TI in Focus: AP® Calculus

2019 AP® Calculus Exam: AB-4/BC-4
Related Rates

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Outline

(1) Background

(2) Strategy for Solving Related Rates Problems

(3) Problems, Problems, Problems

Background Information

(1) Need to know: differentiation rules, implicit differentiation.

(2) Basic concept: compute the rate of change of one quantity in terms of the rate of change of another quantity.

(3) If $y = f(t)$, then $\dfrac{dy}{dt} = f'(t)$ represents the instantaneous rate of change of y with respect to t (time).

(4) Basic premise:

Suppose two or more quantities which depend on t are related by an equation.

We can obtain an expression relating their rates of change with respect to time (called **related rates**) by using the Chain Rule.

Differentiate both sides of the equation with respect to t.

General idea: compute the rate of change of one quantity in terms of the rate of change of another quantity.

Strategy for Solving Related Rates Problems

(1) Read the problem carefully.

(2) Translate the problem into a figure, if possible.

(3) Use some notation; use variables that are all functions of time.

(4) Write the given information in terms of the variables and rates.

(5) Write an equation that relates the various quantities in the problem.

If possible, use the geometry in the problem to eliminate one of the variables by substitution.

(6) Use the Chain Rule to differentiate both sides of the equation with respect to the variable t.

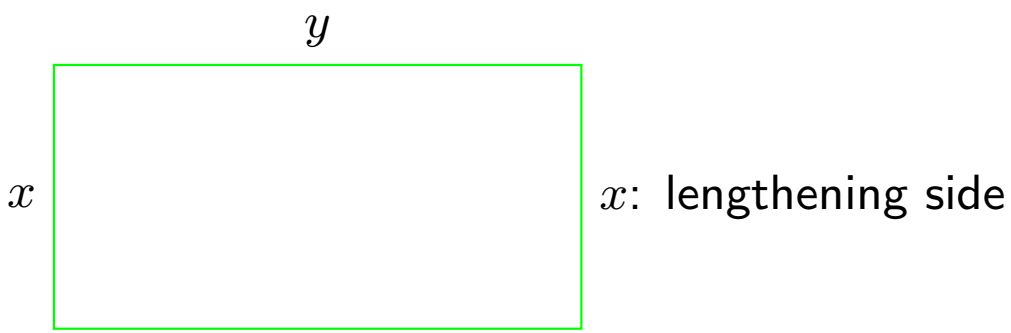
(7) Use the given information to solve for the unknown rate.

Example 1 Related Rates and a Rectangle

A pair of opposite sides of a rectangle are becoming longer at the rate of 5 m/s, while the remaining two sides are decreasing in length such that the figure remains a rectangle with area 1000 m^2. Find the rate of change of the perimeter when the lengthening sides are 40 m.

Solution

Here is a figure with some notation:



Solution (Continued)

Given information:

$x \cdot y = 1000$ Area of the rectangle

$\dfrac{dx}{dt} = +5$ Positive; side x is lengthening, increasing

Perimeter of the rectangle:

$P = 2(x + y)$

Need:

$\dfrac{dP}{dt}$ when $x = 40$

Solution (Continued)

Use the Chain Rule:

$$\frac{dP}{dt} = \frac{d}{dt}\left[2(x+y)\right] = 2\left[\frac{dx}{dt} + \frac{dy}{dt}\right]$$

Given $\frac{dx}{dt}$. Use the expression for area to find $\frac{dy}{dt}$.

$$\frac{d}{dt}[x \cdot y] = \frac{d}{dt}[1000]$$

$$x\,\frac{dy}{dt} + \frac{dx}{dt}\,y = 0$$

Product Rule

When $x = 40$: $\quad 40 \cdot y = 1000 \implies y = 25$

$$40\,\frac{dy}{dt} + (5)(25) = 0 \implies \frac{dy}{dt} = -\frac{125}{40} = -\frac{25}{8} = -3.125$$

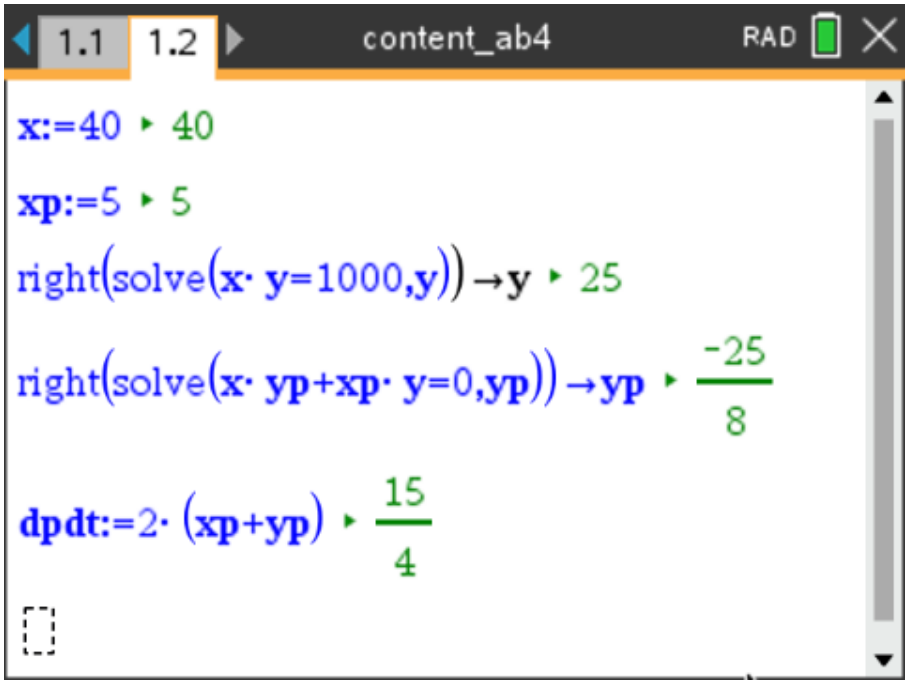
Solution (Continued)

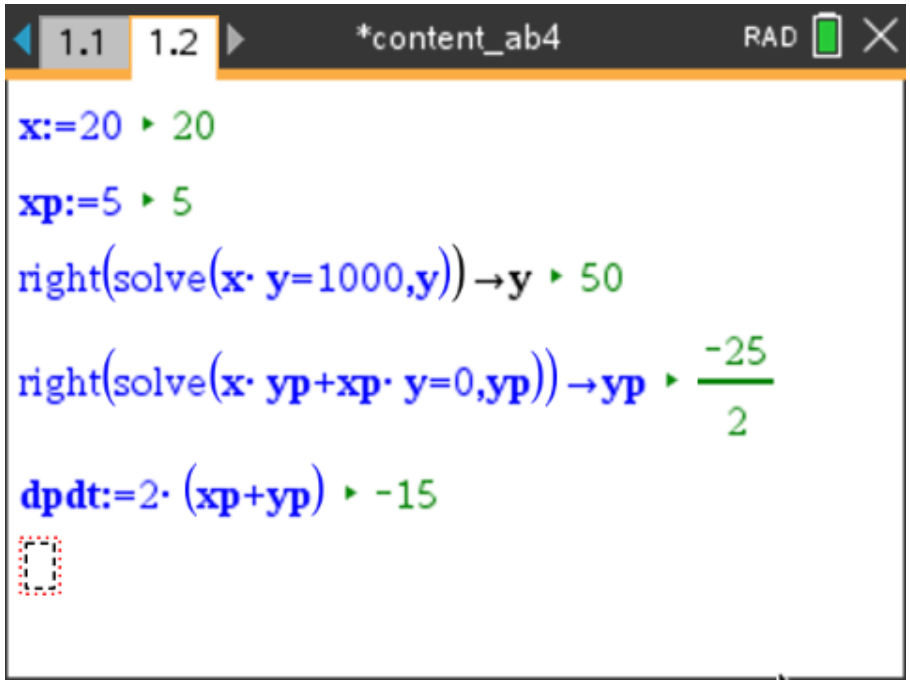
Therefore:

$$\frac{dP}{dt} = 2\left[5 + \left(-\frac{25}{8}\right)\right] = 2 \cdot \frac{15}{8} = \frac{15}{4} = 3.75$$

When $x = 40$, the perimeter is increasing.

Technology





Example 2 The Sliding Ladder

A ladder 13 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 5 feet from the wall?

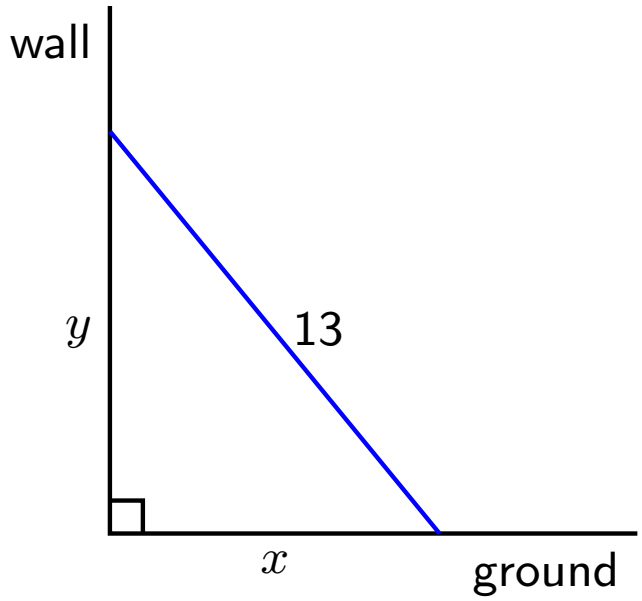
From digg.com

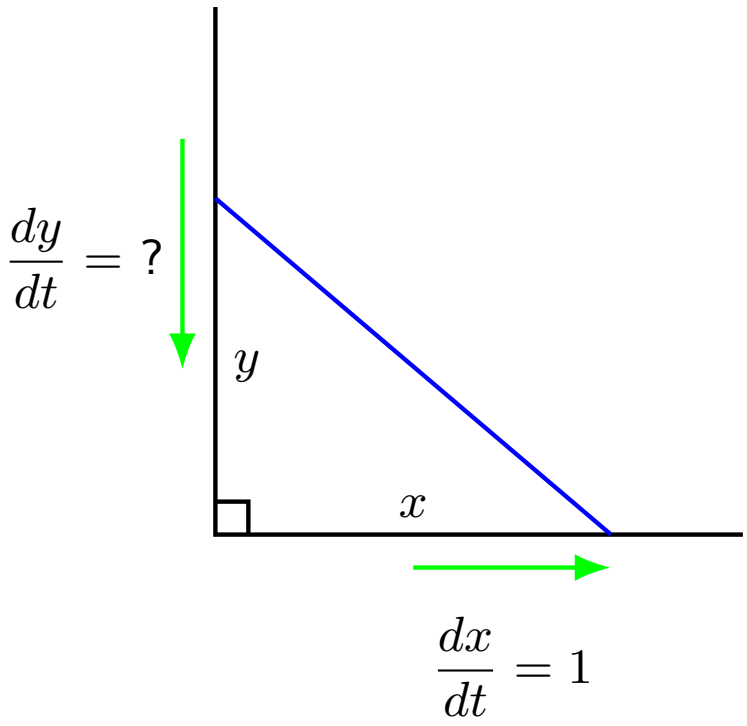
Solution

Draw a diagram.

$x =$ distance from the bottom of the ladder to the wall.

$y =$ distance from the top of the ladder to the ground.





Solution (Continued)

The variables x and y are related using the Pythagorean Theorem.

$$x^2 + y^2 = 13^2 = 169$$

Differentiate each side with respect to t; use the Chain Rule.

$$2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} = 0$$

Solve for the desired rate.

$$\frac{dy}{dt} = -\frac{x}{y}\,\frac{dx}{dt}$$

Solution (Continued)

For $x = 5$: $(5)^2 + y^2 = 169 \implies y = 12$

$$\frac{dy}{dt} = -\frac{5}{12}(1) = -\frac{5}{12}$$

Since $\frac{dy}{dt} < 0$, the distance from the top of the ladder to the ground is decreasing (at a rate of 5/12 ft/s)

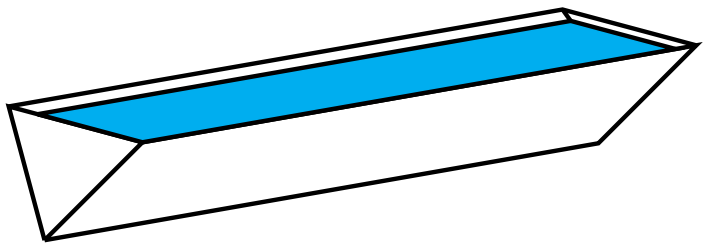
The top of the ladder is sliding the the wall at a rate of 5/12 ft/s.

Example 3 Like Ships Passing in the Night

At noon, ship A is 200 km west of ship B. Ship A is sailing east at 30 km/h and ship B is sailing north at 35 km/h. How fast is the distance between the ships changing at 3:00 pm?

Example 4 You can Lead a Horse to Water

A water trough is in the shape of a triangular prism with length 6 feet. The ends are equilateral triangles of length 2 feet, as shown in the figure.



The volume of a triangular prism of length l and side s is $V = \frac{\sqrt{3}}{4}s^2 l$.

(a) How many cubic feet of water does the trough hold when full?

(b) What is the depth of the water in feet when the trough is half full?

(c) Water is leaking from the bottom of the trough at the rate of 0.25 cubic feet per minute. At what rate is the depth of the water changing, in feet per hour, when the trough is half full?

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