

TI in Focus: AP[®] Calculus

2018 AP[®] Calculus Exam: BC-6

Technology Solutions and Problem Extensions

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Outline

- (1) Free Response Question
- (2) Scoring Guidelines
- (3) Solutions using technology
- (4) Problem Extensions

6. The Maclaurin series for $\ln(1+x)$ is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \frac{x^n}{n} + \cdots.$$

On its interval of convergence, this series converges to $\ln(1+x)$. Let f be the function defined by

$$f(x) = x \ln\left(1 + \frac{x}{3}\right).$$

- (a) Write the first four nonzero terms and the general term of the Maclaurin series for f .
- (b) Determine the interval of convergence of the Maclaurin series for f . Show the work that leads to your answer.
- (c) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Use the alternating series error bound to find an upper bound for $|P_4(2) - f(2)|$.

(a) The first four nonzero terms are $\frac{x^2}{3} - \frac{x^3}{2 \cdot 3^2} + \frac{x^4}{3 \cdot 3^3} - \frac{x^5}{4 \cdot 3^4}$.

The general term is $(-1)^{n+1} \frac{x^{n+1}}{n \cdot 3^n}$.

2 : $\begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

$$(b) \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+2} x^{n+2}}{(n+1)(3^{n+1})}}{\frac{(-1)^{n+1} x^{n+1}}{n \cdot 3^n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{-x}{3} \cdot \frac{n}{(n+1)} \right| = \left| \frac{x}{3} \right|$$

$$\left| \frac{x}{3} \right| < 1 \text{ for } |x| < 3$$

Therefore, the radius of convergence of the Maclaurin series for f is 3.

— OR —

The radius of convergence of the Maclaurin series for $\ln(1+x)$ is 1, so the series for $f(x) = x \ln\left(1 + \frac{x}{3}\right)$ converges absolutely for $\left|\frac{x}{3}\right| < 1$.

$$\left| \frac{x}{3} \right| < 1 \Rightarrow |x| < 3$$

Therefore, the radius of convergence of the Maclaurin series for f is 3.

5 : {

- 1 : sets up ratio
- 1 : computes limit of ratio
- 1 : radius of convergence
- 1 : considers both endpoints
- 1 : analysis and interval of convergence

— OR —

5 : {

- 1 : radius for $\ln(1+x)$ series
- 1 : substitutes $\frac{x}{3}$
- 1 : radius of convergence
- 1 : considers both endpoints
- 1 : analysis and interval of convergence

When $x = -3$, the series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-3)^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{3}{n}$, which diverges by comparison to the harmonic series.

When $x = 3$, the series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{n}$, which converges by the alternating series test.

The interval of convergence of the Maclaurin series for f is $-3 < x \leq 3$.

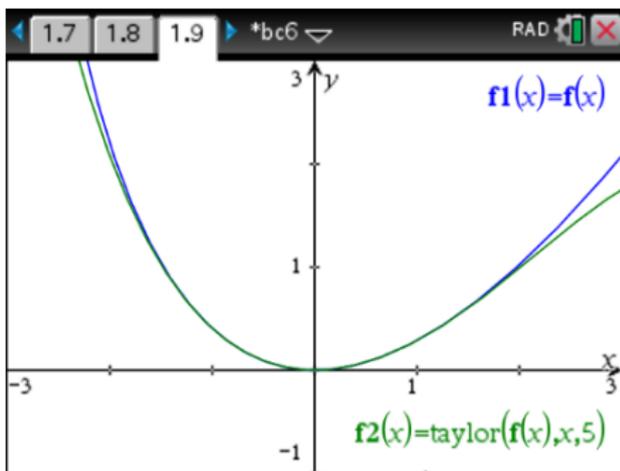
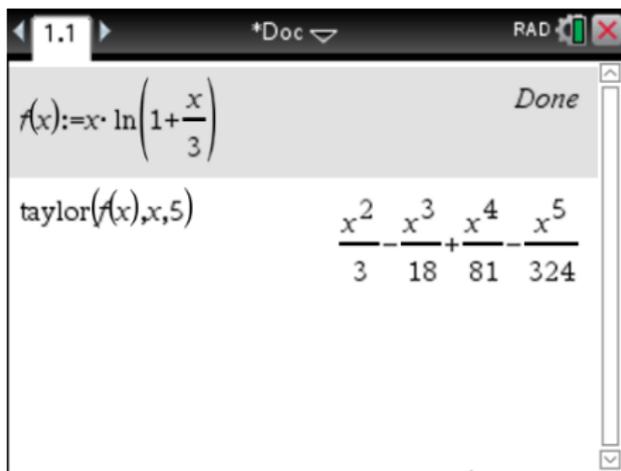
- (c) By the alternating series error bound, an upper bound for $|P_4(2) - f(2)|$ is the magnitude of the next term of the alternating series.

$$|P_4(2) - f(2)| < \left| -\frac{2^5}{4 \cdot 3^4} \right| = \frac{8}{81}$$

- 2 : $\begin{cases} 1 : \text{uses fifth-degree term} \\ \quad \text{as error bound} \\ 1 : \text{answer} \end{cases}$

Part (a)

First for terms and the general term of the Maclaurin series for f .



The general term is $(-1)^{n+1} \frac{x^{n+1}}{n \cdot 3^n}$

Part (b)

- Use the Ratio Test to determine the radius of convergence R .
- Define the n th term of the series.
- Compute the quotient for the Ratio Test.
- Consider the limit associated with the Ratio Test.

The calculator screen shows the definition of the n th term of the series:

$$a(n,x) := \frac{(-1)^{n+1} \cdot x^{n+1}}{n \cdot 3^n}$$

The word "Done" is visible in the top right corner. Below the definition, the ratio test quotient is displayed:

$$\left| \frac{a(n+1,x)}{a(n,x)} \right| = \frac{\left| \frac{n \cdot x}{n+1} \right|}{3}$$

The calculator screen shows the limit of the ratio test quotient as n approaches infinity:

$$\lim_{n \rightarrow \infty} \left(\frac{\left| \frac{n \cdot x}{n+1} \right|}{3} \right) = \frac{|x|}{3}$$

Below the limit, the inequality for convergence is shown:

$$\text{solve} \left(\frac{|x|}{3} < 1, x \right) \quad -3 < x < 3$$

Part (b)

Consider the endpoints.

The calculator screen shows two series:

$$\sum_{n=1}^{\infty} (a(n,-3)) \qquad 3 \cdot \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)$$

$$\sum_{n=1}^{\infty} (a(n,3)) \qquad -3 \cdot \sum_{n=1}^{\infty} \left(\frac{\cos(n \cdot \pi)}{n}\right)$$

The calculator screen shows the sequence of terms for the series:

$$\text{seq}(a(n,-3),n,1,10)$$

$$\left\{ 3, \frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}, \frac{1}{2}, \frac{3}{7}, \frac{3}{8}, \frac{1}{3}, \frac{3}{10} \right\}$$

$$\text{seq}(a(n,3),n,1,10)$$

$$\left\{ 3, \frac{-3}{2}, 1, \frac{-3}{4}, \frac{3}{5}, \frac{-1}{2}, \frac{3}{7}, \frac{-3}{8}, \frac{1}{3}, \frac{-3}{10} \right\}$$

The interval of convergence of the Maclaurin series for f is $-3 < x \leq 3$.

Example 1 Mainstream Maclaurin

Write the first four nonzero terms and the general term of the Maclaurin series for f . Determine the interval of convergence of the Maclaurin series for f .

(a) $f(x) = \frac{x}{3} \ln(1 + x)$

(b) $f(x) = x \ln(1 + 3x)$

(c) $f(x) = x^2 \ln(1 + x^2)$

(d) $f(x) = \ln\left(1 - \frac{x^2}{3}\right)$

Part (c)

By the alternating series error bound, an upper bound for $|P_4(2) - f(2)|$ is the magnitude of the next term of the alternating series.

A screenshot of a TI-84 Plus calculator interface. The top status bar shows the mode set to 'RAD' and the window size as '*bc6'. The main display area shows the expression $|a(4,2)|$ on the left and the fraction $\frac{8}{81}$ on the right. The calculator interface includes a navigation bar with buttons for 1.4, 1.5, and 1.6, and a scroll bar on the right side.

Example 2 Error Bounds

Suppose the first four non-zero terms are used to approximate the sum of the series. Find an upper bound on the error of estimation.

$$(a) \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$$

$$(b) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$$

$$(c) \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 3}{2^n}$$

$$(d) \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n^2)}$$

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