

TI in Focus: AP[®] Calculus

2017 AP[®] Calculus Exam: BC-6
Scoring Guidelines

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Outline

- (1) Free Response Question
- (2) Scoring Guidelines
- (3) Student performance
- (4) Interpretation
- (5) Common errors
- (6) Specific scoring examples

$$f(0) = 0$$

$$f'(0) = 1$$

$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

6. A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f .

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t) dt$.

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the n th-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is

the function defined in part (c). Use the alternating series error bound to show that

$$\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| < \frac{1}{500}.$$

$$\begin{aligned}
 \text{(a)} \quad & f(0) = 0 \\
 & f'(0) = 1 \\
 & f''(0) = -1(1) = -1 \\
 & f'''(0) = -2(-1) = 2 \\
 & f^{(4)}(0) = -3(2) = -6
 \end{aligned}$$

The first four nonzero terms are

$$0 + 1x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

The general term is $\frac{(-1)^{n+1}x^n}{n}$.

$$\text{(b) For } x = 1, \text{ the Maclaurin series becomes } \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

The series does not converge absolutely because the harmonic series diverges.

The series alternates with terms that decrease in magnitude to 0 and therefore the series converges conditionally.

$$3 : \begin{cases} 1 : f''(0), f'''(0), \text{ and } f^{(4)}(0) \\ 1 : \text{verify terms} \\ 1 : \text{general term} \end{cases}$$

2 : converges conditionally
with reason

$$\begin{aligned}
 \text{(c)} \quad \int_0^x f(t) dt &= \int_0^x \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + \frac{(-1)^{n+1} t^n}{n} + \cdots \right) dt \\
 &= \left[\frac{t^2}{2} - \frac{t^3}{3 \cdot 2} + \frac{t^4}{4 \cdot 3} - \frac{t^5}{5 \cdot 4} + \cdots + \frac{(-1)^{n+1} t^{n+1}}{(n+1)n} + \cdots \right]_{t=0}^{t=x} \\
 &= \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots + \frac{(-1)^{n+1} x^{n+1}}{(n+1)n} + \cdots
 \end{aligned}$$

3 : $\begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

(d) The terms alternate in sign and decrease in magnitude to 0. By the alternating series error bound, the error $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right|$ is bounded

by the magnitude of the first unused term, $\left| -\frac{(1/2)^5}{20} \right|$.

Thus, $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| \leq \left| -\frac{(1/2)^5}{20} \right| = \frac{1}{32 \cdot 20} < \frac{1}{500}$.

1 : error bound

Student Performance

- (1) Part (a): pattern recognition good - four terms and general term; some did not know the formula for generating a Maclaurin series; difficulty in generating higher-order derivatives using recursive definition.
- (2) Part (b): if recognition of need for alternating harmonic series and harmonic series, then performed well; conditional convergence required analysis of both series; reasons for convergence or divergence insufficient or incorrect; difference between absolute and conditional convergence.
- (3) Part (c): good with elementary integration techniques to earn first two points; difficulty in integrating the general term; additional constant of integration often added.
- (4) Part (d): recognized the correct term and the use of $x = 1/2$; some considered f rather than g ;
found $\frac{1}{640}$ but did not make the connection to the error.

Part (a) 1: $f''(0)$, $f'''(0)$, and $f^{(4)}(0)$

(1) May be listed:

$$f''(0) = -1; \quad f'''(x) = 2 = 2!; \quad f^{(4)}(0) = -6 = -3!$$

(2) May be combined with factorials as coefficients:

$$C_2 = \frac{-1}{2}; \quad C_3 = \frac{2}{3!}; \quad C_4 = \frac{-6}{4!}$$

(3) Derivatives must be stated numerically.

(4) Values may be embedded in verification.

Part (a) 1: verify terms

Must show all three parts (derivative, factorial, x^k) of the terms in a form which is not yet completely simplified.

$$(1) \quad 1 \cdot x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 \qquad 1 - 1 - ?$$

$$(2) \quad 1 \cdot x - \frac{1}{2!}x^2 + \frac{2!}{3!}x^3 - \frac{3!}{4!}x^4 \qquad 1 - 1 - ?$$

$$(3) \quad x - \frac{1}{2}x^2 + \frac{2}{3 \cdot 2}x^3 - \frac{3 \cdot 2}{4!}x^4 \qquad 1 - 1 - ?$$

$$(4) \quad x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \text{ (alone)} \qquad 0 - 0 - ?$$

$$(5) \quad 1 \cdot x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{6}{4!}x^4 \text{ (alone)} \qquad 0 - 0 - ?$$

Part (a) 1: general term

Examples:

$$(1) \frac{(-1)^{n+1}x^n}{n} \qquad ? - ? - 1$$

$$(2) \sum \frac{(-1)^n x^{n+1}}{n+1} \qquad ? - ? - 1$$

$$(3) \frac{(-1)^{n-1}(n-1)!x^n}{n!} \qquad ? - ? - 1$$

$$(4) -\frac{(-1)^n x^n}{n} \qquad ? - ? - 1$$

$$(5) -\frac{(-x)^n}{n} \qquad ? - ? - 1$$

Part (a) 1: general term

Insufficient:

$$(1) f^{(n)}(0) \frac{x^n}{n!} \qquad ? - ? - 0$$

$$(2) \frac{-(n-1)f^{(n-1)}(0)x^n}{n!} \qquad ? - ? - 0$$

Incorrect:

$$(1) \frac{-1^{n+1}x^n}{n} \qquad ? - ? - 0$$

$$(2) \frac{(-1)^n x^n}{n} \qquad ? - ? - 0$$

$$(3) \frac{(-1)^{n+2}x^n}{n} \qquad ? - ? - 0$$

$$(4) \frac{(-1)^{n+1}x^n}{n!} \qquad ? - ? - 0$$

Part (b) 2: converges conditionally with reason

- (1) Must use $x = 1$ or no points.
- (2) To earn both points, must have three parts:
 - Convergence of $\sum \frac{(-1)^{n+1}}{n}$
 - Divergence of $\sum \frac{1}{n}$
 - Conclusion: conditional convergence.
- (3) Must show work in an analysis that produces conditional convergence.
- (4) Presentation can be with or without \sum .

Part (b) 2: converges conditionally with reason

Sufficient examples for $\sum \frac{(-1)^{n+1}}{n}$

(1) $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, $\frac{1}{n} > \frac{1}{n+1}$ therefore $\sum \frac{(-1)^{n+1}}{n}$ converges.

(2) $\frac{(-1)^{n+1}}{n}$ converges by AST.

(3) $\sum \frac{(-1)^{n+1}}{n}$ is the alternating harmonic and converges.

(4) Alternating harmonic converges.

(5) Alternating harmonic converges conditionally.

Part (b) 2: converges conditionally with reason

Sufficient examples for $\sum \frac{1}{n}$

(1) $\sum \frac{1}{n}$ diverges.

(2) Harmonic series diverges.

(3) Harmonic series.

(4) $\frac{1}{n}$ diverges by p -series.

(5) $\sum \frac{1}{n}$ diverges by Integral Test.

Part (b) 2: converges conditionally with reason

To earn 1/2:

Two of three (reasons and conclusion) correct with at most one mistake (solution must be consistent).

- (1) Both reasons correct and either a missing or incorrect conclusion.
- (2) One reason correct and correct conclusion.
Other reason is missing or insufficient.
- (3) If both conclusions are diverge, or both converge, then only eligible for one point if they present one correct reason and consistent conclusion.

Part (c) 1: two terms

Any two of the four terms.

Usually the first two, but may be the middle two, or the two positive terms.

Part (c) 1: remaining terms

$$(1) \frac{x^2}{2} - \frac{x^3}{3 \cdot 2} + \frac{x^4}{4 \cdot 3} - \frac{x^5}{5 \cdot 4} \qquad 1 - 1 - ?$$

$$(2) \frac{x^2}{2} - \frac{x^3}{3!} + \frac{2!x^4}{4!} - \frac{3!x^5}{5!} \qquad 1 - 1 - ?$$

$$(3) \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{20} \qquad 1 - 0 - ?$$

Part (c) 1: remaining terms

- (1) $+C$ is incorrect and does not earn the second point.

This may appear in various locations.

$$\frac{x^2}{2!} - \frac{x^3}{3!} + \frac{2x^4}{4!} - \frac{6x^5}{5!} + C \qquad 1 - 0 - ?$$

- (2) Sign error on all terms may earn one of the first two points.

$$-\frac{x^2}{2} + \frac{x^3}{3 \cdot 2} - \frac{x^4}{4 \cdot 3} + \frac{x^5}{5 \cdot 4} \qquad 0 - 1 - ?$$

- (3) If more than our four terms are presented, ignore terms with degrees higher than 5.

Part (c) 1: general term

Many acceptable forms:

$$(1) \frac{(-1)^{n-1}x^{n-1}}{(n-1)(n-2)} \qquad ? - ? - 1$$

$$(2) \frac{(-x)^n}{n(n-1)} \qquad ? - ? - 1$$

$$(3) \frac{(-1)^n x^{n+2}}{(n+2)(n+1)} \qquad ? - ? - 1$$

$$(4) \frac{(-1)^n x^{n+2}}{n^2 + 3n + 2} \qquad ? - ? - 1$$

$$(5) \frac{(-1)^n x^n}{n^2 - n} \qquad ? - ? - 1$$

Part (d) 1: error bound

- (1) The student must indicate two items:
 - They are presenting an *error*.
 - The numeric value is less than $\frac{1}{500}$.
- (2) Unsupported $\frac{1}{640}$ does not earn the point.
- (3) There must be an explicit connection to the error.
- (4) A connection to $\frac{1}{500}$ is desired, but not required.

Part (d) 1: error bound

Some sufficient connections to error:

(1) The error is less than $\frac{(1/2)^5}{20}$ 1

(2) The AS error is the first neglected term. ?

(3) error $< |t_5|$ (t_5 is the fifth degree term) ?

(4) The AS error = absolute value of the next term. ?

(5) error $< \frac{1}{2^5 \cdot 20}$ 1

Part (d) 1: error bound

Notes:

- (1) If LaGrange error bound: does not earn the point.
- (2) Common error: use the fifth term rather than the fifth degree term.

$$\frac{1}{6 \cdot 5 \cdot 2^6} = \frac{1}{30 \cdot 64} = \frac{1}{1920} < \frac{1}{500}$$

- (3) Import incorrect general term from part (c):
 - Series must alternate.
 - Error must be the term immediately following the 4th degree term.
 - Error must be less than $\frac{1}{500}$.

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