

TI in Focus: AP[®] Calculus

2019 AP[®] Calculus Exam: BC-2

Arc Length

Stephen Kokoska

Professor, Bloomsburg University

Former AP[®] Calculus Chief Reader

Outline

- (1) Arc Length Derivation
- (2) Application
- (3) Technology
- (4) Examples

The Formula

- Objective: Find the length of a polar curve described by $r = f(\theta)$, $a \leq \theta \leq b$.
- **Theorem:** If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, where f' and g' are continuous on $[a, b]$ and C is traversed exactly once as t increases from a to b , then the length of C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

- Write the parametric equations of the polar curve:

$$x = r \cos \theta = f(\theta) \cos \theta \quad y = r \sin \theta = f(\theta) \sin \theta$$

- Differentiate these expressions with respect to θ .

$$\frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta \quad \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta$$

The Formula (Continued)

- Consider the sum of the squares of these two terms.

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \cos^2 \theta - 2r \frac{dr}{d\theta} \cos \theta \sin \theta + r^2 \sin^2 \theta \\ &\quad + \left(\frac{dr}{d\theta}\right)^2 \sin^2 \theta + 2r \frac{dr}{d\theta} \sin \theta \cos \theta + r^2 \cos^2 \theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2\end{aligned}$$

The Formula (Continued)

- If f' is continuous:

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

- Using the equality, the length of a curve described by the polar equation $r = f(\theta)$, $a \leq \theta \leq b$, is

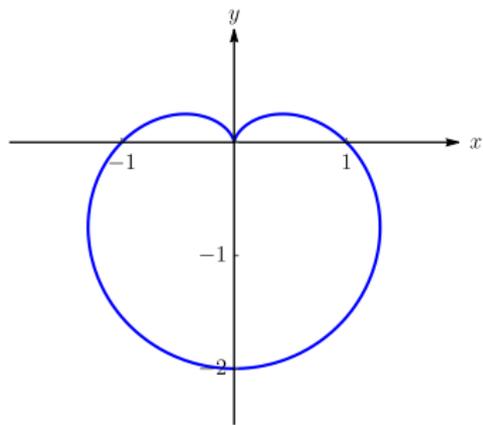
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Example 1 The Shape of my Heart

Find the length of the cardioid described by $r = 1 - \sin \theta$.

Solution

Here is a graph of the cardioid.



The complete graph is traced out for $0 \leq \theta \leq 2\pi$.

Solution (Continued)

$$r = 1 - \sin \theta \quad \Rightarrow \quad \frac{dr}{d\theta} = -\cos \theta$$

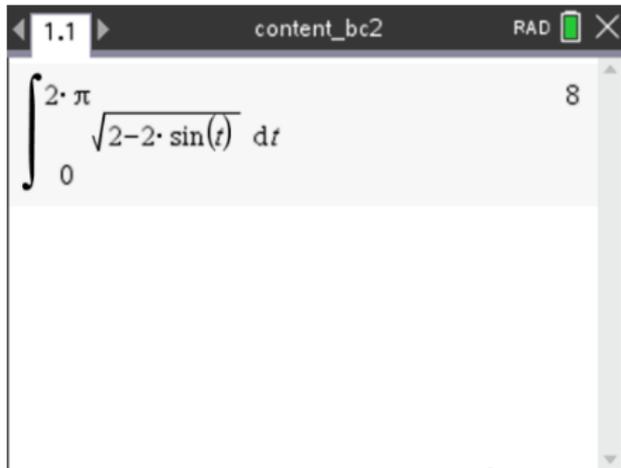
$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_0^{2\pi} \sqrt{(1 - \sin \theta)^2 + (-\cos \theta)^2} d\theta$$

$$= \int_0^{2\pi} \sqrt{1 - 2\sin \theta + \sin^2 \theta + \cos^2 \theta} d\theta$$

$$= \int_0^{2\pi} \sqrt{2 - 2\sin \theta} d\theta$$

$$= \int_0^{2\pi} \sqrt{2}\sqrt{1 - \sin \theta} \cdot \frac{\sqrt{1 + \sin \theta}}{\sqrt{1 + \sin \theta}} d\theta = \dots$$

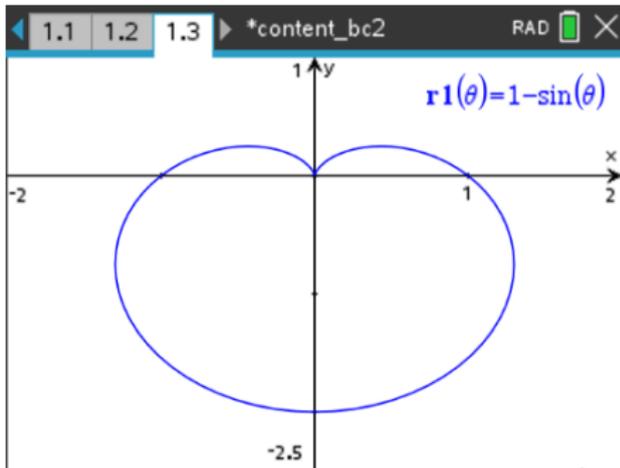
Solution (Continued)



TI-84 Plus calculator screen showing the integral calculation:

$$\int_0^{2\pi} \sqrt{2-2\sin(t)} dt$$

The result of the integral is 8.



Example 2 Closer to the Heart

Consider the graph of the polar curve described by

$$r = \frac{\sin \theta \sqrt{|\cos \theta|}}{\sin \theta + (7/5)} - 2 \sin \theta + 2 \quad 0 \leq \theta \leq 2\pi$$

- (a) Sketch the graph of the polar equation.
- (b) Find the length of the curve described by the polar equation.

Example 3 The Lengths

Find the length of the curve described by the polar equation.

(a) $r = 3 \cos \theta, \quad 0 \leq \theta \leq \pi$

(b) $r = 3^\theta, \quad 0 \leq \theta \leq 2\pi$

(c) $r = \theta^2, \quad 0 \leq \theta \leq 2\pi$

(d) $r = 3 + 2 \cos \theta, \quad 0 \leq \theta \leq 2\pi$

education.ti.com