TI in Focus: AP® Calculus

2020 Mock AP® Calculus Exam

BC-2: Solutions, Concepts, and Scoring Guidelines

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Texas Instruments

BC 2

t	0	2	6	8	10	12
$y'(t)$	4	8	-2	3	-1	-5

A particle moves in the coordinate plane with position $(x(t), y(t))$ at time t, where t is measured in seconds and $x(t)$ and $y(t)$ are twice-differentiable functions, both measured in meters.

For all times t, the x-coordinate of the particle's position has derivative $x'(t) = \dfrac{t}{\sqrt{t^2+25}}$.

Selected values of $y'(t)$, the derivative of $y(t)$, over the interval $0 \le t \le 12$ seconds are shown in the table.

The position of the particle at time $t = 12$ is $(x(12), y(12)) = (4, -3)$.

(c) Using Euler's method, starting at time $t = 12$ with 4 steps of equal size, approximate $y(4)$, the y-coordinate of the position of the particle at $t = 4$.

Key Concepts

Euler's Method

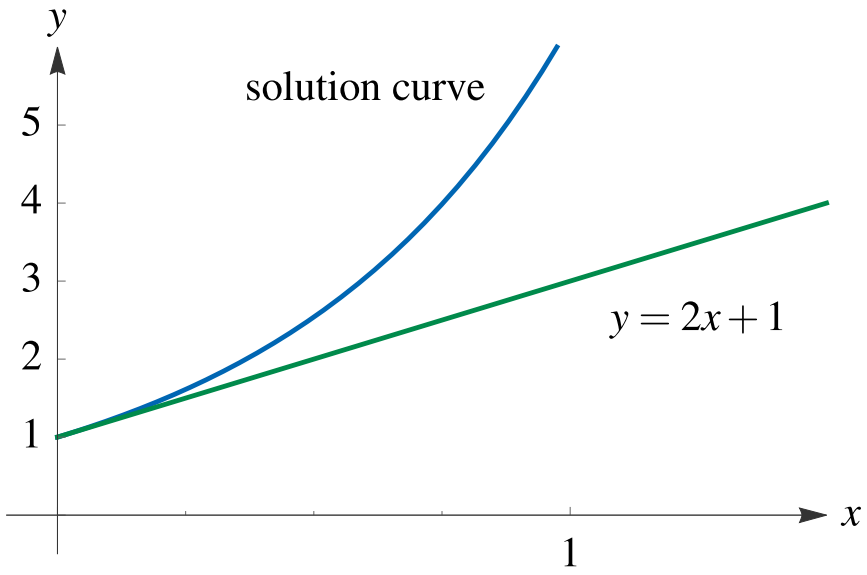
Goal: Find a numerical approximation to the solution of a differential equation.

Consider the initial-value problem: $y' = 2y - x, \;\; y(0) = 1$

Use the DE: $y'(0) = 2(1) - 0 = 2$

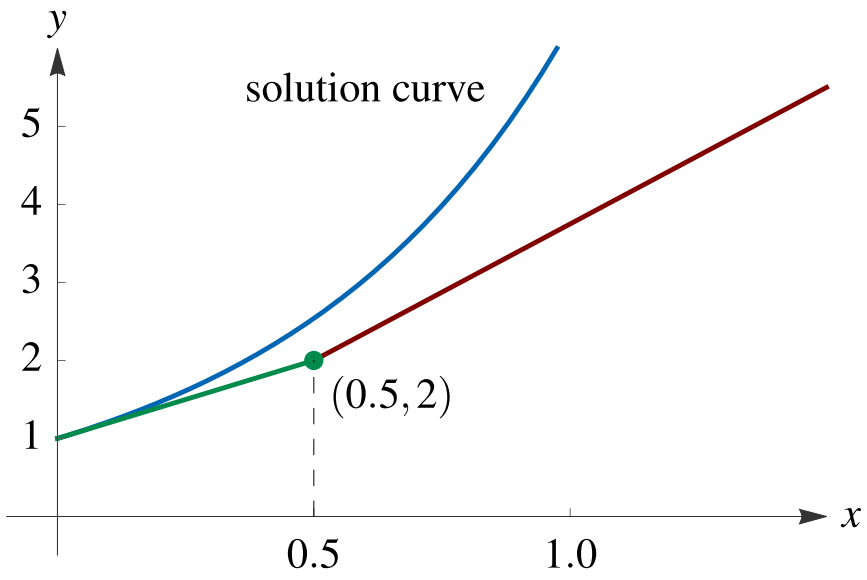
The solution curve as slope 2 at the point $(0, 1)$

A first approximation to the solution curve: $L(x) = 1 + 2(x - 0) = 2x + 1$



Euler's Method

- Improve on this approximation by proceeding only a short distance along this tangent line and then make a midcourse correction.
- Change direction as indicated by the slope field.
- Start out along the tangent line; stop when $x = 0.5$.
  The horizontal distance traveled is the step size.



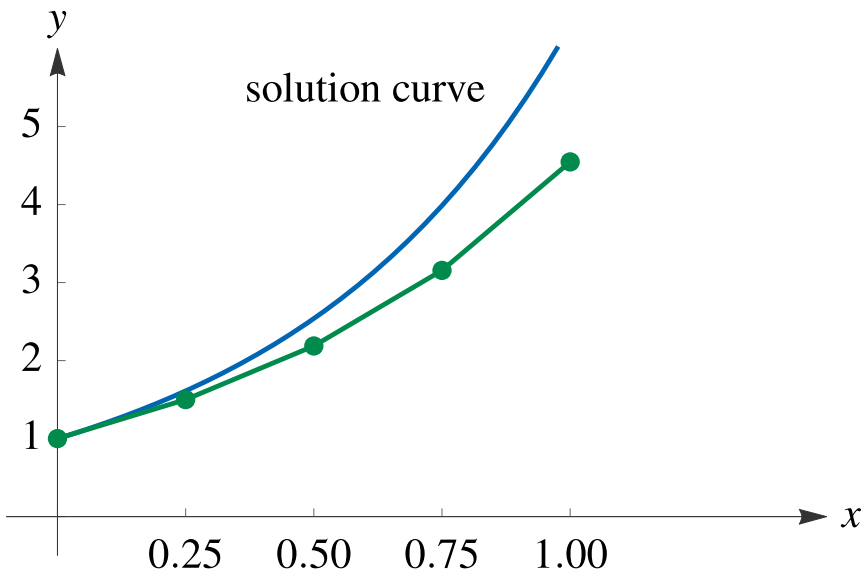
$L(0.5) = 2(0.5) + 1 = 2 \quad \Rightarrow \quad y(0.5) \approx 2$

Take $(0.5, 2)$ as a new starting point; a new tangent line.

$y'(0.5) = 2(2) - 0.5 = 3.5 \quad \Rightarrow \quad L(x) = 2 + 3.5(x - 0.5) = 0.25 + 3.5x$

Euler's Method

- Start at a point given by the initial value and proceed in the direction indicated by the slope field.
- Stop after a short distance; consider the slope at the new location, and proceed in a new direction.
- Continue stopping and adjusting direction.
- Decreasing the step size; we get a better approximation to the exact solution curve.



Euler's Method

Approximate values for the solution of the initial-value problem $y' = F(x, y)$, $y(x_0) = y_0$, with step size h, at $x_n = x_{n-1} + h$, are

$$y_n = y_{n-1} + h \cdot F(x_{n-1}, y_{n-1}) \qquad n = 1, 2, 3, \ldots$$

A Closer Look

1. The step size in Euler's method can be negative.
2. Smaller step size $\Rightarrow$ more accurate approximation.
3. Using Leibniz notation: $h = \Delta x \quad \Rightarrow \quad y_n = y_{n-1} + \Delta x \cdot \dfrac{dy}{dx}$

Solution

Initial value: $y(12) = -3$. $\quad F(x, y) = y'(t)$

We need to get to $t = 4$ in four steps $\quad \Rightarrow \quad h = -2$.

$$y_1 = y_0 + h \cdot y'(t_0)$$

$$y(10) \approx y(12) + (-2) \cdot y'(12) = -3 + (-2)(-5) = 7$$

$$y_2 = y_1 + h \cdot y'(t_1)$$

$$y(8) \approx y(10) + (-2) \cdot y'(10) = 7 + (-2)(-1) = 9$$

$$y_3 = y_2 + h \cdot y'(t_2)$$

$$y(6) \approx y(8) + (-2) \cdot y'(8) = 9 + (-2)(3) = 3$$

$$y_4 = y_3 + h \cdot y'(t_3)$$

$$y(4) \approx y(6) + (-2) \cdot y'(6) = 3 + (-2)(-2) = 7$$

Using Euler's method with four steps, $y(4) \approx 7$.

Scoring Guidelines

(c) $y(10) \approx y(12) + (-2) \cdot y'(12) = -3 + (-2)(-5) = 7$

$y(8) \approx y(10) + (-2) \cdot y'(10) = 7 + (-2)(-1) = 9$

$y(6) \approx y(8) + (-2) \cdot y'(8) = 9 + (-2)(3) = 3$

$y(4) \approx y(6) + (-2)y'(6) = 3 + (-2)(-2) = 7$

$$2: \begin{cases} 1 : \text{First step in Euler's method} \\ 1 : \text{answer} \end{cases}$$

Scoring Notes

- The first point is earned for presenting the first step of Euler's method with at most one error.
- Some possible responses that indicate on error:
  - Positive step: $-3 + (2)(-5) = -13$
  - Uses starting x-value: $4 + (-2)(-5) = 14$
- If an unlabeled table is presented, a correct answer earns both points.
- An incorrect final answer from an unlabeled final answer earns no points.
- The second point is earned only for the correct answer.
- The final answer does not need to be simplified.

(d) Find an equation of the line tangent to the path of the particle at time $t = 12$.

Key Concepts

The slope of the tangent line to the graph of $y = f(x)$ at the point (x_0, y_0) is $f'(x_0)$.

An equation of the tangent line:

$y = f'(x_0)(x - x_0) + y_0$

Suppose the graph is defined by the parametric equations $x = f(t)$, $y = g(t)$.

If f and g are differentiable functions and y is a differentiable function of x, then

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If $\frac{dx}{dt} \neq 0$, then solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Solution

$$\left.\frac{dy}{dx}\right|_{t=12} = \frac{y'(12)}{x'(12)} = \frac{-5}{\frac{12}{\sqrt{12^2+25}}} = -\frac{65}{12}$$

Point: $(4, -3)$ Slope: $-\frac{65}{12}$

An equation of the tangent line is

$$y - (-3) = -\frac{65}{12}(x-4) \quad \Rightarrow \quad y = -\frac{65}{12}(x-4) - 3$$

Scoring Guidelines

(d) $\left.\dfrac{dy}{dx}\right|_{t=12} = \dfrac{y'(12)}{x'(12)} = \dfrac{-5}{\dfrac{12}{\sqrt{12^2+25}}} = -\dfrac{65}{12}$

An equation of the tangent line at the position $(x(12), y(12)) = (4, -3)$ is

$$y = -\frac{65}{12}(x-4) - 3$$

2: $\begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line equation} \end{cases}$

Scoring Notes

1: slope

- The slope does not have to be labeled.
- The slope can appear unannounced as part of a correct tangent line: earns 2 points.
- Examples
  - $y = -\frac{65}{12}(x-4) - 3$ 1 - 1
  - $y = -5.417(x-4) - 3$ 1 - 1
  - $y + 3 = -\frac{65}{12}(x-4)$ 1 - 1

1: tangent line equation

- The second point can be earned with an incorrect slope if

  (1) the line passes through $(4, -3)$, and

  (2) the slope is labeled as $\frac{dy}{dx}$, but not m or *slope*.

Scoring Notes

Examples

(1) $\frac{dy}{dx} = -5 \qquad y + 3 = -5(x - 4)$ 0 - 1

(2) $\frac{dy}{dx} = -3 \qquad y = -3(x - 4) - 3$ 0 - 1

(3) $\frac{dy}{dx} = \frac{12}{\sqrt{12^2 + 25}} = \frac{12}{13} \qquad y = \frac{12}{13}(x - 4) - 3$ 0 - 1

(4) $m = 5 \qquad y = 5(x - 4) - 3$ 0 - 0

(5) $\frac{dy}{dx} = -5.416 \qquad y = -5.416(x - 4) - 3$ 1 - 1

(6) $\frac{dy}{dx} = -\frac{65}{12} \quad \frac{y + 3}{x - 4} = -\frac{65}{12}$ 1 - 0

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