TI in Focus: AP® Calculus

2020 Mock AP® Calculus Exam

BC-2: Solutions, Concepts, and Scoring Guidelines

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Texas Instruments

BC 2

t	0	2	6	8	10	12
$y'(t)$	4	8	-2	3	-1	-5

A particle moves in the coordinate plane with position $(x(t), y(t))$ at time t, where t is measured in seconds and $x(t)$ and $y(t)$ are twice-differentiable functions, both measured in meters.

For all times t, the x-coordinate of the particle's position has derivative $x'(t) = \dfrac{t}{\sqrt{t^2+25}}$.

Selected values of $y'(t)$, the derivative of $y(t)$, over the interval $0 \le t \le 12$ seconds are shown in the table.

The position of the particle at time $t = 12$ is $(x(12), y(12)) = (4, -3)$.

(e) Let $r(t)$ be the distance between the particle and the origin $(0, 0)$ at time t. Find $r'(12)$.

Key Concepts

Distance Formula

The distance between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and $F'(x)$ is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

A Closer Look

1. In words: the derivative of the composition of two functions is the product of the derivative of the *outer* function evaluated at the *inner* function and of the derivative of the inner function.

2. Other notation: $(f \circ g)' = f'(g(x)) \cdot g'(x)$

3. And more notation: $\dfrac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

4. Given $(f \circ g)(x)$, here is a procedure for finding $(f \circ g)'(x)$ using the Chain Rule.
   (a) Identify $f(x)$ and $g(x)$.
   (b) Find $f'(x)$ and $g'(x)$.
   (c) Write the final answer as $f'(g(x))\, g'(x)$.

5. Chain Rule: differentiate the outer function f [at the inner function $g(x)$] and then we multiply by the derivative of the inner function.

$$\frac{d}{dx} \underbrace{f}_{\text{outer function}} \underbrace{(g(x))}_{\text{evaluated at inner function}} = \underbrace{f'}_{\text{derivative of outer function}} \underbrace{(g(x))}_{\text{evaluated at inner function}} \cdot \underbrace{g'(x)}_{\text{derivative of inner function}}$$

Solution

The position of the particle at time t is $(x(t), y(t))$.

The distance between the particle and the origin at time t:

$$r(t) = \sqrt{[x(t)-0]^2 + [y(t)-0]^2} = \sqrt{x(t)^2 + y(t)^2}$$

$$r'(t) = \frac{1}{2}[x(t)^2 + y(t)^2]^{-1/2} \cdot [2x(t)x'(t) + 2y(t)y'(t)]$$ Chain Rule

$$= \frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{x(t)^2 + y(t)^2}}$$ Simplify

$$r'(12) = \frac{x(12)x'(12) + y(12)y'(12)}{\sqrt{x(12)^2 + y(12)^2}}$$ Use $t = 12$

$$= \frac{(4)\left(\frac{12}{13}\right) + (-3)(-5)}{\sqrt{4^2 + (-3)^2}}$$ $x(12)$ and $y(12)$ given; use expression for $x'(t)$ to find $x'(12)$; read $y'(12)$ from the table.

$$= \frac{243}{65}$$ Simplify

Scoring Guidelines

(e) $r(t) = \sqrt{x(t)^2 + y(t)^2}$

$$r'(t) = \frac{1}{2}(x(t)^2 + y(t)^2)^{-1/2}(2x(t)x'(t) + 2y(t)y'(t))$$

$$= \frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{x(t)^2 + y(t)^2}}$$

$$r'(12) = \frac{x(12)x'(12) + y(12)y'(12)}{\sqrt{x(12)^2 + y(12)^2}}$$

$$= \frac{(4)\left(\frac{12}{13}\right) + (-3)(-5)}{\sqrt{4^2 + (-3)^2}}$$

$$= \frac{243}{65}$$

$$3: \begin{cases} 1 : \text{expression for } r(t) \\ 1 : \text{expression for } r'(t) \\ 1 : \text{answer} \end{cases}$$

Scoring Notes

1: expression for $r(t)$

- No additional work necessary.
- If $r(t)$ is not presented explicitly, $r'(t)$ must be correct to earn the first point.
- If $r'(t)$ is not correct: 0/3

1: expression for $r'(t)$

- Earned for a correct derivative.
- Any simplification error: not eligible for the third point.

1: answer

- Must be our answer.
- Functions must be evaluated
- Final numerical answer does not need to be simplified.

(f) Is the particle moving closer or further from the origin at time $t = 12$? Justify your answer.

Key Concepts

Definition

A function f is **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever} \quad x_1 < x_2 \text{ in } I$$

The function is **decreasing** on I if

$$f(x_1) > f(x_2) \quad \text{whenever} \quad x_1 < x_2 \text{ in } I$$

Increasing/Decreasing Test (I/D Test)

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Solution

$$r'(12) = \frac{243}{65} > 0$$ $r'(12)$ from part (e)

Since $r'(12) > 0$ the distance r to the origin is increasing at time $t = 12$.

Therefore the particle is moving away from the origin at time $t = 12$ seconds.

Scoring Guidelines

(f) $r'(12) = \dfrac{243}{65} > 0$ The distance r to the origin is increasing at time $t = 12$. Therefore the particle is moving away from the origin at time $t = 12$ seconds.	1 : answer with reason

Scoring Notes

- Must be considering the function $r'(t)$.
- Must be convinced the student is working with $t = 12$.
- Must commit to *increasing*.
- Do not need a numerical value for $r'(12)$.
- Can import an incorrect value from part (e). Answer must be consistent.
- Minimal examples:
  - $r'(12) > 0$ increasing
  - $r'(12) = 3.739$ increasing

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