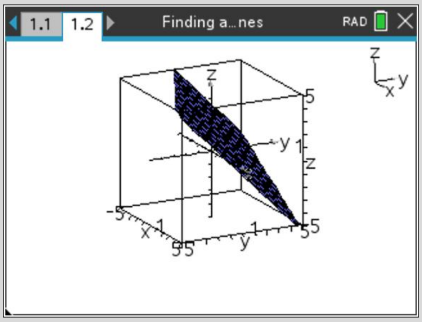


(b) Planes can be plotted using 3D graphing.

On a **Graphs** page:

- Press menu > **View > 3D Graphing**.
- Enter $\boldsymbol{z1(x,y)=\dfrac{4-x-2y}{2}}$.

Answer: Screenshot at right shows the plot.



Note: *Press menu* ***> Actions*** *or* ***View*** *or* ***Range/Zoom*** *to explore a suite of viewing alternatives including the range settings. To rotate the view of the plane, for example, press menu* ***> Actions > Rotate*** *(or press R) and then use the arrow keys.*

(c) Use $x+2y+2z=4$ to state where Π crosses the axes.

Answer: The axis intercepts of Π are $(4,0,0)$, $(0,2,0)$ and $(0,0,2)$.

Showing three planes that do not intersect at a unique point

Question

Consider the planes Π_1, Π_2 and Π_3 defined by the following Cartesian equations:

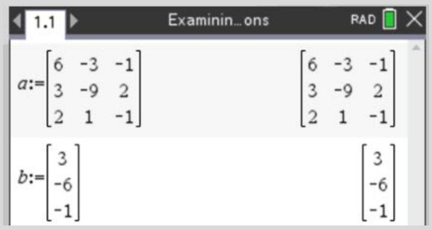
$$\Pi_1 : 6x - 3y - z = 3$$
$$\Pi_2 : 3x - 9y + 2z = -6$$
$$\Pi_3 : 2x + y - z = -1$$

(a) Use the **Reduced Row-Echelon Form** (**rref**) command to show that these three planes do not intersect at a unique point.

(b) Use the **3D Graphing** feature to plot the three planes.

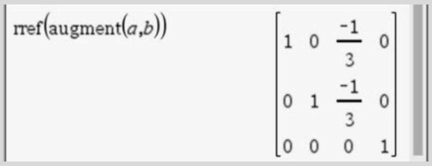
Solution

(a) In matrix form , the system of equations can be expressed as $AX = B$ where $A = \begin{bmatrix} 6 & -3 & -1 \\ 3 & -9 & 2 \\ 2 & 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ -6 \\ -1 \end{bmatrix}$.



On a **Calculator** page, assign A and B as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **m-by-n Matrix** template, fix the dimensions as 3-by-3 and enter as shown.
- Press [catalog] [5], select the **m-by-n Matrix** template, fix the dimensions as 3-by-1 and enter as shown.



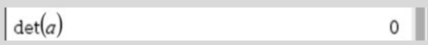
Attempt to solve $AX = B$ using reduced row-echelon form as follows:

- Press [menu] > **Matrix & Vector** > **Reduced Row-Echelon Form**.
- Press [menu] > **Matrix & Vector** > **Create** > **Augment**.
- Enter as shown.

Answer: The third row of the augmented matrix,

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

can be interpreted as the equation $0x + 0y + 0z = 1$.



This system of equations has no solutions and hence these three planes do not intersect at a unique point.

Note: *If* $\det(A) = 0$*, then the planes do not meet at a unique point.*

... continued

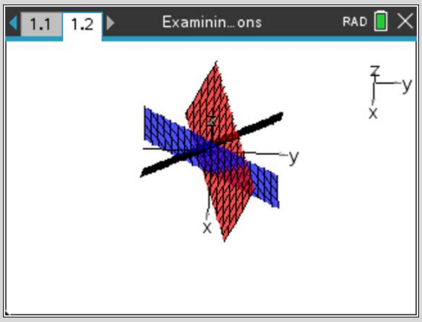
Solution (continued)

(b) Planes can be plotted using the **3D Graphing** feature.

On a **Graphs** page:

- Press menu > **View** > **3D Graphing**.
- Enter $z1(x, y) = -3 + 6x - 3y$.
- Enter $z2(x, y) = -3 - \frac{3x}{2} + \frac{9y}{2}$.
- Enter $z3(x, y) = 1 + 2x + y$.
- Press menu > **Range/Zoom** > **Range Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –5 XMax = 5 XScale = 5
  YMin = –5 YMax = 5 YScale = 5
  ZMin = –5 ZMax = 5 ZScale = 5

Note: *Press* menu *> **Actions** or **View** or **Range/Zoom** to explore a suite of viewing alternatives. To rotate the view of the planes, for example, press* menu *> **Actions** > **Rotate** (or press* R*) and then use the arrow keys.*



For example, for the screenshot shown above right:

- Press menu > **View** > **Hide Box**.

Answer: The three planes do not intersect at a unique point. The three normals are coplanar but not parallel.

Finding the line of intersection of two planes

Two planes that are not parallel will intersect in a line.

Question

Consider the planes Π_1 and Π_2 defined by the following Cartesian equations:

$$\Pi_1 : 2x + 4y - z = 4$$

$$\Pi_2 : x - 2y + z = 3$$

(a) Find a vector equation of the line of intersection of the planes.

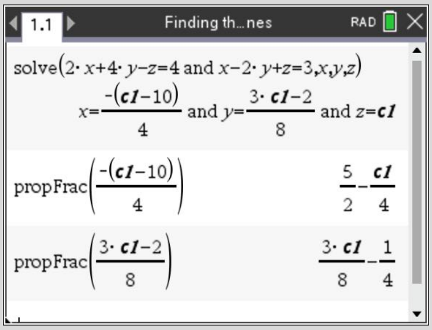
(b) Use the **3D Graphing** feature to plot the two planes.

Solution

(a) There are two equations and three unknowns.

To find the parametric solution to the system of linear equations on a **Calculator** page:

- Press menu > **Algebra** > **Solve**.
- Enter as shown.
- Press menu > **Algebra** > **Fraction Tools** > **Proper Fraction**.



The parametric equations for the line of intersection are

$$x = \frac{5}{2} - \frac{1}{4}\lambda,\ y = \frac{3}{8}\lambda - \frac{1}{4} \text{ and } z = \lambda .$$

... continued

Solution (continued)

Represent these parametric equations as a vector equation**.**

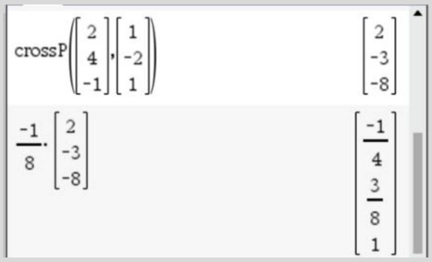
Answer: $\underset{\sim}{r} = \frac{5}{2}\underset{\sim}{i} - \frac{1}{4}\underset{\sim}{j} + \lambda\left(-\frac{1}{4}\underset{\sim}{i} + \frac{3}{8}\underset{\sim}{j} + \underset{\sim}{k}\right)$ where $\lambda \in \mathbb{R}$

Note: *Here TI-Nspire CX II CAS has set* $z = \mathbf{c1}$ *where* $\mathbf{c1} = \lambda$. *If either* $x = \lambda$ *or* $y = \lambda$ *are used, a set of different yet correct parametric equations are generated.*

Alternatively, when $\lambda = 0$, $A\left(\frac{5}{2}, -\frac{1}{4}, 0\right)$ lies on the line.

A direction vector, $\underset{\sim}{d} = \underset{\sim}{n}_1 \times \underset{\sim}{n}_2$, where $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$ are normals to Π_1 and Π_2 respectively can be found as follows:

- Press menu > **Matrix & Vector** > **Vector** > **Cross Product**.
- Press [templates] 5, select the **m-by-n Matrix** template, fix the dimensions as 3-by-1 and enter as shown.
- Multiply each element $\underset{\sim}{n}_1 \times \underset{\sim}{n}_2$ by $-\frac{1}{8}$.



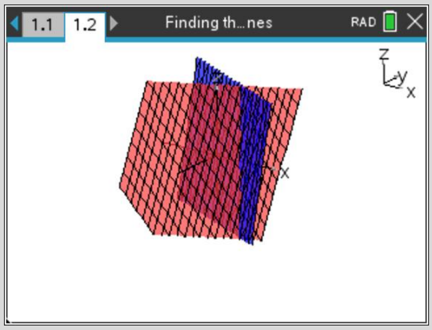
So $\underset{\sim}{r} = \frac{5}{2}\underset{\sim}{i} - \frac{1}{4}\underset{\sim}{j} + \lambda\left(-\frac{1}{4}\underset{\sim}{i} + \frac{3}{8}\underset{\sim}{j} + \underset{\sim}{k}\right)$ where $\lambda \in \mathbb{R}$.

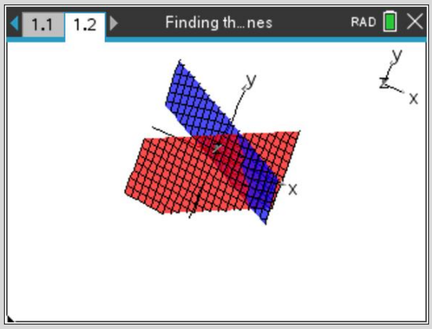
(b) Planes can be plotted using the **3D Graphing** feature.

On a **Graphs** page:

- Press menu > **View** > **3D Graphing**.
- Enter $z1(x, y) = -4 + 2x + 4y$.
- Enter $z2(x, y) = 3 - x + 2y$.
- Press menu > **Range/Zoom** > **Range Settings**.
  In the dialog box that follows, enter the following values:

XMin = –5	XMax = 5	XScale = 5
YMin = –5	YMax = 5	YScale = 5
ZMin = –5	ZMax = 5	ZScale = 5





Note: *Press* menu *>* ***Actions*** *or* ***View*** *or* ***Range/Zoom*** *to explore a suite of viewing alternatives. To rotate the view of the planes, for example, press* menu *>* ***Actions*** *>* ***Rotate*** *(or press* R*) and then use the arrow keys.*

For example, for the screenshots shown above right:

- Press menu > **View** > **Hide Box**.

Answer: The two screenshots above right show a plot of the two planes.

Analysing three cases of three planes

Question

The Cartesian equations of three planes are given by:

$$dx + 2y + z = 3$$
$$-x + (d+1)y + 3z = 1$$
$$-2x + y + (d+2)z = k$$

where $d, k \in \mathbb{R}$.

(a) Given that $d = 0$, show that the three planes intersect at a point.

(b) Find the value of d such that the three planes do not meet at a point.

(c) For the value of d such that the three planes do not meet at a point, find the value of k such that the planes meet in a line and find a vector equation of this line.

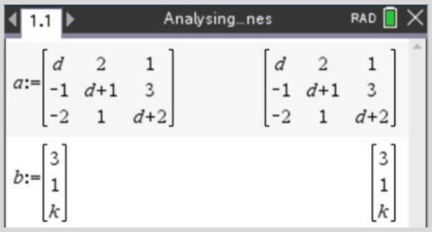
Solution

(a) In matrix form , the system of equations can be expressed as $AX = B$ where

$$A = \begin{bmatrix} d & 2 & 1 \\ -1 & d+1 & 3 \\ -2 & 1 & d+2 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 1 \\ k \end{bmatrix}.$$

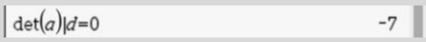
On a **Calculator** page, assign A and B as follows:

- Press [ctrl] [template] to access the **Assign** [:=] command.
- Press [book] [5], select the **m-by-n Matrix** template, fix the dimensions as 3-by-3 and enter as shown.
- Press [book] [5], select the **m-by-n Matrix** template, fix the dimensions as 3-by-1 and enter as shown.



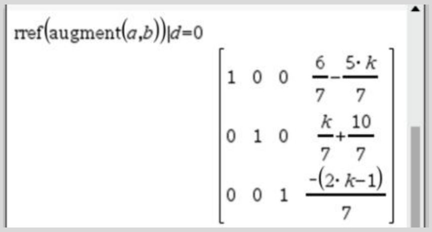
To calculate $\det(A)$ with $d = 0$:

- Press [menu] > **Matrix & Vector** > **Determinant**.
- Press [ctrl] [=] to access the 'with' or 'given' symbol |.
- Enter as shown.



Answer: Since $\det(A) = -7 (\neq 0)$, there is a unique solution to the system of linear equations and so the planes intersect at a point. Alternatively, solve $AX = B$ using reduced row-echelon form as follows:

- Press [menu] > **Matrix & Vector** > **Reduced Row-Echelon Form**.
- Press [menu] > **Matrix & Vector** > **Create** > **Augment**.
- Press [ctrl] [=] to access the 'with' or 'given' symbol |.
- Enter as shown.



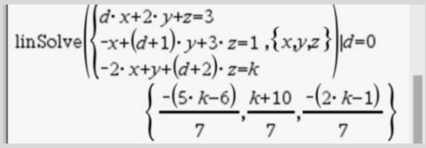
Answer: Using row reduction, the unique solution for a particular value of k is $\left(\dfrac{6-5k}{7}, \dfrac{10+k}{7}, \dfrac{1-2k}{7}\right)$. Hence the three planes intersect at a point.

... continued

Solution (continued)

Alternatively:

- Press [menu] > **Algebra > Solve System of Equations > Solve System of Linear Equations**.
- In the dialog box that follows:
  - For **Number of equations**, enter **3**.
  - For **Variables**, enter ***x*,*y*,*z***.
- Complete the template as shown.
- Press [ctrl] [=] to access the 'with' or 'given' symbol |.



Answer: The unique solution for a particular value of k is $\left(\frac{6-5k}{7},\frac{10+k}{7},\frac{1-2k}{7}\right)$. Hence the three planes intersect at a point.

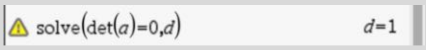
(b) If $\det(A)=0$, then the planes do not meet at a point.

To calculate $\det(A)$:

- Press [menu] > **Matrix & Vector > Determinant**.
- Enter as shown.

To solve $\det(A)=0$ for d:

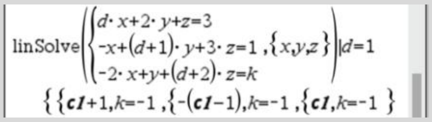
- Press [menu] > **Algebra > Solve**.
- Enter as shown.



Answer: The three planes do not meet at a point when $d=1$.

(c) Consider $d=1$ and solve the system of linear equations as follows:

- Press [menu] > **Algebra > Solve System of Equations > Solve System of Linear Equations**.
- In the dialog box that follows:
  - For **Number of equations**, enter **3**.
  - For **Variables**, enter ***x*,*y*,*z***.
- Complete the template as shown.
- Press [ctrl] [=] to access the 'with' or 'given' symbol |.



Answer: For an infinite number of solutions to exist, $k=-1$. So the planes meet in a line when $k=-1$. A vector equation of this line is $\underset{\sim}{r}=\underset{\sim}{i}+\underset{\sim}{j}+\lambda\left(\underset{\sim}{i}-\underset{\sim}{j}+\underset{\sim}{k}\right)$ where $\lambda\in\mathbb{R}$.

Note: *Using row reduction, the system of linear equations can be represented as the augmented matrix:*

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 4 & 4 & 4 \\ 0 & 0 & 0 & 4+4k \end{array}\right]$$

For an infinite number of solutions to exist, $4+4k=0 \Rightarrow k=-1$.

... continued

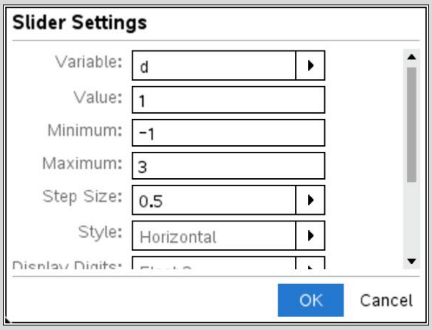
Solution (continued)

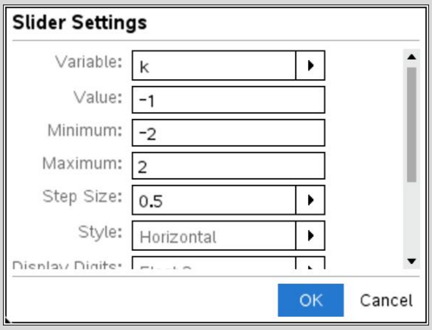
On a **Graphs** page, plot the three planes:

- Press [menu] > **View** > **3D Graphing**.
- Enter $z1(x, y) = 3 - d \cdot x - 2y$.
- Enter $z2(x, y) = \dfrac{1 + x - (d + 1) \cdot y}{3}$.
- Enter $z3(x, y) = \dfrac{k + 2x - y}{d + 2}$.
- Press [menu] > **Range/Zoom** > **Range Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –5 XMax = 5 XScale = 5
  YMin = –5 YMax = 5 YScale = 5
  ZMin = –5 ZMax = 5 ZScale = 5

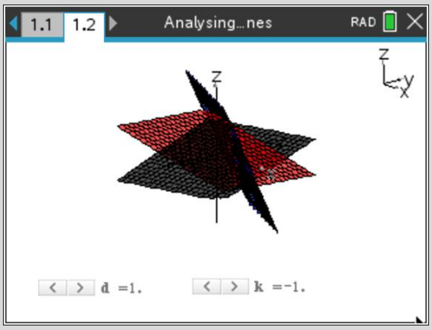
Insert a slider to control the value of *d* and a slider to control the value of *k* as follows:

- Press [menu] > **Actions** > **Insert Slider**.
- Set the **Slider Settings** as shown.
- Click to check the **Minimised** box.





***Note:** Press [menu] > **Actions** or **View** or **Range/Zoom** to explore a suite of viewing alternatives. To rotate the view of the planes, for example, press [menu] > **Actions** > **Rotate** (or press [R]) and then use the arrow keys.*



For example, for the screenshot shown above right with $d = 1$ and $k = -1$:

Press [menu] > **View** > **Hide Box**.

3.5.3 Vector calculus

Finding the Cartesian equation of a particle's path

Given $\underset{\sim}{r}(t)$, the corresponding path of a particle can be found and sketched. Paths include circles, ellipses and hyperbolas in Cartesian or parametric forms (see section 2.4.1).

The equation of a circle is given by:

- $(x-h)^2+(y-k)^2=r^2$

The equation of an ellipse is given by:

- $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$

The equation of a hyperbola is given by:

- $\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1$ or $\dfrac{(y-k)^2}{a^2}-\dfrac{(x-h)^2}{b^2}=1$

Question

Find the Cartesian equation of the path of a particle which moves such that its position vector at time t is given by

$$\underset{\sim}{r}(t)=(1-2\cos(t))\underset{\sim}{i}+3\sin(t)\underset{\sim}{j} \text{ where } t\geq 0.$$

(a) Plot the path of the particle.

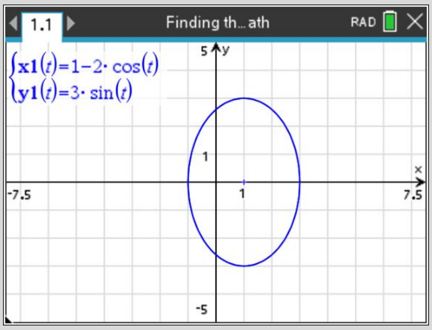
(b) Describe the motion of the body.

Solution

(a) Use parametric graphing mode to plot the path.

On a **Graphs** page:

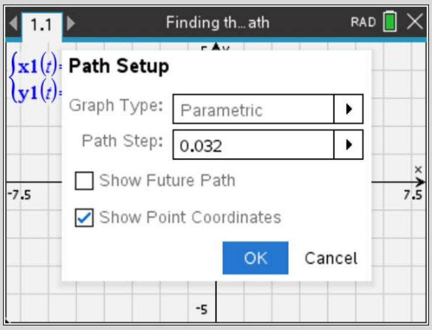
- Press menu > **Graph Entry/Edit** > **Parametric**.
- Enter $\mathbf{x1}(t)=1-2\cos(t)$ and $\mathbf{y1}(t)=3\sin(t)$.
- Enter $0\leq t\leq 6.28$ $tstep=0.032$.
- To add a grid, press menu > **View** > **Grid** > **Lined Grid**.
- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –7.5 XMax = 7.5 XScale = 1
  YMin = –5 YMax = 5 YScale = 1



Answer: Plot of the particle's path shown at right.

(b) To animate the particle's the path:

- Press menu > **Trace** > **Path Plot** > **Path Setup**.
- Complete the required fields as shown.



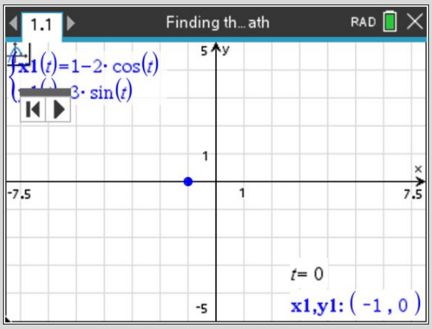
***Note**: If desired, the future path can be shown by checking the **Show Future Path** box.*

... continued

Solution (continued)

To set up the animation of the particle's motion:

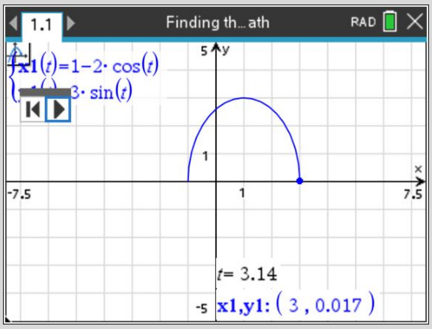
- Press menu > **Trace** > **Path Plot** > **Parametric**.



Note: The coordinates of the particle's path display at the bottom right of the screen. This needs to be accounted for when positioning labels and setting suitable viewing windows.

To start the animation:

- Move the cursor over the animation start button and press enter.



Note: To pause the animation, move the cursor over the animation pause button and press enter. To reset the animation, move the cursor over the animation reset button and press enter.

Answer: The particle starts at $(-1,0)$ and travels in a clockwise elliptical path with a period of $6.28(2\pi)$. The particle finishes its motion at $(-1,0)$. The ellipse has centre $(1,0)$ and domain $-1 \le x \le 3$.

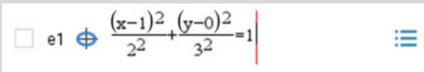
The parametric equations are:

$x = 1-2\cos(t)$ (1) and $y = 3\sin(t)$ (2)

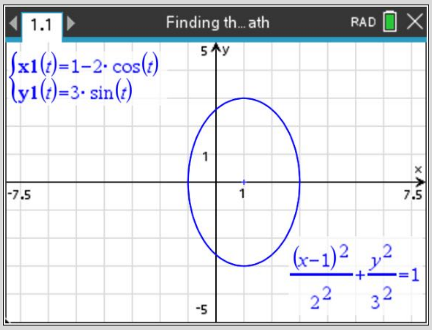
From (1), $\frac{x-1}{-2} = \cos(t)$ and from (2), $\frac{y}{3} = \sin(t)$.

Squaring and adding the above two equations gives:

$$\frac{(x-1)^2}{4} + \frac{y^2}{9} = 1.$$

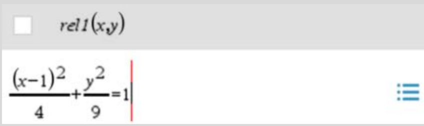


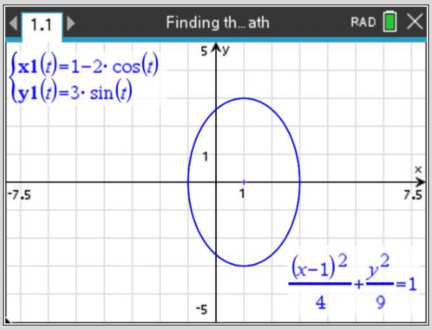
*Note: The use of **Equation Templates** to graph conics such as ellipses is shown in section 2.4.1. Press menu > **Graph Entry/Edit** > **Equation Templates** > **Ellipse** > **Centre form** $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Enter as shown.*



To plot the ellipse in relation graphing mode on a **Graphs** page:

- Press menu > **Graph Entry/Edit** > **Relation**.
- Enter as shown.





Finding when and where two particles meet

Two particles meet when they share the same position at the same time.

Question

The motion of two particles are given by the position vectors

$$\underset{\sim}{r}_1(t)=(2t-3)\underset{\sim}{i}+(t^2+10)\underset{\sim}{j} \text{ and } \underset{\sim}{r}_2(t)=(t+2)\underset{\sim}{i}+7t\underset{\sim}{j}\text{, where } t\geq 0\text{.}$$

(a) Find when the two particles meet.

(b) Determine the coordinates of their meeting point.

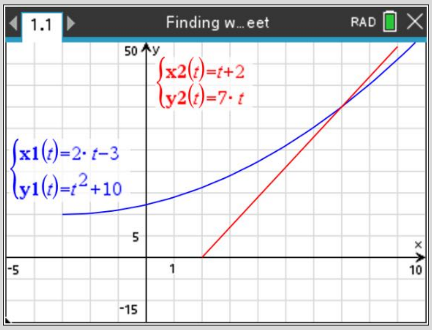
Solution

(a) and **(b)**.

Use parametric graphing mode to plot the two paths.

On a **Graphs** page:

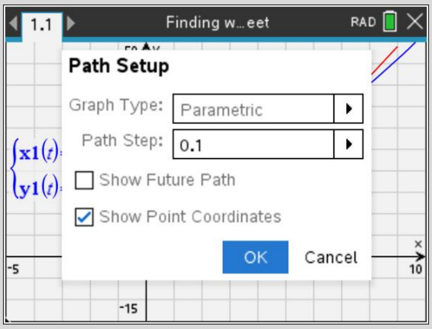
- Press menu > **Graph Entry/Edit** > **Parametric**.
- Enter $\boldsymbol{x1(t)=2t-3}$ and $\boldsymbol{y1(t)=t^2+10}$.
- Enter $\boldsymbol{0\leq t\leq 7}$ $\boldsymbol{tstep=0.1}$.
- Enter $\boldsymbol{x2(t)=t+2}$ and $\boldsymbol{y2(t)=7t}$.
- Enter $\boldsymbol{0\leq t\leq 7}$ $\boldsymbol{tstep=0.1}$.
- To add a grid, press menu > **View** > **Grid** > **Lined Grid**.
- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –5 XMax = 10 XScale = 1
  YMin = –15 YMax = 50 YScale = 5



To animate the particle's path:

- Press menu > **Trace** > **Path Plot** > **Path Setup**.
- Complete the required fields as shown.

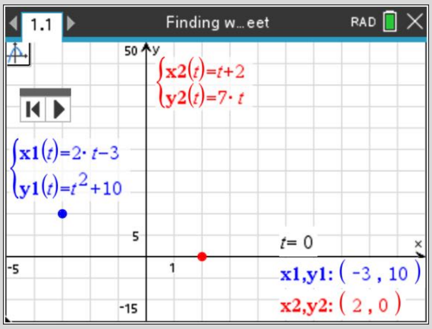
*Note: If desired, the future paths can be shown by checking the **Show Future Path** box.*



To set up the animation of the two particles' motion:

- Press menu > **Trace** > **Path Plot** > **Parametric**.

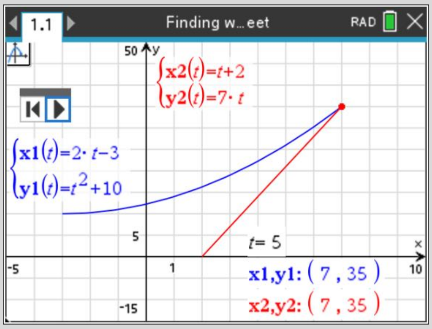
Note: The coordinates of the particles' paths display at the bottom right of the screen. This needs to be accounted for when positioning labels and setting suitable viewing windows.



... continued

Solution (continued)

To start the animation:

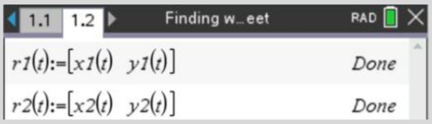


- Move the cursor over the animation start button and press [enter].

Note: To pause the animation, move the cursor over the animation pause button and press [enter]. To reset the animation, move the cursor over the animation reset button and press [enter].

Answer: The animation shows the two particles meeting when $t = 5$ at $(7,35)$.

Alternatively, on a **Calculator** page, assign $\underset{\sim}{r}_1(t)$ and $\underset{\sim}{r}_2(t)$ as row vectors as follows:



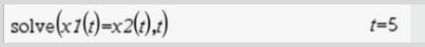
- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **1-by-2 Matrix** template and enter as shown.

*Note: The parametric equations, **x1(t)**, **y1(t)**, **x2(t)** and **y2(t)** defined on the **Graphs** page are recognised on the **Calculator** page provided these two pages form part of the same problem.*

$\underset{\sim}{r}_1(t) = \underset{\sim}{r}_2(t) \Rightarrow 2t - 3 = t + 2$ and $t^2 + 10 = 7t$

Equate the $\underset{\sim}{i}$ components and solve as shown:

- Press [menu] > **Algebra** > **Solve**.



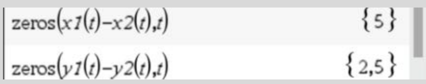
Solving $2t - 3 = t + 2$ for *t* gives $t = 5$.

Equate the $\underset{\sim}{j}$ components and solve as shown:

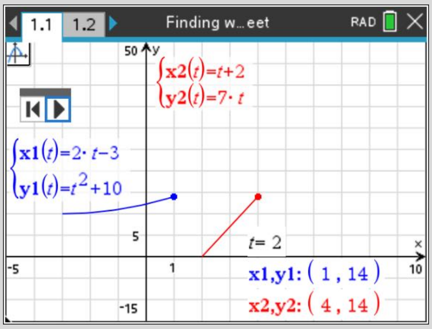
Solving $t^2 + 10 = 7t$ for *t* gives $t = 2, 5$.

Hence the two particles meet when $t = 5$.

*Note: Alternatively, press [menu] > **Algebra** > **Zeros** to use the **Zeros** command. Note that the equation is assumed set to zero.*



Note: The paths of the two particles do not cross at $t = 2$ and hence do not meet at this time. The row vectors and the animation show when $t = 2$, the particles are at $(1,14)$ and $(4,14)$.



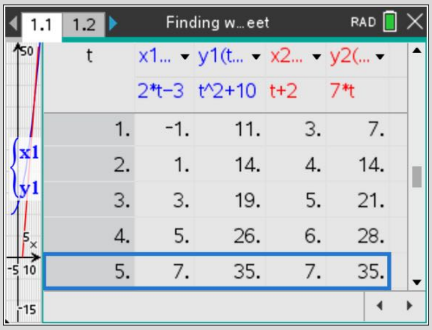
... continued

Solution (continued)

$\underset{\sim}{r}_1(5) = 7\underset{\sim}{i} + 35\underset{\sim}{j}$ and $\underset{\sim}{r}_2(5) = 7\underset{\sim}{i} + 35\underset{\sim}{j}$.

This confirms that the two particles meet at $(7,35)$.

***Note**: If desired, on the **Graphs** page, press ctrl T to view a tabular representation of the parametric equations. To change the width of a column, press menu > **Actions** > **Resize** > **Resize Column Width**. Use the ◀▶ arrows to adjust the width.*



Using vector calculus to analyse the motion of a particle

The position of a particle at time t can be described by the vector function

$$\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j} + z(t)\underset{\sim}{k}.$$

The velocity of the particle at time t is given by

$$\dot{\underset{\sim}{r}}(t) = x'(t)\underset{\sim}{i} + y'(t)\underset{\sim}{j} + z'(t)\underset{\sim}{k}.$$

The acceleration of the particle at time t is given by

$$\ddot{\underset{\sim}{r}}(t) = x''(t)\underset{\sim}{i} + y''(t)\underset{\sim}{j} + z''(t)\underset{\sim}{k}.$$

The velocity vector, $\dot{\underset{\sim}{r}}(t)$, specifies the direction of the particle's motion at time t.

The speed of a particle is given by $\left|\dot{\underset{\sim}{r}}(t)\right|$.

Given that $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$, the distance, l, travelled along a path from $t = t_1$ to $t = t_2$ is

$$l = \int_{t_1}^{t_2} \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\,dt.$$

Question

The position vector, $\underset{\sim}{r}(t)$, at time t of a particle moving in a plane is given by

$$\underset{\sim}{r}(t) = 60t\underset{\sim}{i} + \left(20 + 45t - 5t^2\right)\underset{\sim}{j} \text{ where } t \geq 0.$$

(a) Find the initial position of the particle.

(b) Find the initial velocity of the particle.

(c) Find the particle's speed at $t = 1$. Give your answer correct to one decimal place.

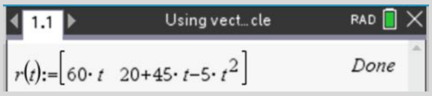
(d) Find when the particle is moving parallel to $\underset{\sim}{i}$.

... continued

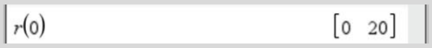
Solution

On a **Calculator** page, assign $\underset{\sim}{r}(t)$ as a row vector as follows:

- Press [ctrl] [template] to access the **Assign** := command.
- Press [book] [5], select the **1-by-2 Matrix** template and enter as shown.



(a) The particle's initial position is given by $\underset{\sim}{r}(0)$.

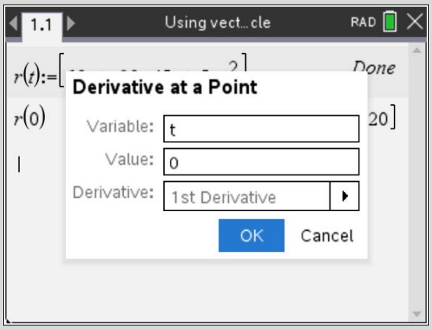


Answer: $\underset{\sim}{r}(0)=20\underset{\sim}{j}$

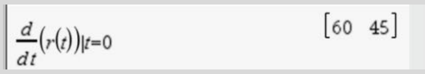
(b) The particle's initial velocity is given by $\dot{\underset{\sim}{r}}(0)$.

To find $\dot{\underset{\sim}{r}}(0)$:

- Press [menu] > **Calculus** > **Derivative at a Point**.
- Complete the required fields as shown.
- Complete the **Derivative at a Point** template as shown.



$\dot{\underset{\sim}{r}}(t)=60\underset{\sim}{i}+(45-10t)\underset{\sim}{j}$



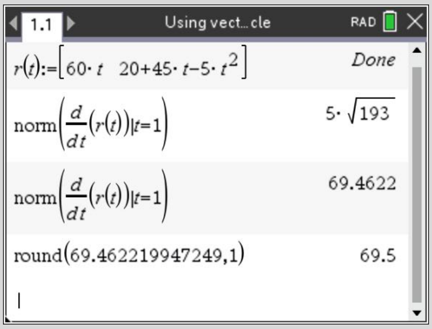
Answer: $\dot{\underset{\sim}{r}}(0)=60\underset{\sim}{i}+45\underset{\sim}{j}$

***Note**: Alternatively, to access the **Derivative** template, press [book] [5]. A more efficient alternative is to press [shift] [−].*

(c) The particle's speed at $t=1$ is given by $|\dot{\underset{\sim}{r}}(1)|$.

To calculate $|\dot{\underset{\sim}{r}}(1)|$ correct to one decimal place, enter as shown, taking note of the following instructions:

- Press [menu] > **Matrix & Vector** > **Norms** > **Norm**.
- Press [shift] [−] to access the **Derivative** template.
- Press [ctrl] [=] to access the 'with' or 'given' symbol |.
- Press [ctrl] [enter] to obtain a decimal answer.
- Press [menu] > **Number** > **Number Tools** > **Round** to give $|\dot{\underset{\sim}{r}}(1)|$ correct to one decimal place.



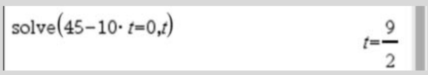
Answer: The particle's speed is 69.5, correct to one decimal place. $\dot{\underset{\sim}{r}}(1)=60\underset{\sim}{i}+35\underset{\sim}{j}$.

$$
\begin{aligned}
|\dot{\underset{\sim}{r}}(1)| &= \sqrt{60^2+35^2} \\
&= 69.4622... \\
&= 69.5
\end{aligned}
$$

(d) To find when the particle is moving parallel to $\underset{\sim}{i}$, equate the $\underset{\sim}{j}$ component of the velocity vector to zero and solve:

The particle is moving parallel to $\underset{\sim}{i}$ when $45-10t=0$.

- Press [menu] > **Algebra** > **Solve**.



Answer: $t=\frac{9}{2}(=4.5)$

***Note**: Students should solve this linear equation by hand.*

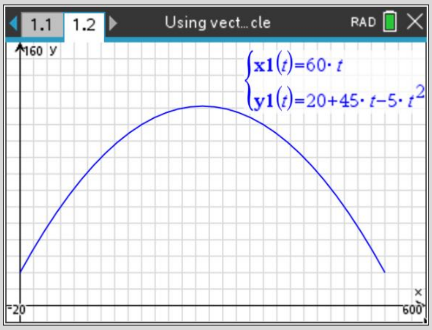
... continued

Solution (continued)

The particle's path can be plotted using parametric graphing.

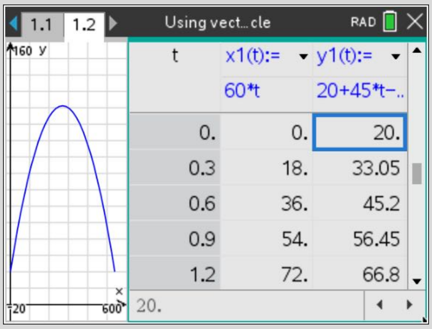
On a **Graphs** page:

- Press [menu] > **Graph Entry/Edit** > **Parametric**.
- Enter $\boldsymbol{x1(t) = 60t}$ and $\boldsymbol{y1(t) = 20 + 45t - 5t^2}$.
- Enter $\boldsymbol{0 \le t \le 9}$ $\boldsymbol{tstep = 0.3}$
- To add a grid, press [menu] > **View** > **Grid** > **Lined Grid**.
- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –20 XMax = 600 XScale = 40
  YMin = –10 YMax = 160 YScale = 20



Notes: *If needed, it can be helpful to press* [menu] *>* ***Window/Zoom*** *>* ***Zoom Fit*** *to obtain a first viewing window of the plotted graph. Otherwise, a useful tactic is to press* [ctrl] [T] *to show a tabular representation of the particle's path. To edit the table settings, press* [menu] *>* ***Table*** *>* ***Edit Table Settings*** *and complete as desired. To resize the table's column widths, press* [menu] *>* ***Actions*** *>* ***Resize*** *and resize as desired.*

The screenshot at right shows the particle initially at $(0, 20)$.

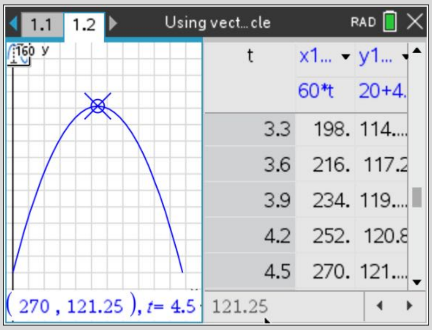


Note: *Press* [ctrl] [tab] *to move between different applications or representations on the same page.*

To find when the particle is moving parallel to $\underset{\sim}{i}$ on a **Graphs** page:

- Press [menu] > **Trace** > **Graph Trace**.
- Press ◀ ▶ to trace along the particle's path.
- If needed, press [menu] > **Trace** > **Trace Setup** to adjust the trace setup.

The particle is moving parallel to $\underset{\sim}{i}$ at $(270, 121.25)$.



Finding a particle's position given its acceleration

Question

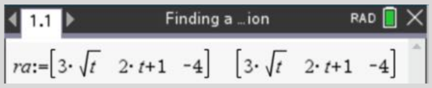
The acceleration of a particle at time t is given by $\ddot{\underset{\sim}{r}}(t) = 3\sqrt{t}\,\underset{\sim}{i} + (2t+1)\,\underset{\sim}{j} - 4\underset{\sim}{k}$.

Find the particle's position at time t if $\underset{\sim}{r}(0) = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ and $\dot{\underset{\sim}{r}}(0) = 5\underset{\sim}{i}$.

Solution

On a **Calculator** page, assign $\ddot{\underset{\sim}{r}}(t)$ as a row vector as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **m-by-n Matrix** template, fix the dimensions as 1-by-3 and enter as shown.



To find $\dot{\underset{\sim}{r}}(t)$:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [menu] > **Calculus** > **Integral**.
- Enter as shown.

So $\dot{\underset{\sim}{r}}(t) = 2t^{\frac{3}{2}}\underset{\sim}{i} + (t^2+t)\,\underset{\sim}{j} - 4t\underset{\sim}{k} + \underset{\sim}{c}_1$.

***Note**: Alternatively, to access the **Integral** template, press [catalog] [5]. A more efficient alternative is to press [shift] [+].*

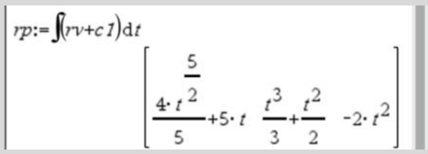
Use $\dot{\underset{\sim}{r}}(0) = 5\underset{\sim}{i}$ to find $\underset{\sim}{c}_1$:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **m-by-n Matrix** template, fix the dimensions as 1-by-3 and enter as shown.
- Press [ctrl] [=] to access the 'with' or 'given' symbol |.

Hence $\underset{\sim}{c}_1 = 5\underset{\sim}{i}$ and so $\dot{\underset{\sim}{r}}(t) = \left(2t^{\frac{3}{2}} + 5\right)\underset{\sim}{i} + (t^2+t)\,\underset{\sim}{j} - 4t\underset{\sim}{k}$.

To find $\underset{\sim}{r}(t)$:

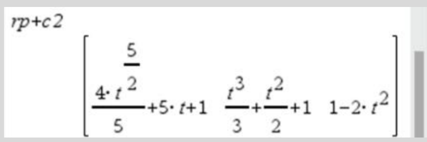
- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [menu] > **Calculus** > **Integral**.
- Enter as shown.



So $\underset{\sim}{r}(t) = \left(\frac{4}{5}t^{\frac{5}{2}} + 5t\right)\underset{\sim}{i} + \left(\frac{1}{3}t^3 + \frac{1}{2}t^2\right)\underset{\sim}{j} - 2t^2\underset{\sim}{k} + \underset{\sim}{c}_2$.

Use $\underset{\sim}{r}(0) = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ to find $\underset{\sim}{c}_2$:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **m-by-n Matrix** template, fix the dimensions as 1-by-3 and enter as shown.
- Press [ctrl] [=] to access the 'with' or 'given' symbol |.



Hence $\underset{\sim}{c}_2 = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$.

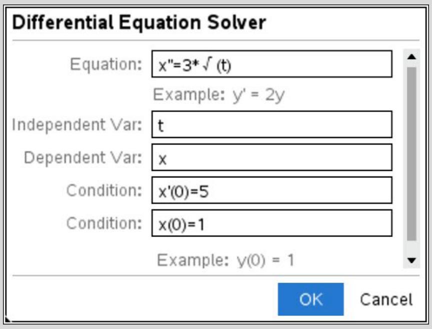
Answer: $\underset{\sim}{r}(t) = \left(\frac{4}{5}t^{\frac{5}{2}} + 5t + 1\right)\underset{\sim}{i} + \left(\frac{1}{3}t^3 + \frac{1}{2}t^2 + 1\right)\underset{\sim}{j} + (1 - 2t^2)\underset{\sim}{k}$

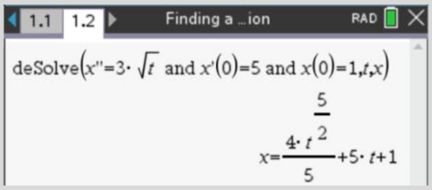
... continued

Solution (continued)

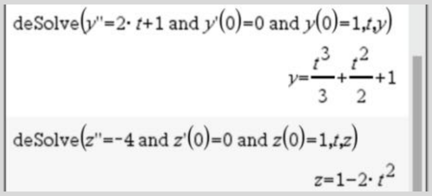
Alternatively on a **Calculator** page, use the **Differential Equation Solver** on each component as follows:

- Press [menu] > **Calculus** > **Differential Equation Solver**.
- Press [?!▸] to select the symbol '.
- Complete the dialog box as shown.





- Repeat for the other two components as shown.



Finding the minimum speed and distance travelled by a particle

Question

Relative to the origin O, the position vector of a particle at time t seconds is given by

$\underset{\sim}{r}(t) = 6\sin\left(\frac{\pi t}{8}\right)\underset{\sim}{i} + \left(\sin\left(\frac{\pi t}{4}\right) - \cos\left(\frac{\pi t}{8}\right)\right)\underset{\sim}{j}$ where $t \geq 0$ and distances are measured in metres.

(a) Plot the path of the particle, indicating the particle's initial position.

In the following parts, where appropriate give your numerical answers correct to one decimal place.

(b) Find when the particle is moving at its minimum speed.

(c) Find the particle's minimum speed.

(d) Suppose that d metres is the distance of 'one lap' travelled by the particle. Find the value of d.

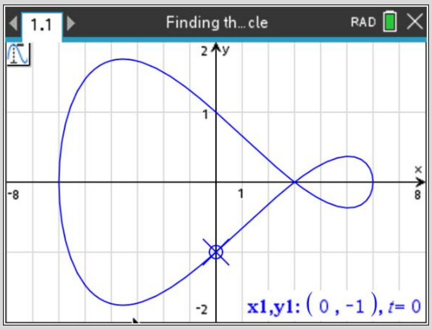
Solution

(a) Use parametric graphing mode to plot the path.

On a **Graphs** page:

- Press [menu] > **Graph Entry/Edit** > **Parametric**.
- Enter $\boldsymbol{x1(t) = 6\sin\left(\frac{\pi t}{8}\right)}$ and

  $\boldsymbol{y1(t) = \sin\left(\frac{\pi t}{4}\right) - \cos\left(\frac{\pi t}{8}\right)}$.
- Enter $\boldsymbol{0 \leq t \leq 16 \;\; tstep = 0.25}$.
- To add a grid, press [menu] > **View** > **Grid** > **Lined Grid**.
- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –8 XMax = 8 XScale = 1
  YMin = –2 YMax = 2 YScale = 1



To locate the particle's initial position:

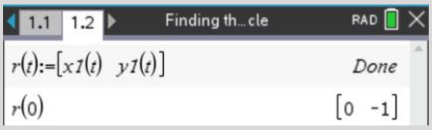
- Press [menu] > **Trace** > **Graph Trace**.

Answer: A plot of the particle's path is shown at right. The particle's initial position is $(0, -1)$.

***Note**: To hide the parametric equations, place the cursor over the label and press [ctrl] [menu] > **Hide**. To reinstate the label, press [menu] > **Actions** > **Hide/Show**. Move the cursor over the ghosted out label and press [enter].*

Alternatively, to find the initial position on a **Calculator** page assign $\underset{\sim}{r}(t)$ as a row vector as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **1-by-2 Matrix** template and enter as shown.



Thus confirming that the particle's position is $(0, -1)$.

... continued

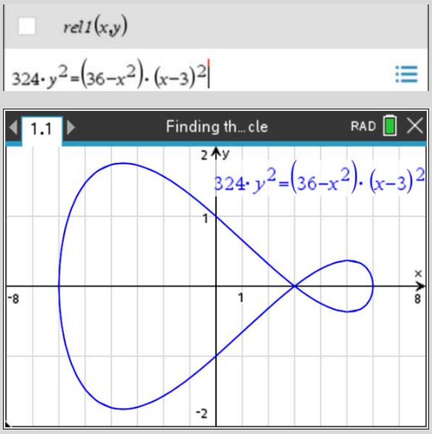
Solution (continued)

***Note**: The parametric equations, $\mathbf{x1}(t)$ and $\mathbf{y1}(t)$, defined on the **Graphs** page are recognised on the **Calculator** page provided these two pages form part of the same problem.*

It can be shown that the Cartesian equation of the path is $324y^2 = (36 - x^2)(x-3)^2$.

To plot the path in relation graphing mode on a **Graphs** page:

- Press [menu] > **Graph Entry/Edit** > **Relation**.
- Enter as shown.



(b) and **(c)** Find when the particle is moving at minimum speed, and its minimum speed.

On a **Calculator** page, assign $\underset{\sim}{v}(t)$ as a row vector as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [shift] [−] to access the **Derivative** template.
- Enter as shown.

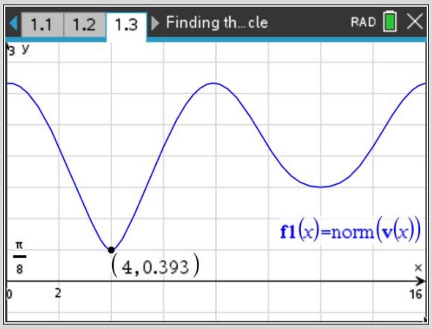
$v(t):=\frac{d}{dt}(r(t))$ Done

On a new **Graphs** page, plot the particle's speed for $0 \le t \le 16$ as follows:

- Press [catalog] [1] [N], scroll down and select **norm(**.
- Enter $f1(x) = \mathbf{norm}(v(x))$.
- To add a grid, press [menu] > **View** > **Grid** > **Lined Grid**.
- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = 0 XMax = 16 XScale = 2
  YMin = –0.5 YMax = 3 YScale = $\frac{\pi}{8}$

To find the coordinates of the first minimum:

- Press [menu] > **Analyse Graph** > **Minimum**.
- Move the cursor to the left of the minimum for a lower bound and press [enter].
- Move the cursor to the right of the minimum for an upper bound and press [enter].



Answer: (b) The particle's speed is a minimum at $t = 4$.

Answer: (c) At $t = 4$, the particle's speed is 0.4 ms^{-1}, correct to one decimal place.

Although not asked for, the particle's minimum speed is $\frac{\pi}{8}$ ms^{-1}. To verify this, on a **Calculator** page:

- Press [catalog] [1] [E], scroll down and select **exact(**.
- Press [menu] > **Matrix & Vector** > **Norms** > **Norm**.
- Press [ctrl] [=] to access the 'with' or 'given' symbol |.
- Enter as shown.

exact(norm(v(t)))|t=4 $\frac{\pi}{8}$

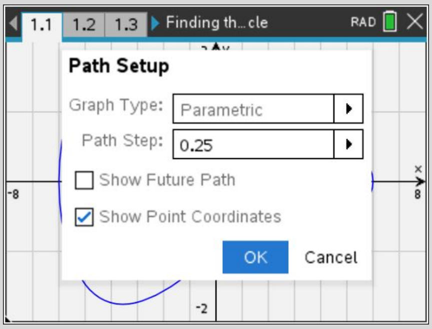
... continued

Solution (continued)

Also not asked for here, the particle moves with minimum speed at $\underset{\sim}{r}(4) = 6\underset{\sim}{i}$ i.e. at $(6, 0)$.

To animate the particle's path:

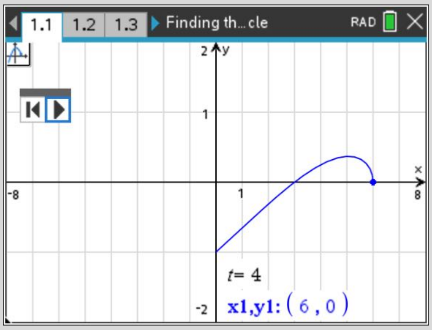
- Press menu > **Trace** > **Path Plot** > **Path Setup**.
- Complete the required fields as shown.



***Note**: If desired, the future path can be shown by checking the **Show Future Path** box.*

To set up the animation of the particle's motion:

- Press menu > **Trace** > **Path Plot** > **Parametric**.



***Note**: The coordinates of the particle's path display at the bottom right of the screen. This needs to be accounted for when positioning labels and setting suitable viewing windows.*

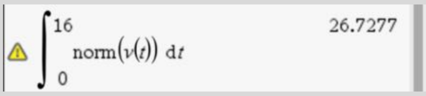
To start the animation:

- Move the cursor over the animation start button and press enter.

***Note**: To pause the animation, move the cursor over the animation pause button and press enter. To reset the animation, move the cursor over the animation reset button and press enter.*

The screenshot shows when and where the particle is moving at minimum speed.

(d) The distance, d, travelled by the particle along the path from $t = 0$ to $t = 16$ is given by $d = \int_0^{16} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$



where $x'(t) = \frac{d}{dt}\left(6\sin\left(\frac{\pi t}{8}\right)\right)$ and

$y'(t) = \frac{d}{dt}\left(\sin\left(\frac{\pi t}{4}\right) - \cos\left(\frac{\pi t}{8}\right)\right)$.

So $d = \int_0^{16} \left|\dot{\underset{\sim}{r}}(t)\right| dt$.

To find the value of d on a **Calculator** page:

- Press menu > **Calculus** > **Integral**.
- Press menu > **Matrix & Vector** > **Norms** > **Norm**.
- Enter as shown.

Answer: $d = 26.7$ (m) correct to one decimal place

Alternatively, to access the **Integral** template, press [template key] 5.
A more efficient alternative is to press ⇧shift +.

3.6 Data analysis, probability and statistics

3.6.1 Distribution of linear combinations of random variables

Simulating the distribution of linear combinations of random variables

Suppose that Annie is an orchardist who grows apples and pears. Let $X \sim \mathrm{N}\left(310,\ 40^2\right)$ denote the net weight in kilograms of a randomly selected bin of apples, and $Y \sim \mathrm{N}\left(270,\ 55^2\right)$ denote the net weight in kilograms of a randomly selected bin of pears.

Annie sells 3 bins of apples and 2 bins of pears.

(a) Conduct a 1000-trial simulation to model the distribution of the total weight of the 5 bins of fruit. Let W be the random variable representing the total weight of fruit. From the simulation, estimate the following, to the nearest kg: **(i)** E(W), **(ii)** sd(W).

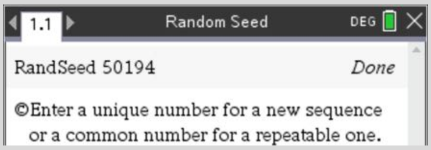
(b) Compare the estimated values of E(W) and sd(W) with the theoretical values for μ_W and σ_W.

Solution

To seed the pseudorandom number generator for variability or reproducibility, on a **Calculator** page:

- Press [menu] > **Probability** > **Random** > **Seed**.

Enter a number – either unique or common.

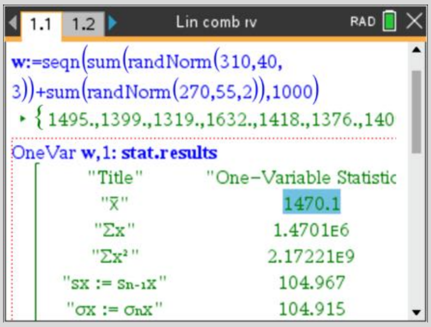


(a) To conduct 1000 trials for total weight, on a **Notes** page:

- Press [ctrl][M] to insert a **Maths Box**.
- Enter ***w*:= seqn(sum(randNorm(310, 40, 3)) + sum(randNorm(270, 55, 2)), 1000)**, as shown, pressing [📖][1][S] for **seqn** and **sum**, and [📖][1][R] for **randNorm**.

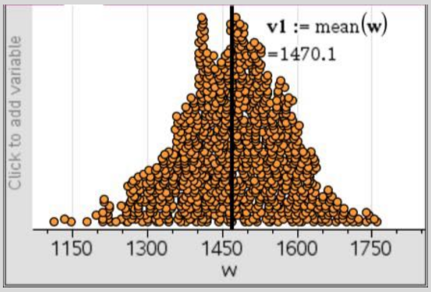
To estimate **(i)** E(W), **(ii)** sd(W) from the simulation:

- Insert a **Maths Box**, press [menu] > **Calculations** > **Statistics** > **Stat Calculations**. Select **One-Variable Statistics**. Enter Num of Lists:**1**, then X1 List: ***w***.



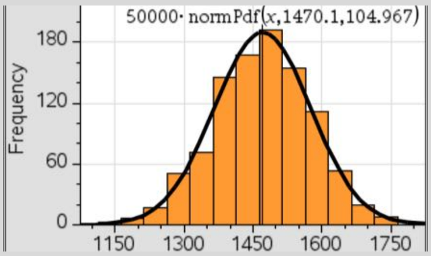
To plot the simulation results, add a **Data & Statistics** page:

- Press [tab] and select ***w*** on the horizontal axis.



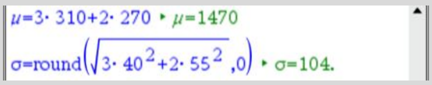
To analyse key features of the simulated distribution:

- Press [menu] > **Analyse** > **Plot Value**. Enter ***v*1:= mean(*w*)**.
- Press [menu] > **Plot Type**. Select **Histogram**.
- Press [menu] > **Analyse**. Select **Show Normal PDF**.



(b) To calculate the theoretical values of μ_W and σ_W:

- Insert **Maths Boxes** and enter as shown.



Answer: (a) (i) E(W) ≈ 1470, **(ii)** sd(W) ≈ 105, very close to **(b)** $\mu_W = 1470$, $\sigma_W \approx 104$. The simulation suggests that $W = X_1 + X_2 + X_3 + Y_1 + Y_2$ is also normally distributed.

Calculating probabilities for linear combinations of normal random variables

As in the previous problem, let $X \sim \mathrm{N}\left(310,\ 40^2\right)$ and $Y \sim \mathrm{N}\left(270,\ 55^2\right)$ denote the net weight (in kg) of randomly selected bins of apples and pears, respectively. Each month, Annie sells exactly 3 bins of apples and 2 bins of pears. She is paid at a rate of \$1500 per tonne for apples and \$1250 per tonne for pears.

(a) Calculate the following, correct to two decimal places. Compare the answers to the simulation: $\Pr(W > 1600)$, **(ii)** $\Pr(W > 1300 \mid W < 1500)$.

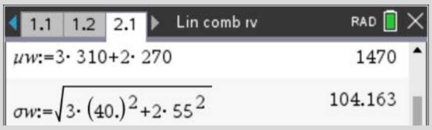
(b) Let C be the random variable representing the total price paid for 3 bins of apples and 2 bins of pears. Find the mean and standard deviation of C, correct to the nearest dollar.

(c) In March, Annie sells 3 bins of apples and 2 bins of pears for a total of \$2000. She sells another 3 bins of apples and 2 bins of pears in April. Assuming the total price in April is a random variable, find the probability, correct to two decimal places, that the absolute difference between the March and April totals exceeds \$200.

Solution

(a) To find μ_W and σ_W, on a **Calculator** page:

- Enter as shown to store the values of μw and σw, pressing [ctrl] [📖] ($\infty\beta°$) to select μ and σ.



(i) To determine $\Pr(W > 1600)$ and compare with simulation:

- Press [menu] > **Probability** > **Distributions** > **Normal CDF**. Enter Lower Bound: **1600**, Upper Bound: **∞**, μ: $\boldsymbol{\mu w}$, σ: $\boldsymbol{\sigma w}$, pressing [π▸] for ∞ and [var] for μw and σw.

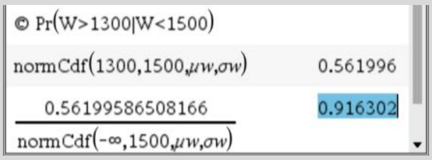
To compare with the previous simulation, on the **Notes** page:

- Insert a **Maths Box** and enter **countIf(*w*, ? > 1600)**, pressing [📖] [1] [C] to select **countIf** and [?!▸] for **?**.

Answer: $\Pr(W > 1600) = 0.11$ (2 decimal places), consistent with simulation result (typical result: $109/1000 \approx 0.11$).

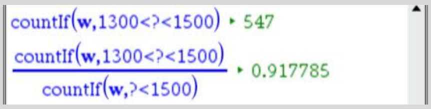
(ii) To find $\Pr(W > 1300 | W < 1500)$, on the **Calculator** page:

- Press ▲ key to select previous input, then press [enter].
- Edit the copied inputs to **normCdf(1300,1500,*μw*,*σw*)**.
- Similarly, enter **Ans/normCdf(– ∞, 1500, *μw*,*σw*)**.



To compare with the previous simulation, on the **Notes** page :

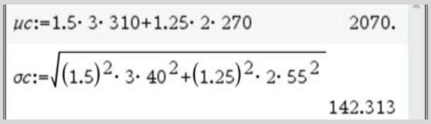
- Similarly to part (i), enter as shown.



Answer: $\Pr(W > 1300 | W < 1500) = 0.92$ (2 decimal places), consistent with simulation result (typical result shown).

(b) To find the mean price μ_C and standard deviation σ_C:

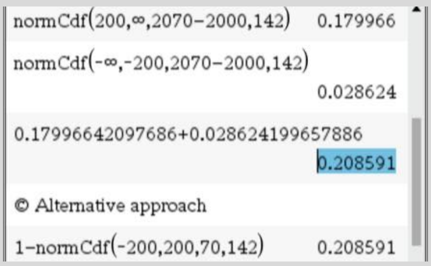
- Enter as shown, with the unit prices in \$/kg.



Answer: Mean price μ_C is \$2070, $\mu_C = 142$.

(c) To find $\Pr\left(|D| > 200\right) = \Pr\left(D > 200\right) + \Pr\left(D < -200\right)$ where $D \sim \mathrm{N}((2070 - 2000), 142^2)$, $D = C_{April} - C_{March}$.

- Enter as shown to calculate the right and left tails.



Answer: Probability $|D| > \$200$ is 0.21 (2 decimal places).

3.6.2 Distribution of the sample mean

Demonstrating variability between samples for the sample mean

Suppose that in a particular year the reported NAPLAN Numeracy results for participating year 9 students in Victoria showed a mean score of 582 and a standard deviation of 85. Assume that the scores for this population are normally distributed.

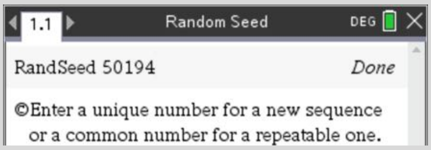
Question

Set up a simulation that repeatedly draws random samples of size $n = 50$ from a normal distribution $N(582, 85^2)$, representing random sampling from this population. Use the simulation to explore the variability and shape of the sampling distribution of the sample mean under repeated sampling.

Solution

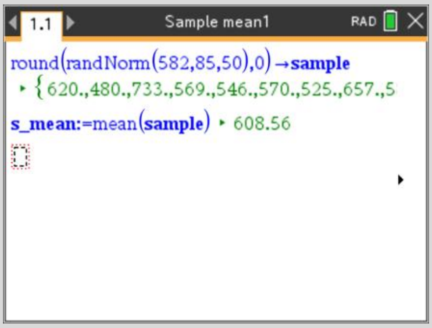
To seed the pseudorandom number generator for variability or reproducibility, on a **Calculator** page:

- Press [menu] > **Probability** > **Random** > **Seed**.
- Enter a number – either unique for a new sequence of random numbers or common for a repeatable one.



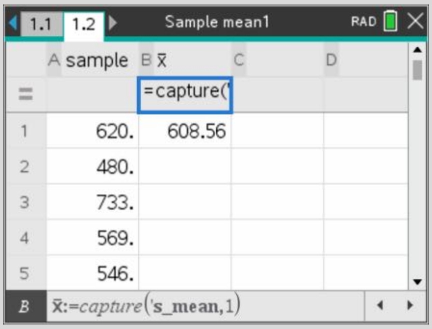
To select random samples of size $n = 50$ from $N(582, 85^2)$ and calculate the sample mean, on a **Notes** page:

- Press [ctrl] [M] to insert a **Maths Box**.
- In the Maths Box, press [book] [1] [S] to select round(), then press [menu] > **Calculations** > **Probability** > **Random** > **Normal** and input **round(randNorm(582,85,50),0)**.
- Press [ctrl] [var] ([sto→]), input **sample** then press [enter].
- Insert a **Maths Box**. Enter ***s_mean*:= mean(*sample*)** by pressing [ctrl] [␣] for underscore, [book] [1] [M] to select **mean** and [var] to select **sample**.
- Click on the first **Maths Box** and press [enter] several times. Observe that after each press a new sample is taken.



To build a model of the sampling distribution by capturing the sample means, one sample at a time, add a **Lists & Spreadsheet** page to the document, then:

- Enter the headings as shown, by pressing [var] > **Link To** to select **sample** and [ctrl] [book] ([∞β°]) to select $\bar{x}$.



To enter the formula to capture the sample means:

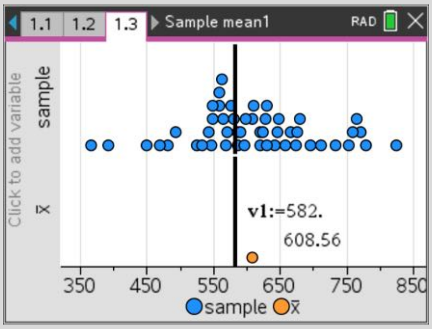
- Navigate to the column B formula cell and press [menu] > **Data** > **Data Capture** > **Auto** and press [var] to select **s_mean**.

To obtain plots of the sample and captured sample means, add a **Data & Statistics** page to the document, then:

- Press [tab] and select **sample** on the horizontal axis.
- Press [menu] > **Plot Properties** > **Add X variable**, select $\bar{x}$.

To display the value of the population mean on the plot:

- Press [menu] > **Analyse** > **Plot Value** and enter $\mathbf{v1 := 582}$.

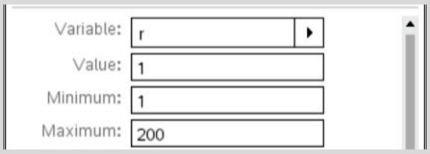


... continued

Solution (continued)

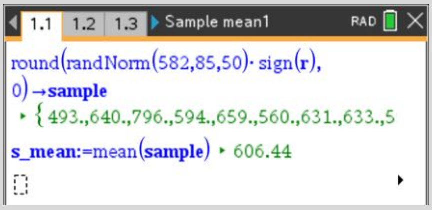
To add a slider that will allow the sampling to be repeated, on the **Data & Statistics** page:

- Press [menu] > **Actions** > **Insert Slider**, then enter the following values:
- Variable: ***r***, Min.: **1**, Max.: **200**, Step:**1**, Minimise: ☑.



To edit the sampling formula so the slider allows repeated sampling, navigate back to the **Notes** page, **page 1.1**.

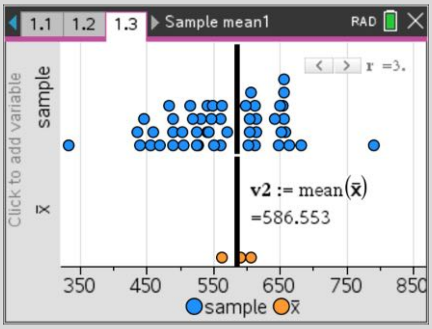
- Edit the first **Maths Box** to:
  $\textbf{round}\left(\textbf{randNorm}\left(582,85,50\right)\cdot\textbf{sign}\left(r\right),0\right)\rightarrow\textbf{sample}$



Note: Press [📖] [1] [S] *to select* ***sign***. *Multiplying by* ***sign(r)*** *doesn't alter the result because* ***sign(r)*** *= +1 for all r > 0. But each time* ***r*** *changes, it triggers a new sample to be taken.*

To add a plot value for the mean of $\bar{X}$ for repeated sampling, navigate back to the **Data & Statistics** page, then:

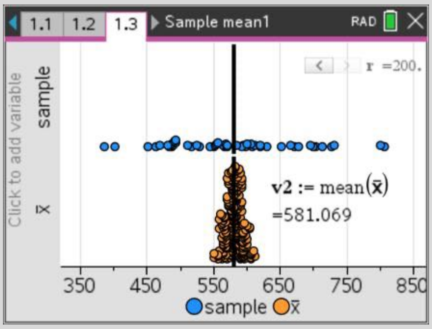
- Press [menu] > **Analyse** > **Plot Value**. Press [📖] [1] [M] to select **mean** and [var] for $\bar{x}$ to enter $\textbf{v2 := mean}\left(\bar{x}\right)$.



Note: The plot value ***v2*** *allows a comparison of* $\mathrm{E}\left(\bar{X}\right)$ *and the population mean* $\mu = 582$*, shown by* ***v1***. *Observe that as the number of samples taken increases,* ***v2*** *approaches* ***v1***.

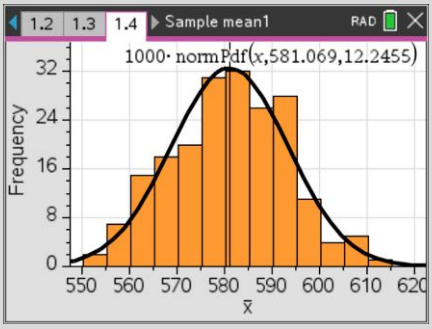
- Click or animate the slider to observe the building up of a model of the distribution of $\bar{X}$ for 200 samples.

Note: To animate, click on the slider. Press [ctrl] [menu] > ***Animate***. *To stop, press* [ctrl] [menu] > ***Stop Animate***.



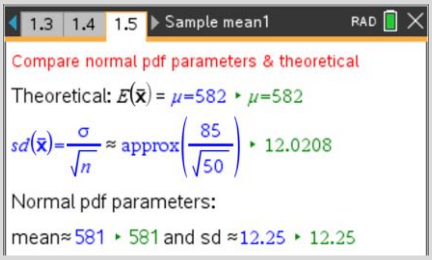
To display the captured values of $\bar{x}$ in a histogram and compare the shape with a normal distribution curve, add a **Data & Statistics** page, then:

- Press [tab] and select $\bar{x}$ on the horizontal axis.
- Press [menu] > **Plot Type** > **Histogram**.
- Press [menu] > **Analyse** > **Show Normal PDF**.



To clear the sample data and reset the simulation, navigate to **Data & Statistics page 1.3**, then:

- Reset the slider value to ***r*** = **1**.
- Navigate to the column B formula cell in **Lists & Spreadsheet** page 1.2. Press [menu] > **Data** > **Clear Data**.



Answer: The normal pdf curve shows a good approximate fit to the histogram for $\bar{X}$. In the example shown, the curve parameters show a mean (581) and standard deviation (12.25) of $\bar{X}$. The theoretical values are mean = 582 and sd ≈ 12.02.

Exploring the effect of sample size on the distribution of sample means

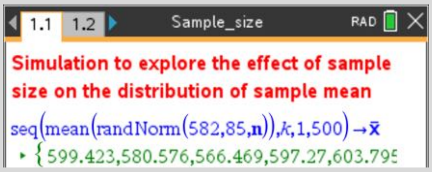
Question

In the previous example, random samples of size $n = 50$ were drawn from a normal population with distribution $N(582, 85^2)$. Set up a simulation that, for each sample size n with $10 \le n \le 100$, repeatedly draws independent samples from this distribution and calculates the corresponding sample means. For each chosen value of n, repeat the sampling 500 times and use the results to analyse the shape and spread of the sampling distribution of the sample mean as n varies.

Solution

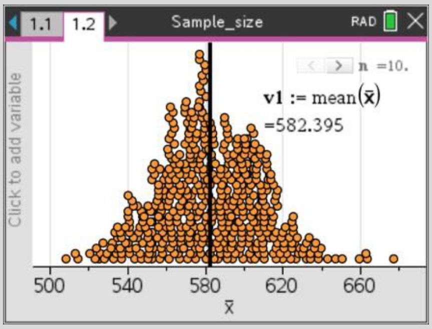
To set up the simulation, carry out the following steps.

- Add a **Notes** page in a **New Document**.
- Add a **Data & Statistics** page and press [menu] > **Actions** > **Insert Slider**. Enter the slider settings:
  Variable: ***n***, Value: **10**, Minimum: **10**, Maximum: **100**, Step Size:**10**, Minimise : ☑.



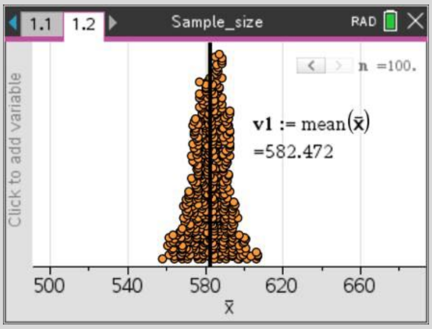
Navigate to the **Notes** page 1.1.

- Press [ctrl] [M] to insert a Maths Box and enter: $\mathbf{seq(mean(randNorm(582,85,}\boldsymbol{n}\mathbf{)),}\boldsymbol{k}\mathbf{,1,500)} \rightarrow \bar{\boldsymbol{x}}$, by pressing [ctrl] [var] ([sto→]) followed by [ctrl] [📖] ([∞β°]) to input $\rightarrow \bar{\boldsymbol{x}}$.



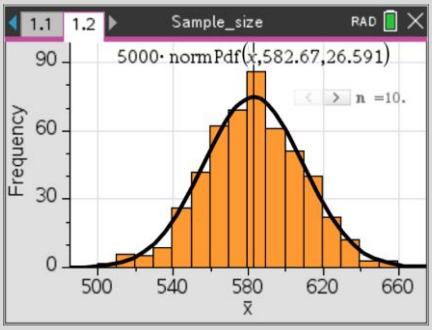
To display the sample means of 500 samples, with the sample size determined by the slider value, on page 1.2:

- Press [tab] and select $\bar{x}$ on the horizontal axis.
- Press [menu] > **Analyse** > **Plot Value**. Enter $\boldsymbol{v1}$**:= mean(** $\bar{\boldsymbol{x}}$ **)** by pressing [📖] [1] [M] to select **mean** and [var] to select $\bar{x}$.
- Change the slider value and observe how the dot plots of the distribution of $\bar{X}$ changes as n increases.



To observe the effect of changing the sample size on the mean and standard deviation of $\bar{X}$, on page 1.2:

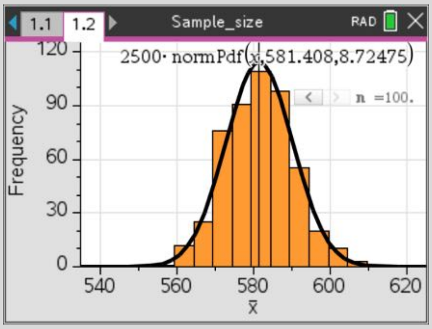
- Set ***n*** **= 10**, then press [menu] > **Plot Type** > **Histogram**.
- Press [menu] > **Analyse** > **Show Normal PDF**.
- Change the value of ***n*** from 10 through to 100.
- As the value of ***n*** changes, it may be necessary to zoom (press [menu] > **Window/Zoom** > **Zoom – Data**) to the data and change **Bin Settings** (via [ctrl] [menu] > **Bin Settings**).



Answer: The normal pdf curve parameters are very close to the theoretical values: $\mathrm{E}(\bar{X}) = \mu = 582$ and

$\mathrm{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{85}{\sqrt{10}} \approx 26.88$ through to $\mathrm{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{85}{\sqrt{100}} \approx 8.5$.

The simulation illustrates that the sampling distribution of the sample means is itself normal with $\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$. The spread of the distribution decreases as the sample size increases, which is consistent with $\mathrm{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}}$ and $n \to \infty$, $\frac{\sigma}{\sqrt{n}} \to 0$.



Sampling from the uniform distribution

Note: *Refer to Section 2.2.1, which provides a detailed treatment of the distribution of the sample mean for random samples drawn from a uniform distribution.*

Sampling from an asymmetric distribution

Question

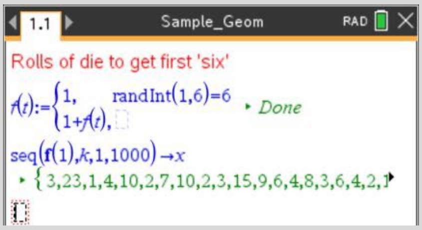
(a) Use simulation to model the random variable representing the number of rolls of a fair six-sided die until the first six occurs, and analyse its distribution.

(b) Using repeated random samples of size n drawn from this distribution, use simulation to explore the sampling distribution of the sample mean.

Solution

(a) To set up a simulation for the number of rolls of a die until the first six occurs, on a **Notes** page:

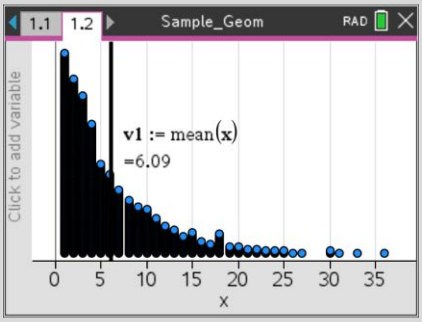
- Press [ctrl] [M] to insert a **Maths Box**.
- In the **Maths Box**, enter
  $$f(t) := \begin{cases} 1, & \text{randInt}(1,6)=6 \\ 1+f(t), & \text{else} \end{cases}$$, pressing [template key] to select the piecewise template and [catalog] [1] [R] for **randInt**.
- In a new **Maths Box**, enter $\textbf{seq}(f(1),k,1,1000) \rightarrow x$, pressing [catalog] [1] [S] for **seq** and [ctrl] [var] ([sto→]) for $\rightarrow$.



Note: *1. The list* ***x*** *stores the number of rolls until the first six occurs. 2. A blank condition (domain) entry in the piecewise template is treated as "****else****" or "otherwise".*

To obtain a plot of the distribution of the random variable, X, representing the number of rolls until the first six occurs, add a **Data & Statistics** page, then:

- Press [tab] and select $\boldsymbol{x}$ on the horizontal axis.
- Press [menu] > **Analyse** > **Plot Value**. Enter $\boldsymbol{v1}$**:= mean(**$\boldsymbol{x}$**)**.

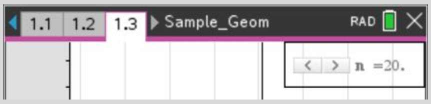


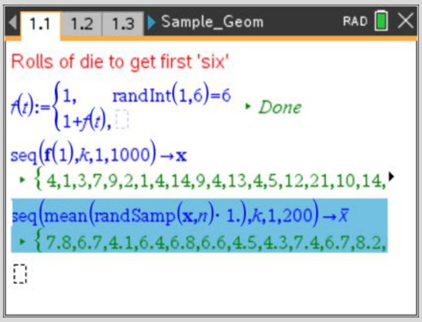
Answer: This simulation models the number of Bernoulli trials required until the first success occurs, (which is described by a geometric distribution) with success probability, $p=\frac{1}{6}$. A plot of the distribution is right-skewed, with probability decreasing exponentially to a long right tail.

$$\mathrm{E}(X)=\frac{1}{p}=6 \text{ and } \sigma=\mathrm{sd}(X)=\frac{\sqrt{1-p}}{p}=\frac{\sqrt{(5/6)}}{(1/6)}\approx 5.48$$

(b) To generate the sampling distribution of the sample mean for samples of size n drawn from the distribution in part **(a)**, add a new **Data & Statistics** page, then:

- Press [menu] > **Actions** > **Insert Slider** and enter the values: Variable: $\boldsymbol{n}$, Value: **20**, Min.: **5**, Max.: **100**, Step Size: **10**.
- Navigate to the **Notes** page 1.1 and in a new **Maths Box** enter $\textbf{seq}(\textbf{mean}(\textbf{randSamp}(x,n)\cdot 1.0),k,1,200) \rightarrow \bar{x}$, press [ctrl] [var] ([sto→]) for $\rightarrow$ and [ctrl] [catalog] ([∞β°]) to select $\bar{x}$.





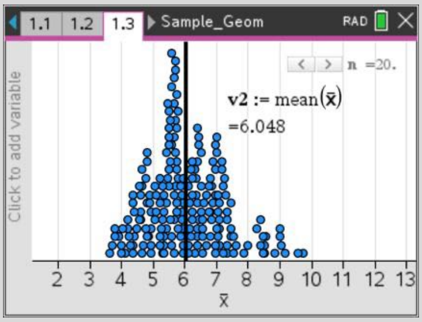
This selects 200 samples of size n (slider value) from list $\boldsymbol{x}$, calculates the sample means and stores the means in list $\bar{x}$.

... continued

Solution (continued)

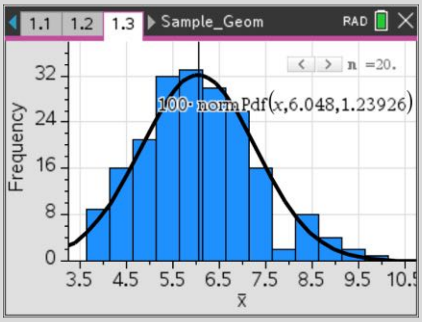
To plot the sampling distribution of sample means taken from the asymmetrical distribution, navigate to page 1.3, then:

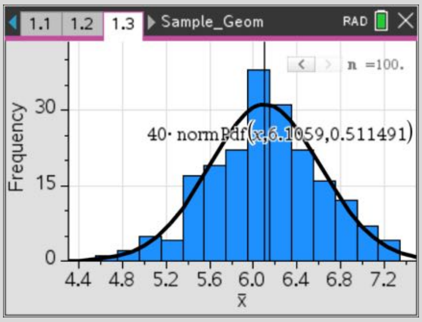
- Click below the horizontal axis and select the variable $\bar{x}$.
- Press menu > **Analyse** > **Plot Value** and enter $\boldsymbol{v2 := \textbf{mean}(\bar{x})}$, selecting $\bar{x}$ from the var list.
- Change the slider value, ***n***, to observe the effect of increasing the sample size.



To observe the approximate normality of the sampling distribution of sample means using, on page 1.3:

- Set the slider value to $n = 20$.
- Press menu > **Plot Type** > **Histogram**.
- Press menu > **Window/Zoom** > **Zoom – Data**.
- Press menu > **Analyse** > **Show Normal PDF**.
- Edit the value of ***n*** to larger sample sizes and observe the effect on the shape and spread of the histogram and the normal pdf parameters.
- Adjust the **Bin Settings** and **Zoom – Data** as necessary.





Answer: This simulation demonstrates that, despite the samples being drawn from a very skewed distribution, the distribution of sample means is approximately normal.
The normal pdf parameters for $n = 20$ should be consistent with the theoretical values: $\mathrm{E}(\bar{X}) = \mu = 6$ and.

$$\mathrm{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{5.48}{\sqrt{20}} \approx 1.23$$

Summarising the findings of the simulations carried out in this section

The suite of simulations carried out in Section 3.6.2 should highlight two related but distinct ideas. First, by the *Law of large numbers*, as the number of repeated samples increases, the empirical mean of the simulated sample means approaches the population mean μ. This supports the theoretical result that $\mathrm{E}(\bar{X}) = \mu$.

Second, the simulations illustrate the sampling distribution of $\bar{X}$ for samples of size n, where the distribution of sample means has mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$.

As the number of repeated samples increases, the simulated standard deviation of the sample means approaches $\frac{\sigma}{\sqrt{n}}$, as predicted by theory.

These simulations also provide clear illustrations of the central limit theorem, which shows that the sampling distribution of the sample mean is approximately normal in many situations, even when the underlying population distribution is not normal.

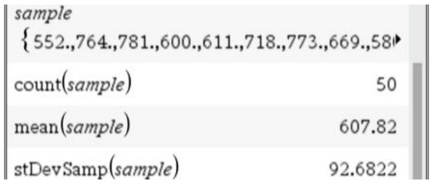
3.6.3 Confidence intervals for the population mean

Finding confidence intervals for the sample mean

Suppose that prior to a full analysis of the year 9 NAPLAN numeracy results, researchers use a random sample of 50 scores to estimate the population mean. The sample mean is a point estimate that is unlikely to be exactly equal to the population mean. It is therefore useful to calculate an interval that is likely to contain the true population mean with a high level of confidence. Assume that the true population parameters are $\mu = 582$ and $\sigma = 85$.

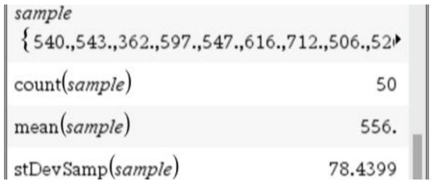
Question

(a) For this sample of size $n = 50$, the sample mean is $\bar{x} = 608$ and the sample standard deviation is $s = 92.7$. Determine an approximate 95% confidence interval for the NAPLAN score, correct to one d.p., using $s = 92.7$ as an estimate of the population standard deviation.



(b) It subsequently becomes known that the true population standard deviation is $\sigma = 85$. Determine an approximate 95% confidence interval for the mean NAPLAN score using $\bar{x} = 608$ and the actual value of the population standard deviation $\sigma = 85$. Compare this interval with the interval determined in part **(a)** above.

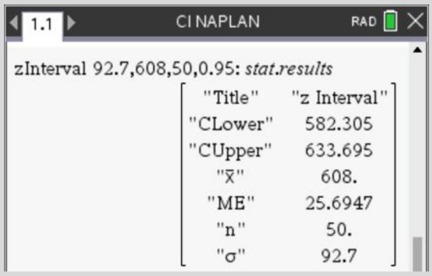
(c) Suppose that another researcher selected a different random sample of size $n = 50$, with sample mean $\bar{x} = 556$ and sample standard deviation $s = 78.4$. Find an approximate 95% confidence interval for the mean NAPLAN score using these known sample statistics and interpret the result.



Solution

(a) To find an approximate 95% confidence interval, on a **Calculator** page:

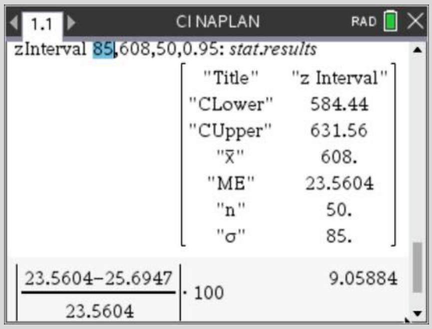
- Press menu > **Statistics > Confidence Intervals > z Interval**.
- Select **Data Input Method: Stats**. In the dialog box enter: $\sigma = 92.7, \bar{x} = 608, n = 50, C\ Level = 0.95$



Answer: An approximate 95% confidence interval is $(582.3, 633.7)$. The margin of error for this interval is 25.7 and it represents the radius of the confidence interval.

(b) To find an approximate 95% confidence interval using the population standard deviation, $\sigma = 85$:

- Press ▲ key to the **zInterval** input then press enter.
- Edit the pasted input as shown.



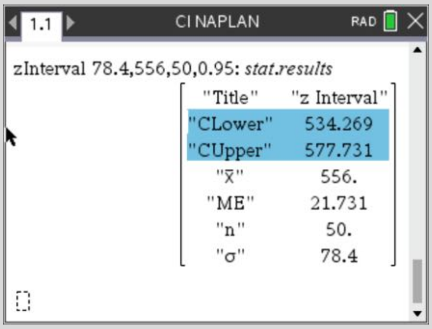
Answer: The approximate 95% confidence interval is $(584.4, 631.6)$ using $\sigma = 85$, compared with $(582.3, 633.7)$ using $s = 92.7$ as an estimate of the population standard deviation. This is a 9.1% difference in the margin of error.

... continued

Solution (continued)

(c) To find an approximate 95% confidence interval for the population mean, given a sample mean, $\bar{x} = 556$, and a standard error, $s = 78.4$:

- Press ▲ key to the **zInterval** input, then press [enter].
- Edit the pasted input as shown.



Answer: The approximate 95% confidence interval using $\bar{x} = 556$, $s = 78.4$ is $(534.3, 577.7)$, which does **not** contain the true population mean $\mu = 582$; i.e. $\mu \notin (534.3, 577.7)$.

Exploring the trade-off between level of confidence and margin of error

Assume that, in a particular jurisdiction, year 9 NAPLAN Numeracy scores have a population mean $\mu = 552$ and population standard deviation $\sigma = 81$.

Question

Use simulation to explore the effect of sampling variability on confidence intervals and the trade-off between confidence level and margin of error.

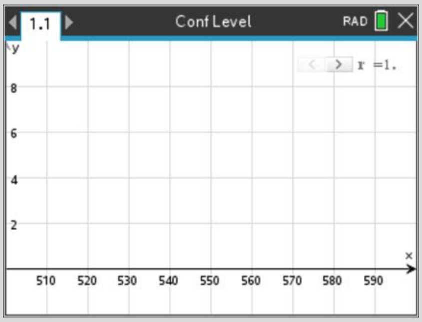
For the simulation, assume that independent random samples of size n = 100 are drawn from a normally distributed population with mean 552 and standard deviation 81. For each sample, construct confidence intervals for the population mean at confidence levels of 99%, 95%, 90%, and 50%. Repeat the sampling many times and compare the resulting confidence intervals.

Solution

To set up a slider that will be used to trigger the taking of a new random sample, on a **Graphs** page:

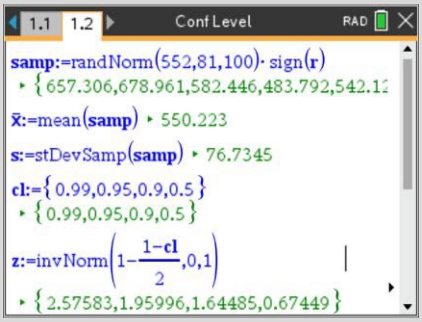
- Press [menu] > **Window/Zoom > Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = 500 XMax = 600 XScale = 10
  YMin = −2 YMax = 10 YScale = 2
- Press [menu] > **Actions > Insert Slider**. Enter the values:

Variable: ***r*****,** Min.: **1,** Max.: **200,** Step Size:**1,** Minimise: ☑.



To set up the first part of the simulation, add a **Notes** page:

- Press [ctrl][M] to add a **Maths Box** and enter
  $\textbf{samp} := \textbf{randNorm}(552, 81, 100) \cdot \textbf{sign}(r)$.
- Enter the following, with each entry in a new **Maths Box**.
- $\bar{x} :=$ $\textbf{mean}(\textbf{samp})$ (select $\bar{x}$ from the [∞β°] menu).
- $s :=$ $\textbf{stDevSamp}(\textbf{samp})$ (press [📖][1][S] for **stDevSamp**).
- $cl := \{0.99, 0.95, 0.90, 0.5\}$ (this sets the confidence level)
- $z :=$ $\textbf{invNorm}\left(1 - \dfrac{1 - cl}{2}, 0, 1\right)$ (z-scores for conf. levels)



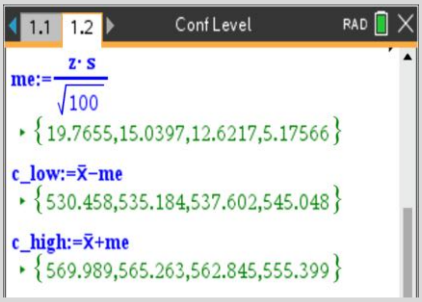
... continued

Note: *For this activity, all required commands can be selected from the catalogue by pressing* [📖][1], *followed by the key for the first letter of the desired command.*

Solution (continued)

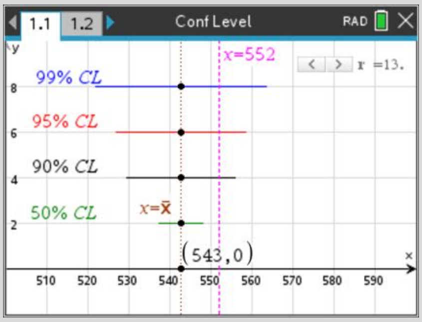
To set up the second part of the simulation on **Notes** page 1.2, enter the following, with each entry in a new **Maths Box**:

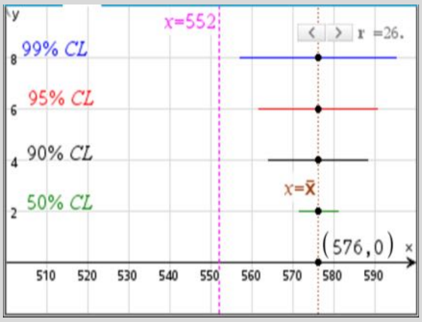
- $\boldsymbol{me} := \dfrac{z \cdot s}{\sqrt{100}}$ (Calculates margins of error for each C.I.)
- $\boldsymbol{c\_low} := \bar{x} - \boldsymbol{me}$ (Lower fence for each C.I.)
- $\boldsymbol{c\_high} := \bar{x} + \boldsymbol{me}$ (Upper fence for each C.I.)



To complete setting up the graphical representations of the confidence intervals, navigate to **Graphs** page 1.1, then:

- Press ctrl G and enter as follows.
- $f1(x) = 8 \mid c\_low[1] \le x \le c\_high[1]$
- $f2(x) = 6 \mid c\_low[2] \le x \le c\_high[2]$
- $f3(x) = 4 \mid c\_low[3] \le x \le c\_high[3]$
- $f4(x) = 2 \mid c\_low[4] \le x \le c\_high[4]$
- Press menu > **Graph Entry/Edit** > **Relation**.
- Enter $x = \bar{x}$ then $x = 552$.
- Press menu > **Geometry** > **Points & Lines** > **Intersection Point(s)**. For each intersection point, click the line $x = \bar{x}$, followed by the horizontal interval.
- Press ctrl menu > **Show/Hide**. Click to hide unwanted labels, then press esc to escape the tool.





Use the slider to draw samples and observe which intervals contain the population mean (i.e. intersect the line $x = 552$).

Answer: The simulation shows that increasing the confidence level increases the margin of error, reflecting a trade-off between confidence and precision. Conversely, reducing the margin of error leads to lower confidence that the interval contains the true population mean μ.

The simulation also illustrates that, for a confidence level of C%, approximately C% of confidence intervals constructed from repeated random samples of a given size will contain the population mean μ, and approximately $(100 - C)\%$ will not.

3.6.4 Hypothesis testing for a population mean

Evaluating evidence: the role of the null hypothesis and p-values

Assume that for a random sample of ten year 5 students at Happyside School the sample mean NAPLAN numeracy score was $\bar{x} = 525$. By comparison, the population mean is $\mu_0 = 503$ and the standard deviation is $\sigma = 65$. Assume that the scores for this population are normally distributed. Is this result statistically significant? A researcher plans to investigate the claim that Happyside students are drawn from a population having a higher mean aptitude than that of the general population. Is the claim supported, or is the observed difference simply the kind of variability that arises from sample to sample?

Question

For the situation described above, answer the following questions to explore whether there is sufficient evidence to suggest that students at Happyside are not drawn from this population.

(a) State the null and alternative hypotheses that could be used to test this claim, for both two-tailed and one-tailed tests.

(b) Simulate taking random samples of **10 students repeatedly** from a population with $\mu_0 = 503$ and $\sigma = 65$. Plot the sample means and use the simulation results to estimate $\mathrm{E}(\bar{X})$, $\mathrm{sd}(\bar{X})$ and the proportion of samples with $\bar{x} \geq 525$.

(c) Use the zTest command to test H_0 against H_1: $\mu > \mu_0$ and determine the associated *p*-value.

Solution

(a) Answer: Null hypothesis: H_0: $\mu = \mu_0 = 503$, where μ is the mean score of the population from which the Happyside students are drawn. The null hypothesis states that the Happyside students come from the same population as all year 5 students. **Alternative hypothesis**: H_1: $\mu \neq 503$ (denial of the null hypothesis). A one-tailed test may be used: H_1: $\mu > 503$ to test whether performance is *better* than the general population.

(b) To simulate drawing 1000 samples of size 10 from $\mathrm{N}(503,\ 65^2)$ and calculating the mean of each sample:

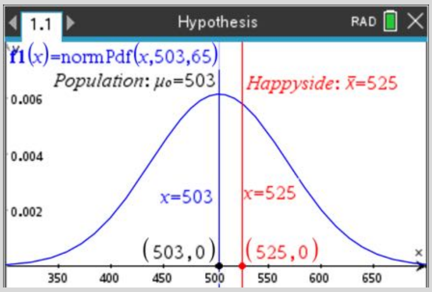
- On a **Notes** page, press [ctrl][M] to insert a **Maths Box** and enter: **seqn(mean(randNorm(503,65,10)),1000)** $\rightarrow \bar{x}$, pressing [ctrl][var] for [sto→], [ctrl][📖] ([∞β°]) for $\bar{x}$, and [📖][1] followed by [S] for **seqn**, [M] for **mean**, and [R] for **randNorm**.

To display the sample means of 1000 samples, add a **Data & Statistics** page, then:

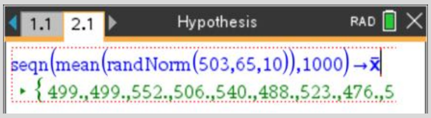
- Press [tab] and select $\bar{x}$ on the horizontal axis.
- Press [menu] > **Analyse** > **Plot Value**. Enter ***v*1:= 525**.

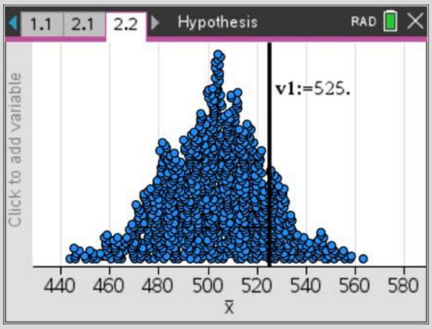
To estimate $\mathrm{E}(\bar{X})$, $\mathrm{sd}(\bar{X})$ and count samples with $\bar{x} \geq 525$:

- On the **Notes** page, press [ctrl][M] to insert a **Maths Box**.
- Enter **countIf(** $\bar{x}$ **,? ≥ 525)**, pressing [📖][1][D] to select **countIf**, [var] for $\bar{x}$, [?!▸] for **?** and [ctrl][=] ([≠≥▸]) for ≥ .



Note: The above can be created by entering the equations and window settings as shown.



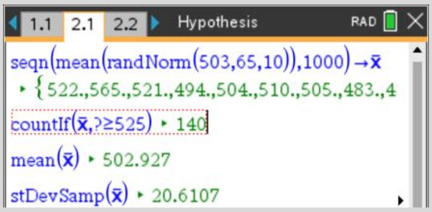


... continued

Solution (continued)

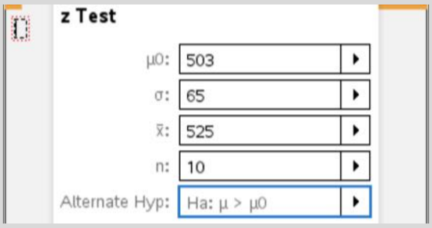
- Enter **mean(** $\bar{x}$ **)** then **stdDevSamp(** $\bar{x}$ **)**, pressing [📖][1] followed by [M] for **mean** or [S] for **stDevSamp.**

Answer: The proportion of samples with $\bar{x} \geq 525$ is approximately 0.14. $\mathrm{E}(\bar{X}) = 503$, $\mathrm{sd}(\bar{X}) \approx 20.6$.



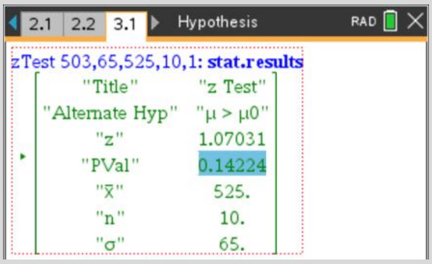
(c) To test H_0 against H_1: $\mu > \mu_0$ using **zTest**, on a **Notes** page:

- Press [ctrl][M] to insert **Maths Box**, then press [menu] > **Calculations > Statistics > Stat Test > z Test**.
- Select Data Input Method: **Stats**. In the dialog box that follows, enter the values as shown.



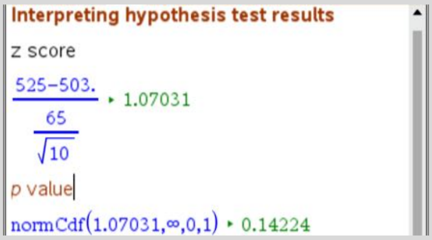
Confirmation of results. z-test is appropriate despite small sample size because the population is normally distributed and therefore $\bar{X}$ is also normally distributed, with:

$\mathrm{E}(\bar{X}) = \mu_0 = 503$ and $\mathrm{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{65}{\sqrt{10}} \approx 20.6$. These theoretical values are consistent with the simulation results.



To confirm the z score and p-value, in **Maths Boxes**:

- Enter $\frac{525-503}{\left(65/\sqrt{10}\right)}$. Therefore $Z \geq 1.07031...$
- Press [📖][1][N] to select **normCdf** and enter as shown, pressing [π▸] to select ∞.



Answer: The p-value ≈ 0.14 is the probability, assuming H_0 is true, of obtaining a result at least as extreme as the observed sample result from Happyside. This is consistent with the simulation: approximately 140/1000 samples had $\bar{x} \geq 525$.

Visualising a two-tailed test: connecting p-values, α and confidence level

In the previous problem, the population is modelled by $N(\mu_0 = 503,\ \sigma^2 = 65^2)$. For the observed sample of size 10, $\bar{x} = 525$. Consider the hypotheses H_0: $\mu = \mu_0 = 503$ and H_1: $\mu \neq 503$.

P-value: A p-value is the probability of observing a value of $\bar{x}$ that is at least as extreme as our observation, if H_0 is true.

Level of Significance, α, is the probability of rejecting H_0 when it is actually true (a Type I error). A common choice is $\alpha = 0.05$. This value sets the threshold for deciding whether to reject H_0.

Decision rule using p-value approach: If $\boldsymbol{p}$**-value** $\leq \boldsymbol{\alpha}$, reject H_0. Otherwise, do not reject H_0.

Critical values, $\pm z_{\alpha/2}$ approach: Using the significance level, α, determine critical values from the sampling distribution. These define the critical region for rejecting H_0. If the test statistic falls in the critical region, reject H_0. Otherwise, do not reject H_0.

Question

(a) Construct a dynamic diagram that shows three key ideas on the graph of N(0,1):
(i) the significance level α, **(ii)** the p-value, **(iii)** the critical values, $\pm z_{\alpha/2}$, for rejecting H_0.

Explain how the choice of α relates to the confidence level and the critical values.

(b) Use the diagram to carry out a two-tailed test for H_0: $\mu = 503$ gainst H_1: $\mu \neq 503$. Assume the population is $N(503, 65^2)$ and that a sample of size $n = 10$ has mean $\bar{x} = 525$.

(i) If $\alpha = 0.05$, decide whether to reject H_0.

(ii) Find the highest confidence level for rejecting H_0.

Solution

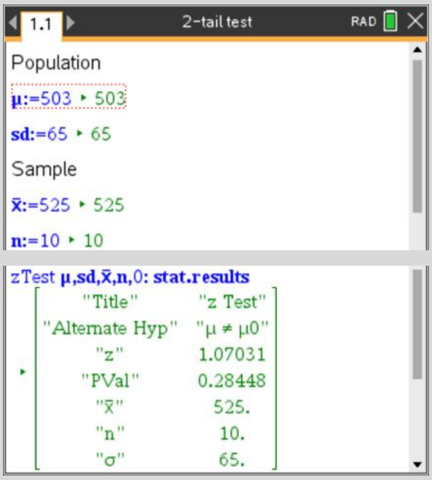
(a) To input population and sample values, on a **Notes** page:

- Press ctrl M to insert **Maths Boxes** and enter the values as shown, pressing ctrl 📖 (∞β°) to select $\boldsymbol{\mu}$ and $\bar{\boldsymbol{x}}$.

To test H_0 against H_1: $\mu \neq \mu_0$ and determine the p-value:

- Press ctrl M to insert a **Maths Box**, then press menu > **Calculations** > **Statistics** > **Stat Tests** > **z Test**.
- Select Data Input Method: **Stats**. In the dialog box, enter the following, pressing var to select the stored variables.

  μ_0: $\boldsymbol{\mu}$, σ: ***sd***, $\bar{x}$ **:** $\bar{\boldsymbol{x}}$, n: $\boldsymbol{n}$, Alternate Hyp: ***Ha***: $\boldsymbol{\mu \neq \mu_0}$.



To plot N(0, 1), add a **Graphs** page to the document, then:

- Enter $\boldsymbol{f1(x)}$ **= normpdf(*x*, 0, 1)**, pressing 📖 1 N to select **normpdf,** then enter XValue: $\boldsymbol{x}$, μ : **0** and σ : **1**.
- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:

XMin = −4	XMax = 4	XScale = 1
YMin = −0.076	YMax = 0.45	YScale = 1

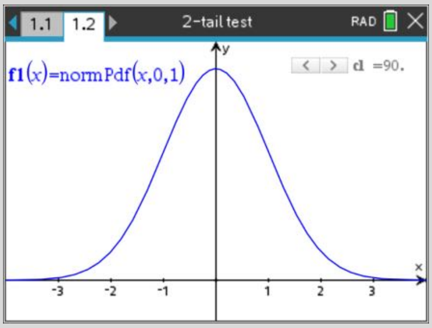
... continued

Solution (continued)

To make the confidence level, ***cl***, dynamically changeable:

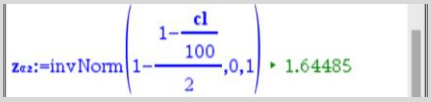
- Press [menu] > **Actions** > **Insert Slider**. Enter the values:

Variable: ***cl***, Value: **90**, Min.: **50**, Max.: **99**, Step Size: **1**.



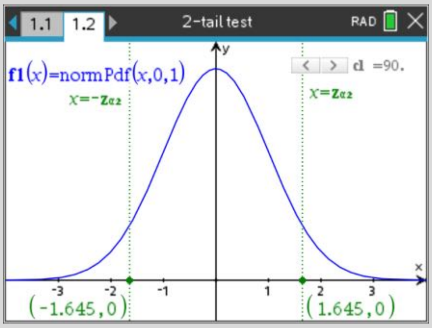
To set the z-test critical values, $\pm z_{\alpha/2}$, associated with ***cl***:

- On the **Notes** page 1.1, insert a **Maths Box** and enter $z_{a2} := \textbf{invNorm}\left(1 - \dfrac{1 - cl/100}{2}, 0, 1\right)$, pressing [subscript key] for subscript, and [catalog] [1] [I] for **invNorm**. Uncheck **Wizards On** to enter using the syntax invNorm(*Area*, *μ*, *σ*).



To plot the critical values, on the **Graphs** page 1.2:

- Press [menu] > **Graph Entry/Edit** > **Relation.** Enter $x = z_{a2}$ then $x = -z_{a2}$. Hover over the vertical lines and press [ctrl] [menu] to customise the colour and attributes.
- Press [P] > **Point**. Click the x-intercepts of the lines.
- Press [esc], hover over the point, press [ctrl] [menu] > **Coordinates and Equations.**

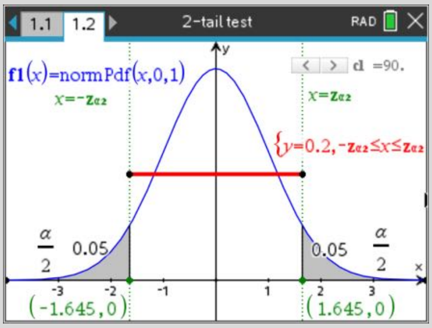


To shade the areas corresponding to the significance level, α:

- Press [menu] > **Analyse Graph** > **Integral**. Click graph ***f*1**, then the point (–4, 0) followed by the point at $(-z_{a2}, 0)$.
- Repeat but click the points $(z_{a2}, 0)$ and (4, 0) instead.

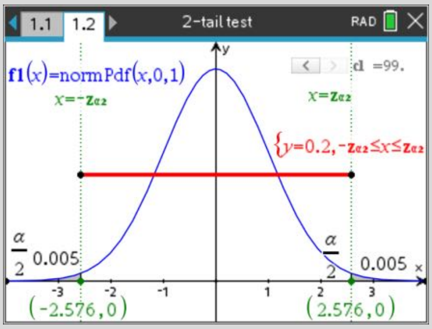
To plot the confidence interval corresponding to ***cl***:

- Press [ctrl] [G] and enter $y = 0.2 \mid -z_{a2} \le x \le z_{a2}$, pressing [ctrl] [=] ([|≠≥▸]) for | and ≤, and [var] to select z_{a2}.



Answer: The relationship between α and cl: A significance level α corresponds to a confidence level, $cl = (1 - \alpha) \times 100$, so that a two-tailed test at, say, $\alpha = 0.1$, is equivalent to constructing a 90% confidence interval for μ.

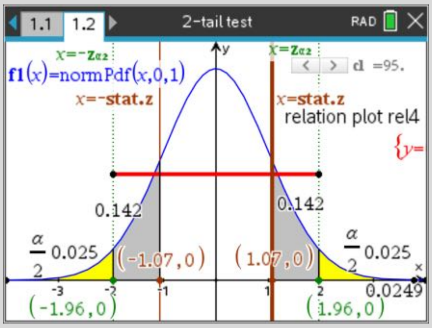
The relationship between α and critical values: For a two-tailed test, the critical regions for rejecting H_0 are shown as the two shaded tails of N(0, 1), with threshold critical values $z = \pm z_{\alpha/2}$, so that each tail has area $\alpha/2$.



... continued

Solution (continued)

(b)(i) To graph the p-value and the corresponding threshold z-scores, which were calculated by the z-test on page 1.1:

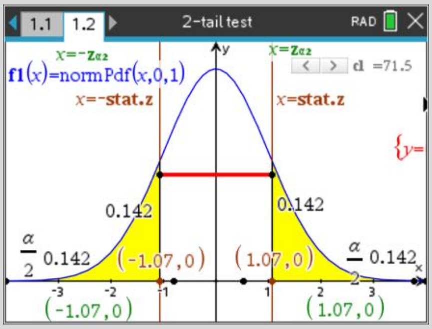


- Press ctrl G and enter x = **stat.z** followed by x = – **stat.z** , pressing var to select **stat.z**.
- Change the slider value, cl, to set the confidence level to **95**, so that $\alpha = 0.05$ (i.e. $\alpha/2 = 0.025$ for each tail).
- Press menu > **Analyse Graph** > **Integral**. Click graph f **1**, then the point (–4, 0) followed by the intersection point of x = – **stat.z** and the x-axis, at (–1.07, 0). Repeat but click the intersection point at (1.07, 0) followed by (4, 0).

Answer: The graph shows that the test statistic, $z \approx 1.07$, does not lie in the rejection region. In particular, the magnitude of the test statistic is less than the critical value, $z_{\alpha/2} \approx 1.96$.

Equivalently, the p-value ≈ 0.284 exceeds $\alpha = 0.05$. Hence, there is insufficient evidence to reject the null hypothesis.

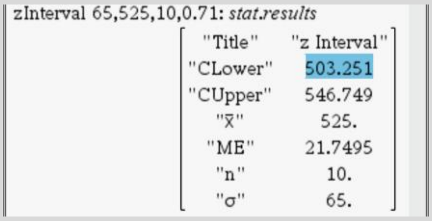
(ii) To find the greatest confidence level for rejecting H_0:



- Adjust/edit the slider value cl so that α is as close as possible to the p-value, and the shaded regions coincide.

Answer: H_0 would be rejected if p-value $\leq \alpha$. Therefore, reject H_0 if $\alpha/2 \geq 0.142\ldots$, and $|z_{\alpha/2}| \leq 1.07$. This occurs when the confidence level, $cl \leq 71$ (correct to the nearest integer).

Therefore, the largest confidence level at which the result would still be statistically significant is approximately 71%. This can be verified by constructing a 71% confidence interval (as shown), which lies entirely above $\mu_0 = 503$.



Interpreting Type I and Type II errors in hypothesis testing

Question

Let $X \sim \mathrm{N}\left(500, 60^2\right)$ be the random variable that represents the scores for a standardised numeracy test. A school claims that its students outperform the general population. A random sample of $n = 36$ students from the school is tested. A hypothesis test is conducted at the 5% significance level, $\alpha = 0.05$: H_0: $\mu = 500$ versus H_1: $\mu > 500$.

(a) Find the critical value of the sample mean, correct to the nearest integer.

(b) Show graphically the probability of making a Type I error and interpret it in context.

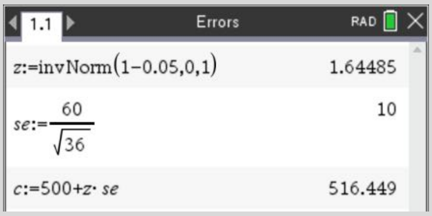
Assume the true mean score of the students at the school is actually $\mu = 520$.

(c) Find the probability of making a Type II error, correct to two decimal places.

Solution

(a) To find the critical value of $\bar{x}$, on a **Calculator** page:

- Input ***z***:=, press [menu] > **Probability** > **Distributions** > **Inverse Normal**. Enter Area: **1 – 0.05**, μ = **0**, σ = **1**.
- To find $\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}}$ (standard error), enter $\boldsymbol{se} := \mathbf{60} / \sqrt{\mathbf{36}}$.
- To find critical value, $c_{\bar{x}}$, enter ***c*****:= 500 +** ***z*** **×** ***se***.



Answer: $c_{\bar{x}} = 516$. Reject H_0 if $\bar{x} > 516$.

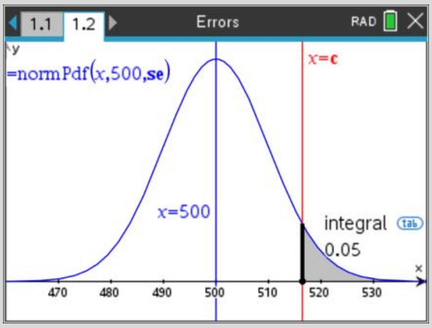
(b) If $X \sim \mathrm{N}\left(500, 60^2\right)$, then $\bar{X} \sim \mathrm{N}\left(500, \left(60/\sqrt{36}\right)^2\right)$.

To plot the distribution of $\bar{X}$, add a **Graphs** page, then:

- Enter ***f*** **1(*x*) = normpdf(*x*, 500, *se*)**.
- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = 460 XMax = 540 XScale = 10
  YMin = −0.007 YMax = 0.043 YScale = 0.1

To graphically verify $\Pr\left(\bar{X} > c_{\bar{x}}\right)$, i.e. Pr(Reject H_0):

- Press [menu] > **Graph Entry/Edit** > **Relation**. Enter ***x*** **=** ***c***.
- Press [menu] > **Analyse Graph** > **Integral**. Click graph ***f*** **1**, then intersection of ***x*** **= c** and the *x*-axis, then point (540, 0).

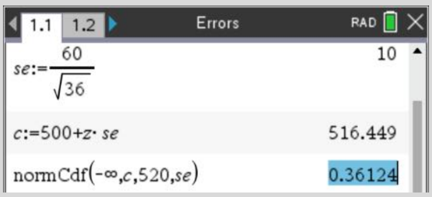


Note: *The entire curve represents outcomes when H_0 is true. The shaded tail shows the values that would lead us to reject H_0. Therefore, the shaded area visually represents making an incorrect rejection of H_0: a Type I error.*

Answer: Pr(Type I error) = α = 0.05. A Type I error is concluding that the school's students outperform the general population when their true mean score is actually 500.

(c) To find $\Pr\left(\bar{X} \le c_{\bar{x}} \mid \mu = 520\right)$, $\bar{X} \sim N\left(520, \left(60/\sqrt{36}\right)^2\right)$:

- On page 1.1, press [📖] [1] [N] > **normCdf.** Enter as shown.



Answer: A Type II error: μ = 520 but fail to reject H_0. Probability of not rejecting H_0 is 0.36.

Appendix: TI-Nspire shortcuts and tips

(Note that a tick (✓) indicates where the shortcut is applicable. MacOS users – substitute CMD for ctrl.)

Shortcut	Handheld	Computer	Glob	Calc	Gra	Geo	L&S	D&S	Note	Data	Prog	Pyth	Result
ctrl A	✓	✓		✓					✓	✓	✓	✓	Select all text
ctrl B	✓	✓									✓	✓	Check Syntax & Store/Save
ctrl C	✓	✓	✓										Copy text
ctrl E	✓	✓							✓				Insert a Chem Box
ctrl F	✓	✓									✓	✓	Find
ctrl G	✓	✓			✓								Toggle Graph entry line
ctrl G	✓	✓					✓				✓	✓	Go to line/cell
ctrl H	✓	✓									✓	✓	Find and replace
ctrl I	✓	✓	✓										Insert page
ctrl J		✓	✓										Take screen capture
ctrl K	✓	✓	✓										Select page (in split screen)
ctrl L	✓	✓		✓	✓	✓			✓		✓		Display variable list
ctrl L	✓	✓					✓						Store or link to variable
ctrl M	✓	✓							✓				Insert Math Box
ctrl N	✓	✓	✓										New document
ctrl O	✓	✓	✓										Open document
ctrl P		✓	✓										Print document
ctrl R	✓	✓					✓						Recalculate
ctrl R	✓	✓									✓	✓	Checks syntax/run program
ctrl S	✓	✓	✓										Save document
ctrl T	✓	✓			✓		✓						Toggle Table/No Table
ctrl T	✓	✓									✓	✓	Comment/Uncomment
ctrl V	✓	✓	✓										Paste text
ctrl W	✓	✓	✓										Close current document
ctrl X	✓	✓	✓										Cut text
ctrl Y	✓	✓	✓										Redo
ctrl Z	✓	✓	✓										Undo

Appendix: TI-Nspire shortcuts and tips (continued)

(Note that a tick (✓) indicates where the shortcut is applicable. MacOS users – substitute CMD for ctrl.)

Shortcut	Handheld	Computer	Global	Calc	Graph	Geom	L&S	D&S	Notes	DataQ	Prog	Python	Result
ctrl 1	✓	✓		✓			✓		✓	✓	✓	✓	Jump to last line
ctrl 2	✓	✓		✓			✓		✓	✓	✓	✓	Jump to end of line/last cell
ctrl 3	✓	✓		✓			✓		✓	✓	✓	✓	Page down
ctrl 4	✓	✓	✓										Merge two pages into split screen.
ctrl 6	✓	✓	✓										Convert split screen into two pages
ctrl 7	✓	✓		✓			✓		✓	✓	✓	✓	Jump to first line
ctrl 8	✓	✓		✓			✓		✓	✓	✓	✓	Jump to start of line/first cell
ctrl 9	✓	✓		✓			✓		✓			✓	Page up
ctrl ␣	✓	✓	✓										Underscore
ctrl tab	✓		✓										Toggle b/w split screen windows
ctrl tab		✓	✓										Toggle b/w open documents
tab	✓	✓		✓	✓		✓	✓	✓	✓	✓		Move through fields or zones
tab	✓	✓			✓								Display graph entry line
tab	✓	✓										✓	Indent
⇧shift ◀▶▲▼	✓	✓	✓										Highlight selected text
⇧shift tab	✓	✓			✓		✓	✓	✓	✓			Move back through fields or zones
⇧shift tab	✓	✓										✓	Remove indent
⇧shift +	✓			✓	✓		✓	✓	✓		✓		Derivative
⇧shift −	✓			✓	✓		✓	✓	✓		✓		Integral
⇧shift ↵	✓			✓	✓			✓	✓		✓		Add a column to a matrix
⇧shift enter		✓		✓	✓			✓	✓		✓		Add a column to a matrix
↵	✓			✓	✓			✓	✓		✓		Add a row to a matrix
Alt enter		✓		✓	✓			✓	✓		✓		Add a row to a matrix
P	✓	✓			✓	✓							Add a point

Appendix: TI-Nspire Shortcuts and Tips (continued)

From the Handheld or Computer Keyboard

To enter this:	Type this shortcut:
π	**pi**
θ	**theta**
∞	**infinity**
≤	**<=**
≥	**>=**
≠	**/=**
⇒ (logical implication)	**=>**
⇔ (logical double implication, XNOR)	**<=>**
→ (store operator)	**=:**
| | (absolute value)	**abs(...)**
√()	**sqrt(...)**
Σ() (Sum template)	**sumSeq(...)**
Π() (Product template)	**prodSeq(...)**
$\sin^{-1}()$, $\cos^{-1}()$, ...	**arcsin(...), arccos(...), ...**
ΔList()	**deltaList(...)**
ΔtmpCnv()	**deltaTmpCnv(...)**

From the Computer Keyboard

To enter this:	Type this shortcut:
e (natural log base e)	**@e**
E (scientific notation)	**@E**
ᵀ (transpose)	**@t**
ʳ (radians)	**@r**
° (degrees)	**@d**
ᵍ (gradians)	**@g**
∠ (angle)	**@<**
▶ (conversion)	**@>**
▶ Decimal, ▶ approxFraction(), and so on.	**@>Decimal, @>approxFraction(), and so on.**
*c*1, *c*2, … (constants)	**@c1, @c2, …**
*n*1, *n*2, … (integer constants)	**@n1, @n2, …**
i (imaginary constant)	**@i**

Useful functions/commands available in the Catalog not available in the menus.

Function/Command name	Function/Command purpose
and	Boolean 'and', useful for specifying restrictions.
domain(expr,var)	Displays the domain of a function.
euler(…)	Generates a table of values using Euler's method.
isPrime(…)	Displays 'true' if prime and 'false' if composite.
true	Displays 'true' if two expressions are equivalent.

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data analysis

$$E(aX_1 + b) = aE(X_1) + b$$

probability

$$Var(X_1 + X_2 + \ldots + X_n) = n\sigma^2$$

statistics