TI in Focus: AP® Calculus

2019 AP® Calculus Exam: AB-4/BC-4
Scoring Guidelines

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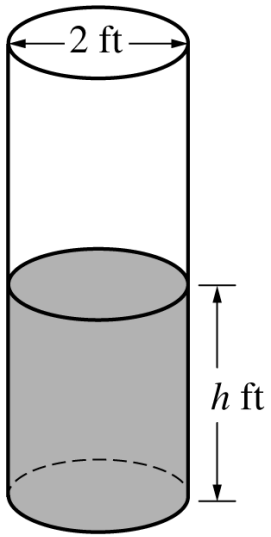
Outline

(1) Free Response Question
(2) Scoring Guidelines
(3) Student performance
(4) Interpretation
(5) Common errors
(6) Specific scoring examples



4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

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(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

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(a) $V = \pi r^2 h = \pi (1)^2 h = \pi h$

$$\left.\frac{dV}{dt}\right|_{h=4} = \pi \left.\frac{dh}{dt}\right|_{h=4} = \pi\left(-\frac{1}{10}\sqrt{4}\right) = -\frac{\pi}{5} \text{ cubic feet per second}$$

$$2 : \begin{cases} 1 : \dfrac{dV}{dt} = \pi \dfrac{dh}{dt} \\ 1 : \text{answer with units} \end{cases}$$

(b) $$\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10}\sqrt{h}\right) = \frac{1}{200}$$

Because $\dfrac{d^2h}{dt^2} = \dfrac{1}{200} > 0$ for $h > 0$, the rate of change of the height is increasing when the height of the water is 3 feet.

$$3 : \begin{cases} 1 : \dfrac{d}{dh}\left(-\dfrac{1}{10}\sqrt{h}\right) = -\dfrac{1}{20\sqrt{h}} \\ 1 : \dfrac{d^2h}{dt^2} = -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{dh}{dt} \\ 1 : \text{answer with explanation} \end{cases}$$

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(c) $\dfrac{dh}{\sqrt{h}} = -\dfrac{1}{10}\,dt$

$$\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10}\,dt$$

$$2\sqrt{h} = -\frac{1}{10}t + C$$

$$2\sqrt{5} = -\frac{1}{10}\cdot 0 + C \Rightarrow C = 2\sqrt{5}$$

$$2\sqrt{h} = -\frac{1}{10}t + 2\sqrt{5}$$

$$h(t) = \left(-\frac{1}{20}t + \sqrt{5}\right)^2$$

$$4 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ \quad\ \text{and uses initial condition} \\ 1 : h(t) \end{cases}$$

Note: $0/4$ if no separation of variables

Note: max $2/4$ [1-1-0-0] if no constant of integration

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Student Performance

Part (a)

- Many students recognized that this was a related rates problem. However, some were not able to correctly differentiate $\pi r^2 h$ with respect to t.
- Common errors:

  Radius $= 2$ (instead of 1)

  Derivative of $\pi r^2 h$ with respect to t: $2\pi r \frac{dh}{dt}$ or $2\pi r h + \pi r^2 \frac{dh}{dt}$

  Use of an incorrect strategy involving proportions: $r = \frac{4}{h}$

Part (b)

- Many students evaluated $-\frac{1}{10}\sqrt{h}$ for $h = 3$.
- Some students only differentiated $-\frac{1}{10}\sqrt{h}$ with respect to h; no chain rule.

Student Performance

Part (c)

- Many students correctly separated the variables, and integrated both sides.
- Most students correctly included a constant of integration.
- Common errors:

$$\int \frac{1}{\sqrt{h}}\,dh = \ln\sqrt{h}$$

$$\left(-\frac{t}{20}+C\right)^2 = \frac{t^2}{400}+C^2$$

Part (a) 1: $\frac{dV}{dt} = \pi \frac{dh}{dt}$

These expressions earn the first point.

- $\pi(1)^2 \dfrac{dh}{dt}$ $\quad \pi \dfrac{dh}{dt}$ $\quad \pi\left(-\dfrac{1}{10}\sqrt{4}\right)$
- $2\pi rh \dfrac{dr}{dt} + \pi r^2 \dfrac{dh}{dt}$ $\quad 2\pi hr' + \pi r^2 h'$
- $\pi \cdot 1 \dfrac{dh}{dt}$ $\quad \pi r^2 \dfrac{dh}{dt}$ $\quad \pi(2)^2 \dfrac{dh}{dt}$
- $\pi\left(-\dfrac{1}{10}\sqrt{h}\right)$ $\quad \pi\, dh$
- $\begin{cases} V = 4\pi h \\ 4\pi \frac{dh}{dt} \end{cases}$ $\quad 4\pi \dfrac{dh}{dt}$ alone not sufficient

Part (a) 1: $\frac{dV}{dt} = \pi \frac{dh}{dt}$

These expressions do not earn the first point.

- $2\pi rh + \pi r^2 \dfrac{dh}{dt}$
- $2\pi r \dfrac{dh}{dt}$

Part (a) 1: answer with units

Must be our answer with correct units.

- $-\dfrac{\pi}{5}$ ft^3/sec $\qquad -\dfrac{3.14}{5}$ ft^3/sec OK
- Bald answer: 0 - 0
- Some reasonable work and answer with units: 0 - 1

Part (a) Examples

(1)	$\pi \dfrac{dh}{dt}$	$\pi - \dfrac{1}{5}$ ft^3/sec	1 - 0
(2)	$-\dfrac{1}{10}\sqrt{4}$	$-\dfrac{1}{5}$ ft^3/sec	0 - 0
(3)	$V = \pi h$	$\dfrac{dV}{dt} = \pi\left(\dfrac{-1}{10}\sqrt{h}\right)$ $\quad \dfrac{dV}{dt} = \pi\left(-\dfrac{1}{10}\sqrt{4}\right)$ ft^3/sec	1 - 1
(4)	$\pi\left(\dfrac{-1}{10}\sqrt{4}\right)$	$V = -\dfrac{\pi}{5}$ ft^3/sec	1 - 0
(5)	$-\dfrac{\sqrt{4}}{10}$	$-\dfrac{\pi}{5}$ ft^3/sec	0 - 1

Part (b) 1: $\frac{d}{dh}\left(-\frac{1}{10}\sqrt{h}\right) = -\frac{1}{20\sqrt{h}}$

This is a power rule point.

$$-\frac{1}{2}\cdot\frac{1}{10\sqrt{h}} \qquad -\frac{1}{20}h^{-\frac{1}{2}} \qquad -\frac{1}{20\sqrt{3}}$$

Part (b) 1: $\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}}\cdot\frac{dh}{dt}$

This is a chain rule point.

These expressions earn the first two points.

$$-\frac{1}{20\sqrt{h}}\cdot\frac{-1\sqrt{h}}{10} \qquad -\frac{1}{20\sqrt{3}}\cdot\frac{-1\sqrt{3}}{10} \qquad -\frac{1}{20\sqrt{h}}\cdot\frac{-1\sqrt{3}}{10}$$

$$\frac{1}{20\sqrt{h}}\cdot\frac{-1\sqrt{h}}{10}$$ 0 - 1

$$-\frac{1}{10\sqrt{h}}\cdot\frac{-1\sqrt{h}}{10}$$ 0 - 1

Part (b) 1: answer with explanation

Eligibility: Earned the second point and the second derivative appears.

Answer: increasing or consistent

Explanation means supporting work.

(1) $\dfrac{1}{200} \qquad -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{-1\sqrt{h}}{10} \qquad \dfrac{d^2h}{dt^2} > 0$

(2) Second derivative positive

(3) Derivative of $\dfrac{dh}{dt}$ is positive

Part (b) Examples

(1) $\dfrac{d^2h}{dt^2} = -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{-1\sqrt{h}}{10}$ increasing 1 - 1 - 1

(2) $\dfrac{d^2h}{dt^2} = \dfrac{1}{20\sqrt{h}} \cdot \dfrac{-1\sqrt{h}}{10}$ decreasing 0 - 1 - 1

(3) $\dfrac{d^2h}{dt^2} = -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{dh}{dt}$ increasing 1 - 1 - 0

(4) $h'' = -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{dh}{dt} > 0$ inc. 1 - 1 - 1

(5) $-\dfrac{1}{10\sqrt{h}} \cdot \dfrac{-1}{10}\sqrt{h}$

Increasing because the second derivative > 0 0 - 1 - 1

Part (c) 1: separation of variables

(1) $\displaystyle\int \frac{-10\,dh}{\sqrt{h}} = \int dt$ 1

(2) $dh\,10h^{-1/2} = -dt$ 1

(3) $\displaystyle\frac{1}{\sqrt{h}}\frac{dh}{dt} = -\frac{1}{10}$ 1

(4) $\displaystyle\int \frac{1}{\sqrt{h}} = \int -\frac{1}{10}$ 1

(5) $\displaystyle\frac{1}{\sqrt{h}} = -\frac{1}{10}$?

(6) $\displaystyle dh = -\frac{1}{10}\sqrt{h}\,dt$ 0/4

Part (c) 1: antiderivatives

Eligibility: $h^{-1/2}\,dh = k\,dt$ where $k = \pm\frac{1}{10},\ \pm 1,\ \pm 10$

(1) $\dfrac{1}{\sqrt{h}}\,dh = -\dfrac{1}{10}\,dt$

$2\sqrt{h} = -\dfrac{1}{10}t$ 1 - 1

(2) $\dfrac{1}{\sqrt{h}}\,dh = -dt$

$2\sqrt{h} = -t$ 0 - 1

(3) $2\sqrt{h} = -\dfrac{1}{10}t$ 1 - 1

Part (c) 1: constant of integration and uses initial condition

Eligibility:

(1) At least one of the antiderivatives correct

(2) h is not logarithmic nor linear

(3) $+C$ must be timely

(4) $t = 0$ and $h = 5$ used before any bad algebra

Part (c) Examples

$-20\sqrt{h} = t + C$

$\sqrt{h} = -\frac{t}{20} + C$

$h = (-\frac{t}{20} + C)^2$

$\sqrt{5} = C$

1 - 1 - 1 - ?

$-20\sqrt{h} = t + C$

$\sqrt{h} = -\frac{t}{20} + C$

$h = \frac{t^2}{400} + C$

$5 = C$

1 - 1 - 0 - 0

$\int \frac{dh}{\sqrt{h}} = -\int \frac{1}{10}\, dt$

$\sqrt{h} = -\frac{1}{10}t + C$

$\sqrt{5} = C$

1 - 0 - 1 - 0

$\int \frac{dh}{\sqrt{h}} = -\int \frac{1}{10}\, dt$

$\ln\sqrt{h} = -\frac{1}{10}t + C$

$\sqrt{h} = e^{-\frac{1}{10}t} + C$

$\sqrt{5} = e^{-\frac{1}{10}(0)} + C$

1 - 0 - 0 - 0

Part (c) 1: $h(t)$

$h(t)$ is an expression for h in terms of t.

The expression must be our answer only.

Some expressions that earn the point:

(1) $\left(-\frac{t}{20}+\sqrt{5}\right)^2$

(2) $\left(\frac{-\frac{1}{10}t+2\sqrt{5}}{2}\right)^2$

(3) $\frac{1}{4}\left(2\sqrt{5}-\frac{t}{10}\right)^2$

This expression does not earn the point:

$-20\sqrt{h}-20\sqrt{5}=t$

Part (c) Examples

$-20\sqrt{h} = C + t$

$\sqrt{h} = C - \frac{t}{20}$

$h = \left(C - \frac{t}{20}\right)^2$

$\sqrt{5} = C$

$\left(\sqrt{5} + \frac{-t}{20}\right)^2$ 1 - 1 - 1 - 1

$-20\sqrt{h} = t + C$

$\sqrt{h} = -\frac{t}{20} + C$

$h = \frac{t^2}{400} + C$

$5 = C$

$h(t) = \frac{t^2}{400} + 5$ 1 - 1 - 0 - 0

$2\sqrt{h} = -\frac{t}{10} + C$

$2\sqrt{5} = -\frac{0}{10} + C$

$2\sqrt{h} = -\frac{t}{10} + 2\sqrt{5}$

$\sqrt{h} = -\frac{t}{20} + \sqrt{5}$

$h = \sqrt{-\frac{t}{20} + \sqrt{5}}$ 1 - 1 - 1 - 0

$\int \frac{1}{\sqrt{h}}\, dh = \int \frac{1}{10}\, dt$

$\sqrt{h} = \frac{1}{10}t + C$

$\sqrt{5} = 0 + C$

$h = \left(\frac{1}{10}t + \sqrt{5}\right)^2$ 0 - 0 - 1 - 0

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